AN ANALYSIS OF VON NEUMANN "GENERAL EQUILIBRIUM"

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Part One*

A. The Nature of the Problem

At first I should confess that the problem under discussion is vast and extremely technical. I am far from being satisfied with the Present work. But there may be other opportunities that I can tackle this topic again. I would like to join the many authors in this field in what they consider to be a tip to the interested speculator in this area. Some experts have pointed out that the exposition of Von Neumann General Equilibrium: "may be of some interest because the original paper is extremely forbidding to the nonmathematician." 1

The Von Neumann General Equilibrium model can be set up as a linear programming over time in which continuous variations are permitted. The basic concept behind the theory of games, namely the Minimax Principle, has been effectively utilized in analyzing this model. The model explores economic equilibrium on the conditions of a steady growth over time. It is the analysis of a closed economic system under "constant return to scale" and with no limits on avalability of primary factors of production. It has applications in a centralized economy as well as a decentralized welfare state economic system.

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1 - Dorfman, Samuelson, and Solow, Linear Programming and Economic Analysis, p. 382

Hamburger, Thompson, and Well[2] have found a computationally feasable method of identifying the expansion rate and other variables of the Von Neumann model.

They also offer a possible planning of expansion rates of different models for quantitative numerical results as a guide for planning and evaluation.

Generally speaking, the Von Neumann model consists of an input matrix and an output matrix from which we can calculate the expansion rate with the associated price and intensity vectors. (These concepts will be explained later in detail.)

One can easily see that Von Neumann model can be utilized in evaluating the performance of an economic system. In fact in some cases it can offer a much more satisfactory and realistic answer to many questions raised in comparative economic studies than the conventional methods. For example, in evaluating the performance of an economic system for comparative purposes, it has been customary to evaluate the rate of change of Gross National Product of the two comparative economies and then draw conclusions about the performances of each system.And then make some projections about their respective performances in the future. As it turns out this method is very crude and leads to no significant results as far as the realities of these two distinct economies are concerned.

2 - Hamburger, Thompson, and Weil, Computation of Expansion Rates for the Generalized Von Neumann Model of an Expanding Economy.

In recent years the attention of many economists has been focused on the potential and actual performances of two different systems, one being the capitalist economies and the other, the socialist or the communist economies. Here the Von Neumann model can offer the best possible answer. It can provide the planner or any other interested party with the actual and potential growth rate.

The advantages of the new approach over the comparisons of the rate of change of GNP is quite clear because the comparative studies of GNP only reveals the observed phenomena, but not the potential. In fact in a study of this kind, the potential growth is what we maybe interested in. Thus Von Neumann General Equilibrium Model provides a better analytical tool for a comperative economic analysis.

B. Historical Background

The systematic study of linear models of production under the titles like Input-Output Analysis, Linear Programming, and Activity Analysis are of recent development. Actually the Walrasian model originally introduced the concept of simultaneous approach to the overall evaluation and systematic study of the General Equilibrium problems. But it was not until recent years that the economists have actually devised new methods to investigate the general economic activities at the so-called equilibrium position.

Early economic theorists had tried to tackle the production problems with linear approaches. In fact, Ricardo's theory of comparative cost was an early attempt to use the linear programming methods in a crude sense.

As was pointed out earlier, the Walrasian system was another attempt to find the "existence and the uniqueness" of the general equilibrium by utilizing the simultaneous approach to this problem. But Von Neumann was the first one to utilize the linear programming concept, before this method reaches its recent development. His celebrated paper, "A Model of General Equilibrium," first appeared in a lecture delivered at Princeton University in 1932. Later it was translated into English in 1945.[3]

An interesting attempt was made by Hamburger, Thompson, and Weil, who set up an algorithm in the language of the game theory by which they attacked the problem of finding the maximum or minimum rates of growth in a Generalized Von Neumann model.[4] Their approach was that they chose limits like upperbound and lowerbound for each of the variables of the Von Neumann model. These boundaries are applied to prices, intensities, output, and input. These constraints insure a maximum available quantity or a minimum desired quantity of each variable. Now by selecting the upperbound and lowerbound of each variable, they can set up a linear programming problem corresponding to these constraints.

Part Two

The Application of Linear Programming (Dynamic Aspects)

A. A Single Linear Dynamic Process (Ramesy's Model)

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3 - Von Neumann, Review of Economic Studies, 1945-46, pp. 1-9.

4 - Ibid. 2

The following assumptions have been Used in this model.

1 - We have one single commodity (X_t)

2 - We use this commodity as an input in the time period (t) to produce itself at the time period t+1.

3 - The output at time period t+1 has two parts:

(a) C_{t+1} = consumption

(b) X_{t+1} = the input of the next period

4 - If A is some unit of input, then through the process of production, A will give rise to B (units of output) in one unit of time.

5 - We assume that $B > A$ or $B = \alpha A$
We call α the technical coefficient of constant return to scale that prevails through time. In the language of Walrasian system, α is the coefficient of fabrication.

B. The Solution to the Model

We can write the production relation at any given time period as

$$C_{t+1} + X_{t+1} \leqslant X_t \qquad \text{(I)}$$

The left side presents the total output at period t+1 whereas the right side presents the input of the period preceding t+1 that is at t.

The reason for $<$ sign is that people may because of ignorance and disadvantage get less output than is technically feasible. That is : potential output may ba longer than actual output.

This is a linear production function. Let's assume that the initial stock of capital is K units at t=0. We have the following relationship:

at t=0

(consumption, input) $\leq$ (initial capital outlay)

$$C_o + X_o \leq K$$

at t=1 :

$$C_1 + X_1 \leq \alpha X_o = \alpha (K - C_o)$$

at t=2 :

$$C_2 + X_2 \leq \alpha X_1 = \alpha\{\alpha(K - C_o) - C_1\}$$

$$\vdots$$

(II) at t=n

$$C_n + X_n \leq \alpha^n . K - \alpha^n . C_o - \alpha^{n-1} . C_1 - \alpha^{n-2} . C_2 \cdots\cdots - \alpha . C_{n-1}$$

We have a production function.

Now if we should change the inequality sign to equality, then equation (II) will be at optimum, and this equation will give us a feasible solution. This means a set of consumptions and its progressive frontiers. We can also calculate (MRS) marginal rate of substitution in respect to each time period:

$$C_1 + X_1 = \alpha (K - C_o)$$

$$C_1 + \alpha C_o - \alpha K + X_1 = 0$$

to find the MRS of consumption in periods one and two , we formulate:

$$\frac{-dC_1}{dC_o} = \alpha = MRS$$

Where: C_o is the consumption at t=0

C_1 is the consumption at t=1

K is the initial input

X_1 is the input at the end of t=1

α is the technical coefficient of constant return to scale which is the same thing as the Walrasian coefficient of fabrication.

Part Three

The Von Neumann Economy

A. Explanation of Some Useful Concepts

1. The meaning of joint production

Before we introduce Von Neumann Economy in a formal way, we need to develop a small vocabulary which is important in the understanding of the Von Neumann Model. One of these important concepts is the "joint production Phenomenon". When goods are produced not only from natural factors of production but from each other, then the production is called a joint phenomenon. This includes land and labor as well. Another meaning of joint production is that every commodity appears in every process as an input or output. This property has been relaxed by Kemeny, Morgenstern, and Thompson.[5] But this has been achieved at the cost of having more than one equilibrium position at which α (the expansion rate) and β (the interest factor) are equal. A very closely related concept to the above notion is the "circular production function". We can illustrate this concept by saying that if good A is produced with

5 - Kemeny, Morgenstern, and Thompson, "A Generalization of the Von Neumann Model of an Expanding Economy", Econometrica, 1956, pp. 115-135.

the help of good B and good B itself is produced with the aid of good A, then we say that the production phenomenon is circular. Another meaning of joint production is that each activity of process involves every commodity either as an input or as an output. A simple way to put this phenomenon is to say that there are no finished goods and every good is an intermediate good.

2. Open versus closed production models

The essence of a closed system is that the output of one period is the input of the next period. When there is nothing entering the system from the outside and nothing going out of the system from the inside, then we can say that we have a closed system. This requires that income over and above the necessities of life will be reinvested.

R. G. D. Allen says that a general equilibrium model is a closed production model in the sense that all variables are determined together and endagenously from a joint condition.[6] An open system is dependent on forces outside the production phenomena, such as consumer demands and prices of other products. So it may lead to a partial analysis.

B. Assumptions and the properties of the Von Neumann General Equilibrium

1. The following assumptions are important properties of Von Neumann Model.

a. $m > n$ (m=number of processes, n=number of goods). This simply says we cannot solve the problem by just simultaneous approach

[6] R. G. D. Allen, Mathematical Economics, pp.600-607.

b. There are constant returns to scale.

c. There are unlimited supplies of the factors of production.

d. Consumption is a part of production phenomena.

e. a_{ij} represents the input matrix and b_{ij} represents the output matrix and both a_{ij} and b_{is} are capital goods.

f. Production takes place in m different processes. Each process has a unit time duration.

g. Each process has an intensity equal to X_i.

2. A Statement of the objectives

The following are the unknowns of Von Neumann model:

a. The vector $X_1 = (x_1, x_2 \ldots x_m)$ named as the intensities vector.

b. The α or the coefficient of expansion of the whole economy.

c. The price vector $Y_j = (y_1, y_2 \ldots y_n)$.

d. The last unknown of the system is β called the interest factor. In a simple language, we wish to establish the following:

(1) Which processes out of the m processes presented by vector P_i will be used.

(2) What would be the relative velocity by which the total quantity of goods will increase That is:What would be the growth rate of this economy?

(3) What prices will be charged?

(4) What would the rate of interest be in an economy in which the goods can be produced from each other with the possibility of $m > n$ and other properties that we have already mentioned in Part III, B. 1. from a. to g.

C. A Formal Presentation of Von Neumann's Economy

Von Neumann model is linear model because for all $\lambda > o$ where λ is a scalar $\lambda \binom{A}{B} = \binom{\lambda A}{\lambda B}$. This model is also a closed model because the output of the time period t becomes the input of the next period t+1. No other good is allowed into the system. The stock of goods at the period t=0 is given. Also there may be free disposal of goods in the case of over production. In reality there is no money flow in the system, one good is picked up as a numeraire and its price is defined to be one.

The input matrix a_{ij} is used at the beginningof the period t=0 and the output matrix b_{ij} is produced at the end of the time period t=1. The result of this activity according to Roman Weil[7] is a profit..Weil defines the profit as the value of input of the next period. That is $b_t - a_{t+1}$ where a_{t+1} is the end of the first period input, and its value is equal to $r.Y_{t+1}$ (Y_{t+1} is the price of input and r is the interest factor.)

7 - Roman Weil, "An Algorithm for the von Neumann Economy".

Table 1. The Model $\underline{M}$ presented by $\binom{A}{B}$ in which $\underline{A}$ is the matrix of input and where B is the matrix of output.

		X_1	X_2	X_m	
Input Prices	Y_1	a_{11}	a_{12}	a_{1m}	
	Y_2	a_{21}	a_{22}	a_{2m}	
	⋮	⋮	⋮	⋮	Inputs $a_{ij} \geqslant 0$
	Y_n	a_{n1}	a_{n2}	a_{nm}	
Output Prices	Y_1	b_{11}	b_{12}	b_{1m}	
	Y_2	b_{21}	b_{22}	b_{2m}	
	⋮	⋮	⋮	⋮	Output $b_{ij} \geqslant 0$
	Y_n	b_{n1}	b_{n2}	b_{nm}	

Now assuming that the above process of production and consumption has an intensity vector of X_i we can represent this intensity vector as $X_i = {}^1(X_1, X_2 \ldots x_n)$ where $x_i > 0$. The column vector X_i represents the intensities at which the processes of model M are operating.

There is one condition for x_i's or the vector X_i and that is x_i's are normalized so that $\sum x_i = 1$ and $X_i > 0$ for all j. In Fact, we should not consider the above condition as a restriction on this model because any set of x_i's say x'_i 's can be specified as an intensity vector x_i and normalized to vector x'_i by the relation[1]

$$X_i' = \frac{X_i}{X_j}$$

The vector $Y_i = (Y_1 \quad Y_2 \ldots Y_n)$ is the set of prices for the n goods[1] in the economy where $Y_i > 0$ for all i.

In general the process of normalization of a vector is achieved by introducing an equation in which the sum of all the elements of a vector is equal to one. For example, if we have a matrix equation $|A - \lambda I| = 0$ where A is the matrix and λ is a scalar where $\lambda > 0$) and I is an identity matrix, then if we multiply a vector x by the above expression, then we get $(A - \lambda I)x = 0$. Now if :

$$(A- \quad I) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

and $(A- \quad I)x = 0$, then we get

$$(1) \qquad x_1+2x_2+3x_3=0$$

$$(2) \qquad 3x_1+4x_2+x_3=0$$

$$(3) \qquad 2x_1+x_2+3x_3=C$$

The normalization of a vector x will come about by introducing another equation in which the sum of the elements of vector x_1 is equal to unity.

$$x_1+x_2+3x_3=0 \quad \text{or:} \sum X_i=1$$

Having the following notational procedures in mind:

1. $(X_i)=(X_1, X_2, X_3....X_m)$=Number of Intensities of the production Processes
2. $P_i=(P_1,P_2, P_3, ...P_m)$ = Number of processes
3. $G_j=(G_1,G_2, G_3, ...G_n)$ = Number of goods
4. (a_{ij})= Number of inputs or input matrix
5. (b_{ij})= Number of outputs or output matrix
6. $(y_j) = (Y_1,Y_2, Y_3,.....Y_n)$=Number of prices for N goods in the economy.

Now that,we have some appreciation of the elements of Von Neumann model we can proceed to introduce the Von Neumann economy by introducing the following eleven equations.

$$(I) \qquad \sum_{J=1}^{n} a_{ij} \xrightarrow{\quad P_i \quad} \sum_{J=1}^{n} b_{ij}$$

This equation says that input a_{ij} after going through the processes of productions called P_i with a given intensity X_i will be transformed to output called b_{ij}.

Let us take a look at the next equation.

$$\text{(II)} \qquad Y_j \geqq 0$$

This equation indicates that in the market place prices of goods and services can not be negative.

Our next equation is:

$$\text{(III)} \qquad X_i \geqq 0$$

Equation III indicates that the intensities of production processes can not be negative. And since a solution with:

$$X_i = x_1 = x_2 = x_3 \ldots = x_m = 0$$

and

$$Y_j = y_1 = y_2 = y_3 \ldots y_n = 0$$

would be meaningless, then we have to have:

$$\text{(IV)} \qquad (X_i) = (x_1, x_2, x_3 \ldots x_m) > 0$$

$$\text{(V)} \qquad (Y_j) = (y_1, y_2, y_3, \ldots y_n) > 0$$

We have assumed the constant return to scale which gives us a constant ratio of intensities, $\frac{x_i}{x_j} = K = \alpha$

Where α is the coefficient of expansion of the whole economy. With these in mind, we can now, write the processes of production as follows:

$$\text{(VI)} \qquad \alpha(X_i)\ (a_{ij}) \leqq (X_i)\ (b_{ij})$$

or the same thing as:

$$\alpha \sum_{i=1}^{m} x_i a_{ij} \leqq \sum_{i=1}^{m} x_i b_{ij}$$

The meaning of equation VI. is that always the production capacity is more than consumption capacity, or it is impossible to consume more than what is being produced. We can also say that input (consumption) cannot exceed the output and on the other hand, if less is consumed, that is, if there is excess production $b_{ij} > \alpha a_{ij}$ then the excess good G_i will be a free good: that is $y_i = 0$. We can also derive another equation or relationship from the outline of the problem.

Our next equation is:

$$\text{(VII)} \quad \beta(a_{ij})(y_j) \geqslant (b_{ij})(y_j)$$

Equation VII indicates that in equilibrium there can be no profit on any of the processes contained in P_i. If there is a profit then the prices Y_j or the interest rate would have to rise and this will bring about equilibrium condition again. On the other hand, if there is a loss (meaning that a process P_i is unprofitable), then P_i will not be used and the intensity of the process P_i will be equal to zero, that is $X_i = 0$. Keeping in mind, that because there is constant return to scale the interest ratio is given by:

$$r = \frac{y_i}{y_j}$$

we like to show that this is "uniquely determined". We should remember that always:

$$\text{(VIII)} \quad a_{ij} + b_{ij} > 0$$

Now given the system of equation (the model) from I to VIII, we can solve for α and β .

(IX)
$$\alpha = \text{Min} \left(\frac{\sum b_{ij} x_i}{\sum a_{ij} x_i} \right) = \text{Min} \frac{\left[b_{ij} \ x_i \right]}{\left[a_{ij} \ x_i \right]}$$

We need to minimize α over this ratio, based on what we have previously said about equation VI*. This means that we can never have overproduction or under-consumption. If we do, then we must have the following relationships: $Y_j = 0$ or $X_i = 0$. Thus to minimize α, we need to maximize a_{ij} in the denominator of equation IX.

But when a_{ij} is at its maximum, then in equation VI we have a new relationship and we call this equation:

(VI)'
$$\alpha \sum_{i=1}^{m} a_{ij} x_i = \sum_{i=1}^{m} b_{ij} x_i .$$

This is the case where :

(IX)
$$\alpha = \text{Min} \frac{\sum b_{ij} x_i}{\sum a_{ij} x_i}$$

Our next equation can be derived directly from equation VII. where β is equal to:

(X)
$$\beta = \text{Max} \left(\frac{\sum b_{ij} y_j}{\sum a_{ij} y_j} \right) = \text{Max} \frac{(b_{ij}) \ (y_j)}{(a_{ij}) \ (y_j)}$$

To maximize β we need to minimize the denominator of equation X. that is we need to minimize a_{ij}. But this can not be achieved because in equation IX where we minimized α ratio, we were forced to maximize a_{ij}. Thus a_{ij} can not be maximized and minimized at the same time. These two processes are mutually exclusive.

* See page 94 of this article

Thus we can only achieve our goal when in equation VII. we get β at its minimum or, $b_{ij}\ y_j$ at its maximum. This again leads to a new interpretation of equation VII which we call it VII':

$$(VII)' \quad \alpha \sum_{j=1}^{n} a_{ij} y_j = \sum_{j=1}^{n} b_{ij} y_j$$

We can summarize this by saying that the process of adjustment in equation X. will come about when β the interest factor is at its minimum, that is we get the lowest interest rate.

As we can see the intensity vector X_i and the price vector Y_i uniquely determine α (expansion rate) and β (the interest rate). These answers are unique because the right side of equations IX. and X. can never become $\frac{0}{0}$ or other nonsensical or undetermined quantitities.

We are about to prove that not only α and β are unique answers, but they are also equal to each other. Consider the following equations IX' and X':

$$(IX)' \quad = \text{Min} \frac{(b_{ij})(x_i)}{(a_{ij})(x_i)} = \frac{(b_{ij})(x_i)(y_j)}{(a_{ij})(x_i)(y_j)}$$

$$(X)' \quad = \text{Max} \frac{(b_{ij})(y_j)}{(a_{ij})(y_j)} = \frac{(b_{ij})(x_i)(y_j)}{(a_{ij})(x_i)(y_j)}$$

Comparing these last two equations, we can see that:

$$(XI) \quad \text{Min}\ \alpha = \beta\ \text{Max}$$

As we can see, α and β are values of a saddle point in a two person zero sum game. Von Neumann reaches this point with the following properties:

a) $\alpha_1 = \alpha_2 = \alpha_3 = \ldots\ldots = \alpha_n$

$\beta_1 = \beta_2 = \beta_3 = \ldots\ldots = \beta_n$

and $\alpha_n = \beta_n$ where $\alpha \leqslant 1$

b) we can consider or characterize α in two independent ways;

(1) Considering a state of economy purely on technical grounds which is expanding with a growth rate α_1 per unit of time, we get:

$$1 \qquad \alpha_1 \sum_{i=1}^{m} a_{ij} x_i \leqslant \sum_{i=1}^{m} b_{ij} x_i$$

Here we are neglecting prices altogether, and only the intensities of production gives us the proper growth rate.

(2) Considering an economy with a system of prices y_j where interest factor β_1 allows for no profit, then we get:

$$1 \qquad \beta \sum_{j=1}^{n} a_{ij} y_j \geqslant \sum_{j=1}^{n} b_{ij} y_j$$

In this case, we are neglecting the intensities of production (X_i) altogether.

Part IV

A Comparison between Leontief Dynamic Model and Von Neumann Economy:

There are a number of differences between Von Neumann General Equilibrium Model and Leontief's Dynamic Input-Output Model.

The following are the main differences between these two models:

a. In Von Neumann General Equilibrium model we have the process of joint production.

The absence of joint production in Leontief's model is a limitation of that model.

b. Von Neumann Economy is a closed model of General Equilibrium whereas Leontief's model is an open system.

c. Von Neumann model does not differentiate between the flow of goods and their stock whereas Leontief's model differentiates between the flow of goods and the stock of those goods. In Leontief's model, the stocks are presented, but are not being utilized in production phenomena.

d. The Leontief's model considers only the flow of goods from one industry to the other industries and to the outside world. It does not determine as Von Neumann's does, the intensities of operation of various processes and the prices at which these goods are disposed -- that is the growth rate and the interest factor which are Uniquely part of Von Neumann General Equilibrium Model.

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