AN OPTIMAL TRANSFORMATION FRONTIER

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The optimal transformation frontier has been defined by Burmister and Kuga (1970) as being the maximum attainable level of per capita consumption when various gross rates of growth are assigned exogenously. Bruno (1969) has also treated this problem in deriving what he calls the real factor price frontier. In this note I will derive an analogue of this frontier, taking as the starting point an input-output model of a type developed by Oscar Lange (1962), and used by him to discuss economic planning.

The model takes on the following appearance:

$$
(1)\qquad \begin{array}{l}
X_1 = C_{11} + \cdots\cdots + C_{1n} + I_{11} + \cdots\cdots + I_{1n} \\
\vdots \\
X_n = C_{n1} + \cdots\cdots + C_{nn} + I_{n1} + \cdots\cdots + I_{nn}
\end{array}
$$

The output of each sector is intended for consumption and investment in all sectors, to itclude itself. For expository purposes this scheme will be simplified to two sectors. It thus takes on the following appearance, where each sector specializes to a particular category of output.

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$$X_1 = I_{11} + I_{12}$$

$$X_2 = C_{21} + C_{22}$$

Sector one produces investment goods (machines), with investment in each sector related to the change in output of the relevant sector, or $I_{1j} = b_{1j} \dot{X}_j$. b_{1j} is thus an incremental output/capital ratio. The second sector produces food, with C_{2j} being food going to sector 'j', and $C_{2j} = a_{2j}X_j$. The a_{ij}'s will be interpreted later. We thus have:

$$(2) \quad X_{1t} = b_{11} \dot{X}_1 + b_{12} \dot{X}_2 = b_{11}(X_{1,t+1} - X_{1t}) + b_{12}(X_{2,t+1} - X_{2t}) \quad X_{2t} = a_{21} X_{1t} + a_{22} X_{2t}$$

Notice that if a global growth rate existed we could write $X_{i,t+1} = X_{it}(1+g)$. We can thus obtain for the above expression:

$$(3) \quad \begin{aligned} X_{1t} &= gb_{11} X_{1t} + gb_{12} X_{2t} \\ X_{2t} &= a_{21}X_{1t} + a_{22}X_{2t} \end{aligned} \quad \text{(or)} \quad \begin{bmatrix} 1-gb_{11} & -gb_{12} \\ -a_{21} & 1-a_{22} \end{bmatrix} \begin{bmatrix} X_{1T} \\ X_{2T} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This has been written as a homogeneous system, and the condition for the solution of such a system is that the determinant of the matrix is zero. Before writing this determinant out, however, the units of the second equation of (2) will be examined. This gives:

$$X_2 = \text{Food} = \frac{\text{Food}}{\text{Machine}} \text{Machines} + \frac{\text{Food}}{\text{Food}} \text{Food}$$

This can be rewritten as:

$$\text{Food} = \frac{\text{Food}}{\text{Labor}} \cdot \frac{\text{Labor}}{\text{Machine}} \text{Machines} + \frac{\text{Food}}{\text{Labor}} \cdot \frac{\text{Labor}}{\text{Food}} \text{Food}$$

$$= wd_{21} \cdot \text{Machines} + wd_{22} \cdot \text{Food}$$

Food/Labor is w, which is the wage measured in units of food; while d_{ij} is a labor/output ratio, and thus $a_{ij} = wd_{ij}$. The use of the d_{ij}'s, incidentally, make clear that in this type of model labor is an intermediate input. The relevant determinant is then:

$$\begin{vmatrix} 1-gb_{11} & -gb_{12} \\ -wd_{21} & 1-wd_{22} \end{vmatrix} = (1-gb_{11})(1-wd_{22})-gwd_{21}b_{12} = 0$$

This can be solved to give

$$(4)\quad g=\frac{(1-wd_{22})}{b_{11}(1-wd_{22})+b_{12}wd_{21}}$$

Equation (4) is the optimal transformation frontier for this type of problem, although in the present formultion it is the per capita consumption, or w, that we would expect to find given. The following observations are then relevant. If, for example, $w=1/d_{22}$, we have g=0. The interpretation in this case is simple:the consumption of each unit of labor is equal to its productivity in the production of food, and thus nothin is left over for employees in the machine producing sector. Similarly, if W=0, then $g=1/b_{11}$, and what we have is a result equivalent to that of a one sector model of the Harrod-Domar type type where g=s/b, and s=1. (This case might be relevant in a situation where consumption goods could be produced in some sort of 'primitive' or 'traditional' sector, whose characteristics vis a vis the rest of the model would resemble a <u>deux ex machina</u>). In fact we can observe from the right hand expression in equation (4) that the term in parenthesis modifies the Harrod-Domar output/capital coefficient to allow for the additional (indirect) capital requirements necessary to raise production in the food sector. This concept applies to the model in this note regardless of the number of sectors it contains, providing that <u>only</u> one sector produces investment goods.

Two last remarks. As indicated (but not gone into explicitly) in Banks (1974) a model of this type can be used to discuss situations in which there is a different wage in each sector. Also, the above analysis is in terms of flows while Bruno, for example, employed <u>stocks</u> of capital and used labor as a primary resource. It does no harm to admit that this arrangement makes for a more elegant problem in mathematical economics, but it causes some difficulties when dealing with so called planning models. Both the two sector and four sector planning models for India, as developed by Mahalanobis (1955), were of th the above type, with only one investment good sector.

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