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Pricing Embedded Options Using Fast Fourier Transform to Compare Variance Gamma and Black-Scholes-Merton Model Efficiency

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Abstract

Embedded options are virtually new instruments identical to options in many aspects except their non-tradable nature. Testing the efficiency of the Variance Gamma and Black-Scholes-Merton model on these instruments would provide a vision of transitioning from the classical model with its deficiency to more intricate models. Considering the complicated nature of the Variance Gamma stochastic process to price options, the Fast Fourier Transform (FFT) method is used in conjunction with the Nelder-Mead Simplex method to calibrate models. This research uses the Fast Fourier Transform (FFT) to price four embedded options with the ticker symbols Hefars912, Heghadir912, Heksho208, and Hetrol911 under the two models. The result approves that the Variance Gamma process is more efficient than the Black-Scholes-Merton model in pricing embedded options. Consequently, the variance gamma process would generate fewer errors in pricing those options that can be used in a practical sense.

Keywords: Embedded Options, Option Pricing, Stochastic Processes, Fast Fourier Transform, Variance Gamma process, Black-Scholes-Merton model.

Introduction

Options are one of the fundamental tools of financial derivatives which can be utilized to hedge portfolio risk. It is evident that pricing options assume importance when the demand for these tools soared in recent years. The most famous option pricing model, Black-Scholes-Merton, proposed in 1973, is still the most ubiquitous model in financial markets. Indeed, it is the most convenient model because of its closed-form formula by which investors can easily calculate option prices. However, some slight defects in its assumptions make the model less precise regarding pricing. Firstly, it is assumed that the underlying asset follows the log-normal distribution while the asset shows a fat-tailed distribution in fundamental stock markets. The other unrealistic assumption in the BSM model is the invariability of volatility during the option's life. However, the volatility for the deep-in-the-money and out-of-the-money prices shows an increase, which is known as a volatility smile. After this model, many attempts have been made to modify it and make the assumptions more realistic. For instance, the Heston model was proposed by Steven Heston in 1993 to modify the BSM model in terms of volatility. The model's most characteristic feature was that the underlying asset's volatility followed its own stochastic process. As a result, Heston proposed a model with five parameters instead of just one parameter in the BSM model.

Moreover, in contrast to the BSM model, Heston introduced a method to allow the interest rate to be a stochastic process, which made the model appropriate to model bond options and foreign currency options (Steven et al., 1993). In addition to the aforementioned points, another phenomenon is observed in fundamental financial markets, which should be noted in modeling; it is "jump" in reality, which was not considered in Brownian motion. In fact, jump assumes importance in modeling financial events such as stock price, which has both continuity and jump nature. In this sense, an important question may be asked: "Does adding jump movement into our models make it more efficient?". This crucial question is answered in this paper. For this purpose, the Variance Gamma process was chosen to model embedded options to compare the “pure jump process” efficiency with Brownian motion, which has a continuous nature without a jump. The Variance Gamma process is defined as the Brownian motion process with a replaced time by gamma time (Weilong, Hirsa, 2019).

Regarding embedded options, customized options have been made in recent years in order to fulfill the demand for hedging in financial markets. The options Market in Iran commenced in 2016 with only eight option chains. Currently, about 50 option chains are tradable with call-and-put options. However, there are other kinds of options, namely embedded options, which are not allowed to trade and are just for hedging the investment portfolio from declining. These derivatives have recently assumed great importance in Iran because they are used as corporate financing instruments. For instance, the embedded option with the ticker symbol Heksho, issued by Pakshoo Co., has been traded for 3 years.

Literature Review

In order to price options in financial markets in a practical sense, investors use the Black-Scholes-Merton model because of its non-stochastic nature in its formula; as a result, it facilitates option pricing. One of the most critical assumptions in deriving the Black-Scholes formula is that it does not contain jumps in its stochastic process, which leads to not considering large jumps in market prices. In addition, in the Black-Scholes formula, the only parameter that controls the distribution of the prices in a Brownian motion process is volatility, which leads to missing out control of kurtosis and skewness of distribution in price modeling of the underlying. However, the price jumps in conjunction with the fat-tailed distribution in financial markets. The variance Gamma model refines the two aforementioned flaws in the Black-Scholes formula in that it controls the variance, skewness, and kurtosis of distribution by three parameters; moreover, the Variance Gamma process is a pure jump process that includes price jumps happening in reality. The Variance Gamma process can be advantageous when pricing options since it allows for broader modeling of skewness and kurtosis than the Brownian motion does (Madan et al., 1990).

Hirsa and Madan derived the partial integro-differential equation (PIDE) for pricing American options under the Variance Gamma process. They developed a numerical algorithm to solve for values of American options under the variance gamma model. (Hirsa, B.Madan, 2001)

Since the VG process is one of finite variation, it can be written as the difference of two increasing processes, the first of which accounts for the price increases, while the second explains the price decreases. In the case of the VG process, the two increasing processes that are differenced to obtain the VG process are themselves gamma processes (Madan et al., 1998):

$$X(t;\sigma,\upsilon,\theta) = \gamma_p\left(t,\mu_p,\upsilon_p\right) - \gamma_n(t;\mu_n,\upsilon_n) \quad (1)$$

The VG process $X(t;\sigma,\upsilon,\theta)$ is defined by:

$$X(t;\sigma,\upsilon,\theta) = b(\gamma(t;1,\upsilon);\theta,\sigma) \quad (2)$$

Which is obtained by evaluating Brownian motion (with constant drift and volatility) at a random time change given by a gamma process (Madan et al., 1990).

The characteristic function of the VG process is as follows:

$$\phi_{X(t)}(u) = \left(\frac{1}{1-i\upsilon\theta u+\frac{\sigma^2\upsilon}{2}u^2}\right)^{\frac{t}{\upsilon}} \quad (3)$$

The random process used by Black and Scholes to model the underlying stock price was Geometric Brownian motion. The characteristic function of that is as follows:

$$\phi(u) = e^{i\left(LnS_0+\left(r-q-\frac{\sigma^2}{2}\right)T\right)u-\frac{\sigma^2\upsilon^2}{2}T} \quad (4)$$

Fast Fourier Transform (FFT) for option pricing

We used the Fast Fourier Transform (FFT) method to price the put options using the characteristic VG process and Brownian motion functions.

By transforming the log-normal space, we define the strike and underlying asset prices at maturity as K = e k and ST = e ST, respectively. Then we would have for the price of a European put option:

$$\mathbb{E}_t[(K-S_T)^+] = \int_0^K (K-S_T)f(S_T)dS_T$$

$$= \int_{-\infty}^{K} \left(e^{k} - e^{s_T}\right) q(s_T) ds_T$$

$$= \int_{-\infty}^{K} \left(e^{k} - e^{s}\right) q(s) ds_T$$

$$= P_T(k) \tag{5}$$

We define control parameter α and then $p_T(k)$ as:

$$p_T(k) = e^{\alpha k} P_T(k) \tag{6}$$

By utilizing the Fourier transform, we have:

$$\psi_T(v) = \int_{-\infty}^{\infty} e^{ivk} p_T(k) dk \tag{7}$$

By substitution of $p_T(k)$ from (5) and (6) and solving the integral of inverse Fourier transform, we would have:

$$\psi_T(v) = \frac{1}{(\alpha+iv)(\alpha+iv+1)} \phi(v - (\alpha+1)i) \tag{8}$$

For every random process, we have the characteristic function of $\phi(v)$, and then by using the inverse Fourier transform, we can calculate the put option price:

$$P_T(k) = \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-ivk} \psi_T(v) dv \tag{9}$$

Research Methodology

This paper applied the VG and Black-Scholes model to four embedded options issued over two years in the Iran Fara Bourse Securities Exchange (IFB). Both embedded and plain vanilla options are traded in IFB. The Special feature of the embedded options, which differentiates them from plain vanilla options, is that they are not tradable during their life; besides that, the primary purpose of purchasing them is to protect the underlying asset from declining in a specific period. In contrast to vanilla put options, which have an option chain with different strike prices, embedded options have just one strike price. Moreover, in order to purchase the embedded option, you must have the underlying asset.

Choosing appropriate options to price

In order to choose option contracts among others, we considered two factors to judge whether option prices are fair and comparable to the theory price from our models. Firstly, the volume of the trades in a specific trading day was maximum, among other options. Secondly, the number of trades on that day was maximum as well. These two conditions ensure that the options were trading between small-scale investors, which is an acceptable indicator to suggest that the price of those trades was fair. Between all embedded options, we chose four of them with the ticker symbols “Hefars912”, “Heksho208”, “Hetrol911”, and “Heghadir912”. Table 1 shows information about the four chosen embedded options.

Table 1. Information on Embedded Options

Embedded Option (Ticker Symbol)	Company Name	Strike Price
Hefars912	Persian Gulf Petrochemical Industries Co.	14,750
Heghadir912	Ghadir Investment Company	17,380
Heksho208	Pakshoo Industrial Group	9,979
Hetrol911	Iranian Petrochemical Investment Group	14,730

Source: rahavard365.com

In figures 1-4, we can see the price fluctuations of the four embedded options in terms of underlying asset price and time to maturity. It is obvious in the figures that there are some intervals in which the option price goes near almost zero. They are the days the options were traded between investors who compromised on 1 IRR option price. It is conventional in the market to reach an agreement on a fixed price without considering the theoretical and intrinsic value of the option. However, we tried to choose options with the least compromised prices in terms of their time interval, and this would not impose any restriction on our research except that it would increase the error function by just 5-7%. Because the 5-7% shift in error function would be done on both of our pricing models, it would not debunk our comparison and judgment about the models. As a result, we did not trim our data because of the compromised prices, and we kept the origin data at the expense of increasing the error function by the aforementioned percentage.

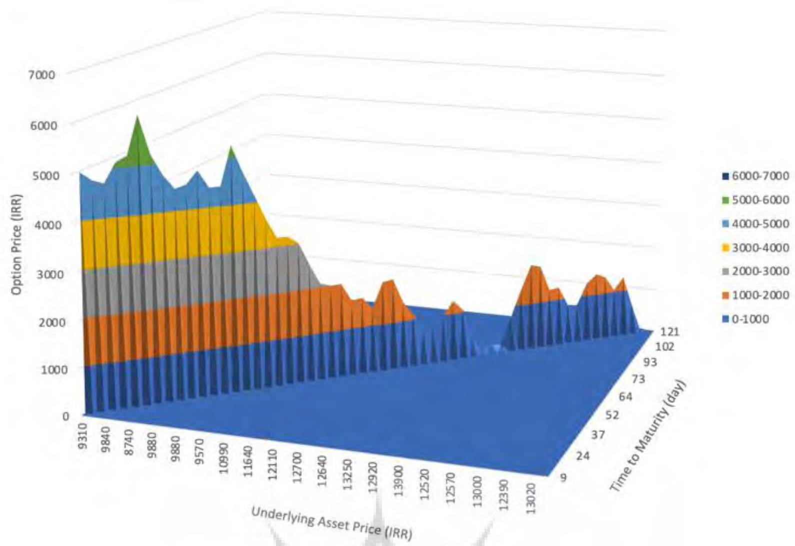

Figure 1. Price Fluctuations of "Hefars912" in terms of underlying asset price and time to maturity

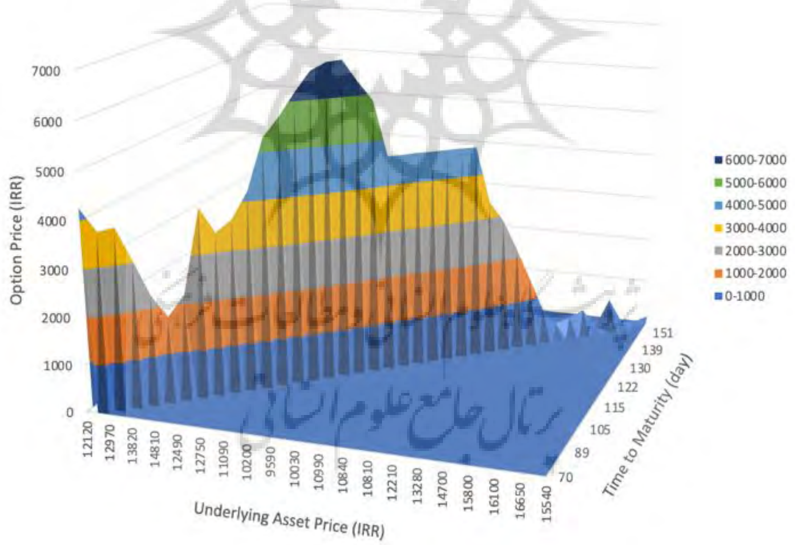

Figure 2 Price Fluctuations of "Heghadir912" in terms of underlying asset price and time to maturity

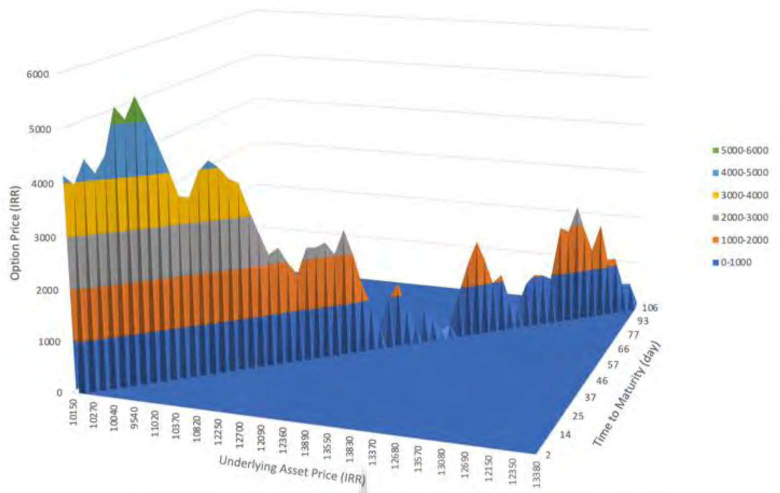

Figure 3. Price Fluctuations of "Hetrol911" in terms of underlying asset price and time to maturity

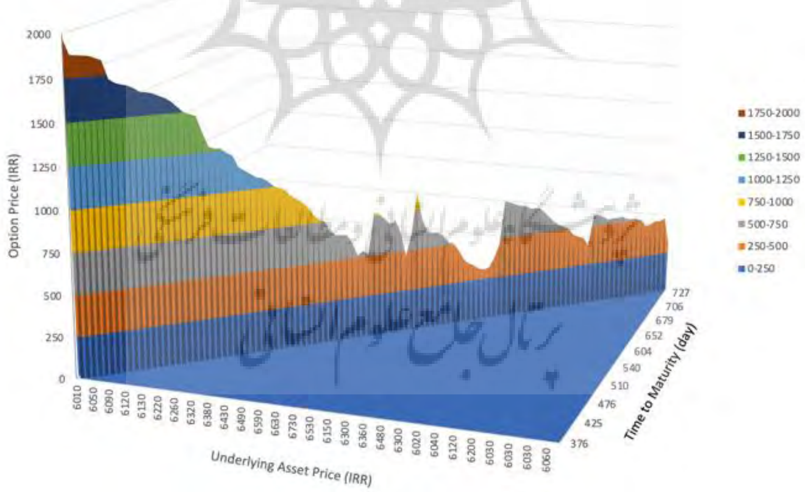

Figure 4. Price Fluctuations of "Heksho208" in terms of underlying asset price and time to maturity

Calibration

In this research, we utilized three error functions to calibrate our models with market prices and find the best parameters; in addition, using three error functions gives us the edge to have a broader point of view to compare the VG and Black-Scholes models. Moreover, we used the Nelder-Mead Simplex algorithm to minimize the error functions because of the non-smooth behavior of the error functions. Three error functions are as follows:

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}|C_i^{Model}(0.\phi) - C_i^{Market}(0)|^2} \quad (10)$$

$$APE = \frac{\sum_{i=1}^{n}|C_i^{Model}(0.\phi) - C_i^{Market}(0)|}{\sum_{i=1}^{n} C_i^{Market}(0)} \quad (11)$$

$$AAE = \frac{1}{n}\sum_{i=1}^{n}|C_i^{Model}(0.\phi) - C_i^{Market}(0)| \quad (12)$$

Because of its time-consuming nature, we did not utilize the Brute-Force search through all surface error functions. To mention the expense of time process, optimizing just one error function through our Python code using the Brute-Force algorithm took 13 hours! Consequently, we sacrificed complete reliability because it would lengthen the optimization process. However, the Nelder-Mead algorithm has its own advantage if we choose the appropriate initial points, which were chosen with more scrutiny in the research.

Results

In this section, we have the results of the pricing models after the calibration process with their optimum parameters. In the Black-Scholes model, we have just one parameter, "s," whereas in the VG model, we have three parameters, σ, υ, and θ. In the tables below, we have the results of those parameters and their error measures for every error function. It is worth mentioning that the order of RMSE and AAE error functions are the same as the order of original market data, while APE is reported as a percentage error.

Black-Scholes model

Table 2. Error and parameter measures of the Black-Scholes model for embedded options

	Error Function	Error Measure	Parameter Measure
Hefars912	RMSE	417.6886236	0.051375
	APE	0.1714622	0.0151475
	AAE	357.67257329	0.015194
Heghadir912	RMSE	502.210961	0.0634
	APE	0.12829155	0.037625
	AAE	431.3775292	0.0375918
Heksho208	RMSE	117.52303089	0.006
	APE	0.13117901	0.006
	AAE	114.53567111	0.006
Hetrol911	RMSE	430.41682553	0.13494727
	APE	0.1739215	0.13172656
	AAE	379.59991467	0.13169922

Source: Python

Table 3. Error and parameter measures of Variance-Gamma model for embedded options

	Error Function	Error Measure	Parameter Measure		
			σ	υ	θ
Hefars912	RMSE	379.48556568	0.03	-1	0.001
	APE	0.118820114	1.31499639	-1.3616335	1.90103423
	AAE	309.11430378	-1.34127728	-0.36891742	0.89972708
Heghadir912	RMSE	463.3343814	0.0154506	-5.03167047	-0.09824263
	APE	0.1207623	-0.11016	-0.71328875	-0.3272998
	AAE	406.05970225	-0.11058	-0.713270952	-0.32732291
Heksho208	RMSE	115.1154737	0.02209838	-1.2191	0.1465
	APE	0.12179541	0.02034583	-1.07853724	0.14597878
	AAE	106.34261675	0.09035	-1.08	0.143019
Hetrol911	RMSE	429.76484261	0.05524295	0.05159103	-0.6727051
	APE	0.17193363	0.01809469	2.38577114	-0.25025168
	AAE	375.607186	0.10357405	1.02174333	-0.26282209

Source: Python

Comparing Variance-Gamma and Black-Scholes model

Table 4. Error and parameter measures of Variance-Gamma model for embedded options

	Error Function	Error Measure for VG model	Error Measure for BSM model
Hefars912	RMSE	379.48556568	417.6886236
	APE	0.118820114	0.141762275
	AAE	309.11430378	357.6725733
Heghadir912	RMSE	463.33438135	502.210961
	APE	0.120762297	0.12829155
	AAE	406.0597022	431.37752924
Heksho208	RMSE	115.1154737	117.5230312
	APE	0.1217954093	0.131179
	AAE	106.34261675	114.53567
Hetrol911	RMSE	429.7648426	430.4168255
	APE	0.17193363	0.173921471
	AAE	375.6071869	379.5999146

Source: Python

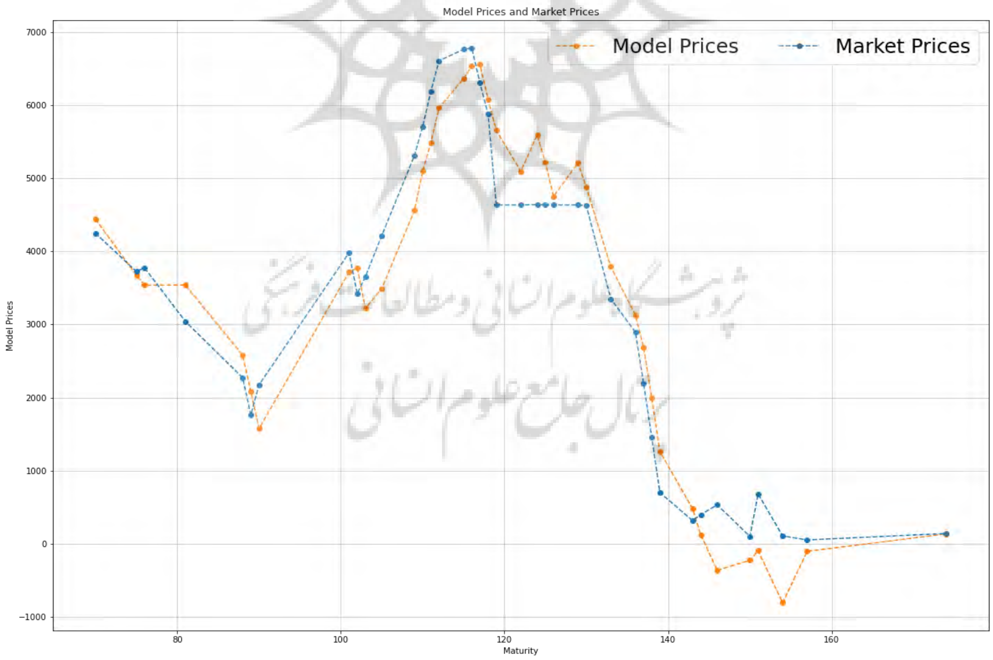

Figure 5. Comparing the VG model with market prices, "Hefars912."

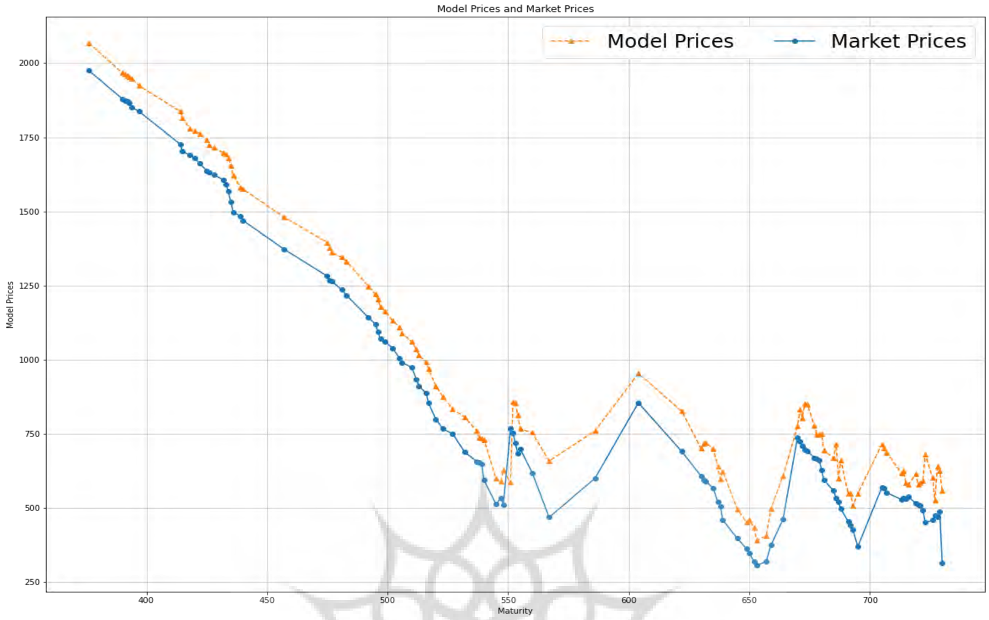

Figure 6. Comparing the VG model with market prices, "Heghadir912."

Source: Python

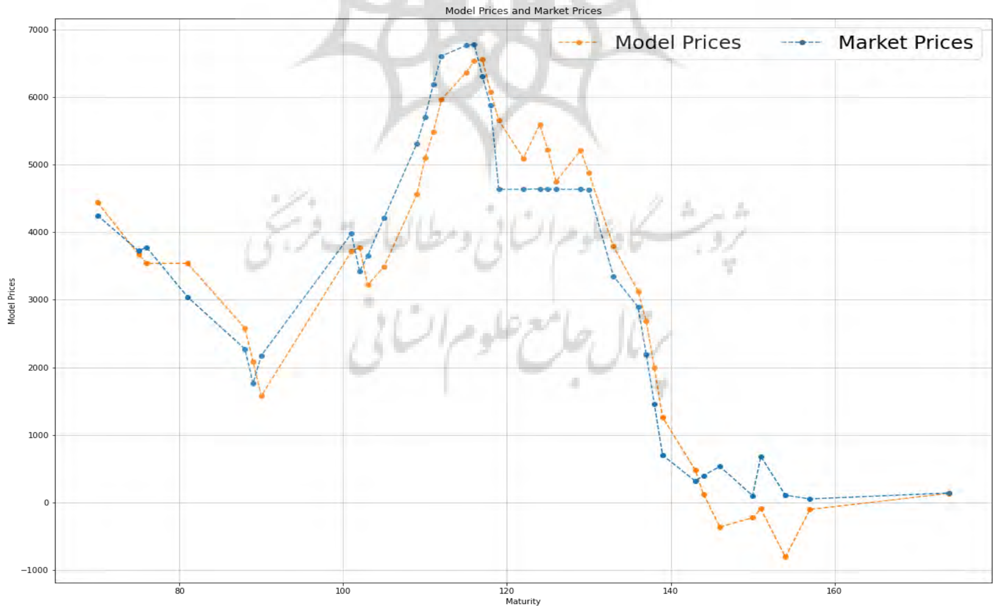

Figure 7. Comparing the VG model with market prices, "Heksho208."

Source: Python

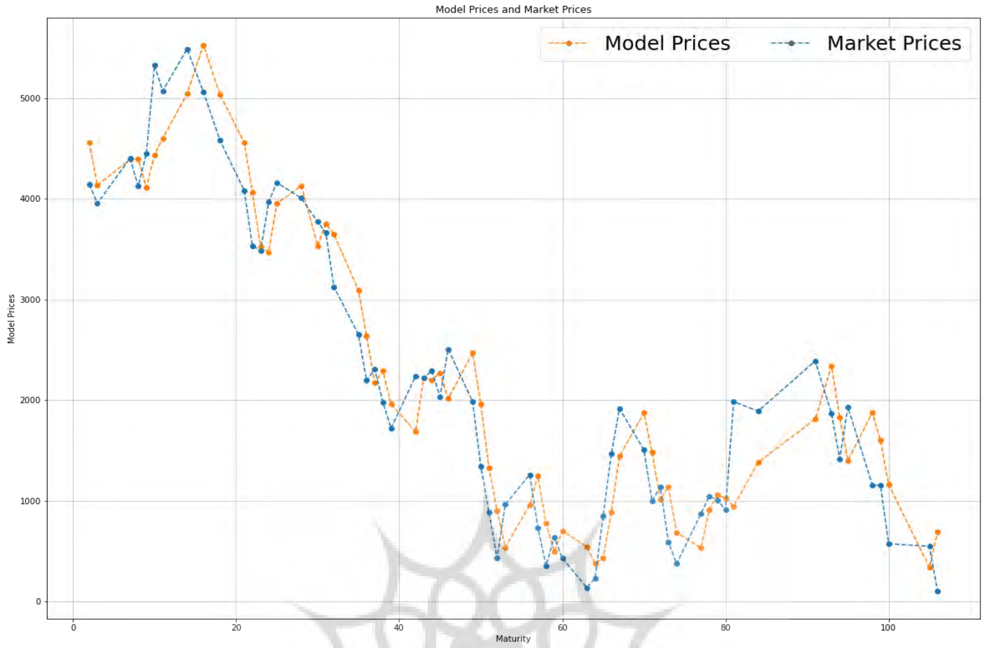

Figure 8. Comparing the VG model with market prices, "Hetrol911."

Source: Python

Discussion and Conclusion

Generally, for all four embedded options, the VG model produced fewer error measures for every error function. In conclusion, the VG model would be more precise in pricing embedded options, but it is more complicated to solve and find the option price. However, because of our powerful computational tools, we can substitute the Black-Scholes model for the VG model because of its precision. In order to compare our results, a virtually identical article was published on embedded options by Jenabi, O., & Dahmarde Ghaleno, N. (2019). We should have a consistent criterion to compare the results with that article; the APE error function is appropriate because of its percentage results. In the article of Jenabi, O., & Dahmarde Ghaleno, N. (2019), the APE error function was 59% to 93%, while in this paper, the APE is 11% to 18%. Nevertheless, we should consider the actual market price of those options in practice. In the research date of that article, the options were traded at an agreed price, which impacted the result of the article.

In the following sections, the results of this research for VG model and BSM model were reported and compared:

Hefars912

As shown in Table 10, the error measure of the VG model for "Hefars912" is less than the amount of the BSM model in the RMSE error function by approximately 37. While the APE error function is %11.9 for VG, it is %14.18 for the BSM model. The error measure of AAE for VG is less than that of the BSM model. As a result, the VG model showed better performance in option pricing by producing fewer error measures.

Heghadir912

Table 11 provides us with the same result as "Hefars912". The error measures of RMSE for the VG and BSM models are 463.33 and 502.21, respectively. The spread between the two error measures of the APE function is far narrower than that of "Hefars912"; nevertheless, the VG model produced fewer error measures. The AAE function for the two models showed 25 differences in favor of the VG model.

Heksho208

In Table 12, while we can notice that the difference between VG and BSM models for the RMSE function is just 2.4, it is 8 for the AAE function. The one percent spread between the two models in the APE function still favors the VG model.

Hetrol911

Finally, it is evident that for every three error functions, the gap between the two models is narrower than the other options. However, the VG model is still better than BSM for all three error functions.

Declaration of Conflicting Interests

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