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# Proposing a Developed Gram–Schmidt Algorithm to Construct Orthogonal Unit Vectors (A Mathematical and Practical Approach)

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A R T I C L E I N F O

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A B S T R A C T

It is no secret that mathematics has always been a guide to solving many problems in human life. Humans, in their decision-making processes, constantly deal with various variables and indicators, and one of the approaches to analyzing these variables is the use of multi-criteria decision-making methods. However, a significant challenge in applying these techniques is the independence of criteria or indicators relative to one another. One of the modern solutions for achieving independence among variables (or indicators) is the use of the Gram-Schmidt method; however, the accuracy of this algorithm might decline when it is implemented on large-scale vectors. This paper proposes a Developed Gram–Schmidt Algorithm (DGSA). The Schmidt vectors obtained from the proposed algorithm are prone to a lower error rate than those resulting from the Gram–Schmidt algorithm. To demonstrate the superiority of the proposed algorithm, several different numerical examples have also been used. At the end of the research, a case study based on the proposed approach was also conducted at Shazand Oil Refinery (SOR) to demonstrate the applicability of this approach in a real-world example. The findings have shown that the proposed approach has relatively high accuracy.

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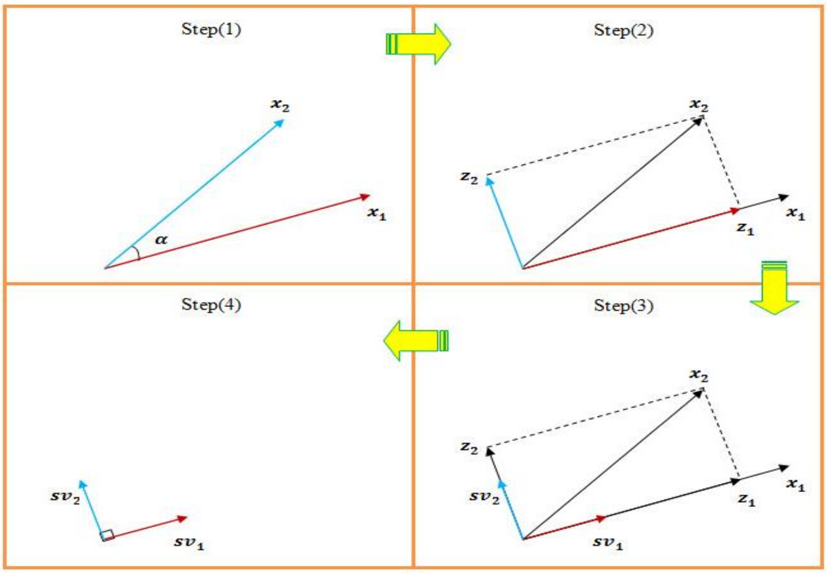


*Figure 1 .A Developed Gram–Schmidt Algorithm to Construct Orthogonal Unit Vectors*

## 1.Introduction

In diverse areas of applied mathematics, use is made of orthogonal sets of vectors, or one needs to make a vector orthogonal to the linear span of a given set of vectors (Ruhe, 1983). The Gram-Schmidt algorithm is instrumental for these purposes (Havlicek & Svozil, 2018). In fact, the Gram–Schmidt algorithm should immediately be used whenever it is necessary to develop orthogonal and unit vectors out of several vectors (Paige et al., 2006). Mathematical modeling is one of the most widely used research methods (Jafari et al., 2023a; Momeni et al., 2022). It employs variables and their relationships instead of relying on judgmental descriptions and personal opinions (Jafari et al., 2023b; Jafari et al., 2023c). High accuracy in describing and analyzing real-world problems distinguishes this method (Alamian & Ayeshgar, 2024). More generally, mathematics is the discipline that uncovers and explains the order hidden within seemingly complex situations. This field provides tools based on principles that enable us to illustrate this order (Erdik, 2019). Since the present study is no exception to this rule, it uses mathematical modeling to address challenges related to decision-making issues. Additionally, MATLAB software has been employed to provide accurate results in the analysis of numerical examples.

MATLAB have high accuracy and speed in data processing and allow for visual interpretation of the model results (Abouzari et al., 2024; Khalilzade & Ahrabi, 2024). In addition to its high computational accuracy, this software provides high-quality and comprehensible graphical outputs (Jafari et al., 2023d; Jafari et al., 2024).

Performance refers to employees' efforts to accomplish their tasks or achieve set goals. Every job comes with responsibilities that must be performed according to defined standards (Pourmahmoud & Norouzi, 2022). The process of defining, measuring, and scoring performance standards is called performance appraisal. This helps to determine the competence, characteristics, quality, skills, capacity, and potential of employees for the future (Izadi & Amiri, 2022). Based on performance appraisal, the best employees are rewarded to encourage progress in others (Ghalayini et al., 1997). Today, with technological advancements and the role of factories and service organizations in human life, we constantly witness the formation of various and newer organizational units. When an organization comprises numerous units, the first fundamental question that arises in the minds of senior managers is, which unit performs better? And in what state is the performance of other units?

One of the solutions to address this challenge is to use MADM methods (Jafari & Ehsanifar, 2020). Additionally, one of the new challenges is the discussion of correlation among decision indicators, and this research demonstrates how it can be mitigated to some extent using a developed mathematical approach (the extended Gram-Schmidt method).

## 2.Gram–Schmidt Method

The Gram–Schmidt (GS) method is a mathematical approach that can be employed to develop an orthogonal unit basis out of an arbitrary basis. Known as a transform in the vector space, the GS method is mostly adopted to reduce the correlation of datasets. It can also be employed to identify unimportant vectors and delete them from the calculation process if necessary. This method was introduced by a scientist named Gram–Schmidt (Salehi & Raisee Dehkordi, 2016).

## 3.Gram–Schmidt Algorithm

1. The following steps should be taken to execute the GS algorithm on the data matrix (*i.e.* $X = [X_1, X_2, \dots, X_n]_{m\times n}$) (Witteveen & Bijl, 2006):
2. A vector (*e.g.*$X_1$) is considered equal to $Z_1$ and used as the first Schmidt vector:

$$Z_1 = X_1 \quad (1)$$

3. The second Schmidt vector is developed through the following equation:

$$Z_2 = X_2 - \frac{X_2 Z_1}{Z_1 Z_1} Z_1 \quad (2)$$

4. The third Schmidt vector is constructed as below:

$$Z_3 = X_3 - \frac{X_3 Z_1}{Z_1 Z_1} Z_1 - \frac{X_3 Z_2}{Z_2 Z_2} Z_2 \quad (3)$$

5. The above procedure can be continued to easily calculate all Schmidt vectors through the following equation:

$$Z_j = X_j - \sum_{k=1}^{j-1} \frac{X_j Z_k}{Z_k Z_k} Z_k \quad (4)$$

6. End.

The following equation can be employed to unitize Schmidt vectors:

$$sv_{i,j} = \frac{z_{i,j}}{\sqrt{\sum_{p=1}^{m} z_{p,q}^2}} \; ; i = 1,2,..,m \; and \; j = 1,2,..n \quad (5)$$

## 4. Developed Gram–Schmidt Method

This method consists of the following steps:

**Step (1):** Put $k = 1$.

**Step (2):** Out of the vectors in matrix $X$, find the vector that has the highest correlation with the other vectors and put it into vector $Z_k$

**Step (3):** Delete that vector with the highest correlation from matrix $X$.

**Step (4):** Put $k = k + 1$.

**Step (5):** Use the following equation to orthogonize all the remaining vectors in $X$ on vector $Z_{k-1}$:

$$X_j = X_j - \frac{X_j Z_{k-1}}{Z_{k-1} Z_{k-1}} Z_{k-1} \; ; \; j = 1,2,\dots,n-k+1 \quad (6)$$

**Step (6):** Out of the vectors in matrix $X$, find the vector with the highest variance and put it into vector $Z_k$.

**Step (7):** Delete that vector with the highest variance from matrix $X$.

**Step (8):** If $k \leq (n-1)$, repeat step (4); otherwise, go to step (9).

**Step (9):** End.

The following equation can be adopted to unitize Schmidt vectors:

$$sv_{i,j} = \frac{z_{i,j}}{\sqrt{\sum_{p=1}^{m} z_{p,j}^{2}}} \; ; i = 1,2,..,m \;\; and \; j = 1,2,..n \tag{7}$$

## 5. Schmidt Vector Error Calculation

If matrix $SV = \left(sv_{i,j}\right)_{m\times n}$ contains all Schmidt vectors, the calculation error of this matrix can be determined through the following equation:

$$Error = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left| \sum_{k=1}^{m} (sv_{k,i} sv_{k,j}) \right| \tag{8}$$

It should be noted that Equation (8) calculates the absolute summation of the dot product of Schmidt vector pairs. The closer this value is to zero, the more accurate Schmidt vectors are.

### Numerical Example (1)

Table 1 presents the data of matrix $X_{10\times10}$.

*Table 1 . Data of Matrix $X_{10\times10}$*

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| 44 | -36 | -81 | 1 | 77 | -35 | 46 | -9 | 10 | -53 |
| 2 | -20 | 17 | -1 | 6 | 33 | -61 | -49 | 83 | 87 |
| -35 | 76 | 70 | 73 | -82 | 0 | -9 | 7 | 11 | 41 |
| -9 | 91 | -54 | 14 | -3 | 72 | 4 | -82 | 72 | 35 |
| -41 | -100 | -61 | -66 | -36 | 50 | -71 | 12 | -9 | -21 |
| 37 | 20 | -93 | 82 | 9 | -21 | 57 | 71 | -68 | 30 |
| 85 | 57 | 66 | 33 | 92 | 3 | -67 | 46 | -53 | -10 |
| -42 | 28 | -95 | 1 | -27 | 17 | -1 | 8 | 22 | 98 |
| -5 | -77 | -27 | 65 | -95 | 2 | -48 | 34 | -13 | -82 |
| -22 | 42 | -65 | -43 | 48 | 47 | 24 | 72 | 22 | -6 |

After the GS algorithm is executed, the matrix contains Schmidt vectors, as presented in Table 2.

*Table 2 . The Matrix Containing Schmidt Vectors Through the Baseline Approach*

| $SV_1$ | $SV_2$ | $SV_3$ | $SV_4$ | $SV_5$ | $SV_6$ | $SV_7$ | $SV_8$ | $SV_9$ | $SV_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.350 | -0.228 | -0.417 | -0.044 | 0.399 | -0.078 | 0.013 | -0.399 | -0.009 | -0.570 |
| 0.016 | -0.105 | 0.097 | 0.022 | 0.336 | 0.684 | 0.027 | -0.452 | 0.012 | 0.439 |
| -0.278 | 0.426 | 0.317 | 0.489 | 0.321 | 0.138 | -0.074 | 0.016 | -0.321 | -0.413 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| -0.072 | 0.479 | -0.330 | -0.031 | -0.592 | 0.272 | 0.246 | -0.372 | 0.005 | -0.177 |
| -0.326 | -0.479 | -0.154 | -0.195 | -0.233 | 0.341 | -0.209 | 0.172 | -0.562 | -0.208 |
| 0.294 | 0.069 | -0.515 | 0.466 | 0.064 | -0.037 | 0.182 | 0.259 | -0.425 | 0.373 |
| 0.676 | 0.215 | 0.159 | -0.087 | -0.163 | 0.333 | -0.503 | 0.240 | -0.014 | -0.120 |
| -0.334 | 0.184 | -0.430 | 0.073 | 0.088 | -0.128 | -0.745 | -0.133 | 0.184 | 0.183 |
| -0.040 | -0.394 | -0.056 | 0.620 | -0.233 | 0.255 | 0.000 | 0.153 | 0.527 | -0.188 |
| -0.175 | 0.238 | -0.323 | -0.325 | 0.357 | 0.350 | 0.219 | 0.549 | 0.297 | -0.123 |

Now if the proposed GS algorithm is executed, the matrix will contain Schmidt vectors, as presented in Table 3.

***Table 3. The Matrix Containing Schmidt Vectors Through the Novel Approach***

| $SV_1$ | $SV_2$ | $SV_3$ | $SV_4$ | $SV_5$ | $SV_6$ | $SV_7$ | $SV_8$ | $SV_9$ | $SV_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| -0.308 | -0.473 | 0.333 | -0.116 | -0.151 | -0.343 | 0.176 | 0.334 | -0.250 | -0.460 |
| 0.290 | 0.158 | 0.087 | -0.242 | 0.631 | -0.064 | 0.461 | 0.427 | 0.168 | 0.003 |
| 0.000 | 0.340 | -0.414 | 0.466 | -0.012 | 0.157 | 0.083 | 0.216 | -0.097 | -0.639 |
| 0.633 | -0.097 | 0.060 | 0.389 | -0.247 | -0.530 | 0.240 | -0.180 | 0.054 | 0.030 |
| 0.439 | -0.181 | -0.154 | -0.561 | 0.009 | 0.187 | -0.076 | -0.388 | -0.078 | -0.488 |
| -0.185 | -0.499 | -0.023 | 0.290 | 0.131 | 0.357 | 0.422 | -0.299 | 0.463 | -0.070 |
| 0.026 | 0.327 | 0.536 | 0.066 | -0.122 | 0.370 | 0.424 | -0.254 | -0.452 | 0.005 |
| 0.149 | -0.422 | -0.167 | 0.290 | 0.465 | 0.109 | -0.156 | -0.002 | -0.633 | 0.188 |
| 0.018 | -0.126 | -0.527 | -0.244 | -0.443 | 0.138 | 0.475 | 0.248 | -0.209 | 0.316 |
| 0.413 | -0.208 | 0.298 | 0.115 | -0.270 | 0.493 | -0.281 | 0.511 | 0.169 | 0.017 |

The error rate of Schmidt vectors was reported as $1.36 \times 10^{-14}$ in the baseline approach, whereas it was $5.62 \times 10^{-15}$ reported as in the novel approach. Therefore, the novel approach yields a lower error rate. It can then be concluded that Schmidt vectors are more accurate in the novel approach.

**Numerical Example (2)**

Table 4 presents the data of matrix $X_{30\times10}$.

***Table 4. Data of Matrix $X_{30\times10}$***

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| 71 | 78 | 73 | 10 | 31 | 77 | 4 | 16 | 39 | 54 |
| 32 | 43 | 34 | 53 | 11 | 64 | 40 | 33 | 71 | 72 |

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| 71 | 14 | 33 | 11 | 54 | 39 | 40 | 80 | 54 | 78 |
| 5 | 23 | 25 | 23 | 57 | 56 | 69 | 7 | 43 | 75 |
| 32 | 6 | 24 | 38 | 71 | 58 | 90 | 9 | 4 | 85 |
| 33 | 19 | 90 | 65 | 67 | 80 | 24 | 19 | 17 | 4 |
| 12 | 42 | 5 | 79 | 72 | 90 | 80 | 56 | 84 | 13 |
| 89 | 42 | 61 | 74 | 59 | 66 | 5 | 25 | 54 | 66 |
| 38 | 32 | 46 | 21 | 83 | 46 | 30 | 68 | 46 | 79 |
| 37 | 40 | 11 | 32 | 55 | 5 | 66 | 28 | 20 | 34 |
| 71 | 30 | 42 | 87 | 67 | 10 | 45 | 69 | 86 | 20 |
| 39 | 52 | 65 | 28 | 65 | 78 | 50 | 23 | 33 | 52 |
| 52 | 90 | 87 | 14 | 63 | 83 | 21 | 61 | 42 | 3 |
| 59 | 65 | 48 | 67 | 83 | 61 | 12 | 12 | 75 | 4 |
| 3 | 33 | 85 | 26 | 88 | 42 | 44 | 85 | 72 | 85 |
| 37 | 52 | 90 | 30 | 30 | 87 | 79 | 41 | 40 | 37 |
| 87 | 35 | 42 | 1 | 9 | 8 | 58 | 33 | 57 | 71 |
| 54 | 34 | 33 | 51 | 73 | 35 | 41 | 16 | 86 | 22 |
| 84 | 39 | 9 | 40 | 62 | 2 | 13 | 45 | 60 | 55 |
| 85 | 67 | 76 | 15 | 16 | 54 | 13 | 41 | 24 | 81 |
| 69 | 33 | 38 | 64 | 44 | 52 | 89 | 6 | 66 | 57 |
| 49 | 57 | 69 | 49 | 85 | 2 | 88 | 80 | 51 | 84 |
| 75 | 77 | 50 | 23 | 13 | 54 | 39 | 82 | 88 | 90 |
| 1 | 11 | 43 | 36 | 60 | 33 | 69 | 71 | 28 | 14 |
| 68 | 16 | 24 | 23 | 18 | 3 | 51 | 43 | 45 | 3 |
| 10 | 86 | 13 | 53 | 57 | 89 | 69 | 72 | 26 | 83 |
| 5 | 88 | 7 | 5 | 56 | 12 | 88 | 79 | 7 | 14 |
| 17 | 23 | 77 | 55 | 40 | 50 | 4 | 61 | 11 | 9 |
| 43 | 26 | 31 | 4 | 60 | 78 | 32 | 7 | 7 | 65 |
| 1 | 76 | 61 | 78 | 41 | 62 | 43 | 12 | 90 | 16 |

After the GS algorithm is executed, the matrix contains Schmidt vectors, as presented in Table 5.

***Table 5. The Matrix Containing Schmidt Vectors Through the Baseline Approach***

| $SV_1$ | $SV_2$ | $SV_3$ | $SV_4$ | $SV_5$ | $SV_6$ | $SV_7$ | $SV_8$ | $SV_9$ | $SV_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.247 | 0.148 | 0.019 | -0.284 | -0.109 | 0.130 | -0.184 | -0.167 | 0.015 | 0.012 |
| 0.111 | 0.109 | -0.013 | 0.165 | -0.286 | 0.162 | 0.098 | 0.131 | 0.165 | 0.247 |
| 0.247 | -0.200 | -0.004 | -0.091 | 0.218 | 0.138 | 0.045 | 0.373 | 0.121 | -0.024 |
| 0.017 | 0.106 | 0.064 | 0.054 | 0.211 | 0.170 | 0.179 | -0.192 | 0.215 | 0.208 |
| 0.111 | -0.092 | 0.057 | 0.152 | 0.264 | 0.223 | 0.275 | -0.187 | -0.245 | 0.203 |
| 0.115 | -0.025 | 0.426 | 0.174 | -0.016 | 0.093 | -0.096 | -0.088 | -0.259 | -0.155 |
| 0.042 | 0.182 | -0.147 | 0.408 | 0.084 | 0.321 | 0.076 | 0.223 | 0.176 | -0.278 |
| 0.310 | -0.118 | 0.031 | 0.189 | -0.112 | 0.069 | -0.272 | -0.025 | -0.179 | 0.183 |
| 0.132 | 0.026 | 0.089 | -0.049 | 0.334 | 0.019 | -0.141 | 0.173 | 0.096 | 0.150 |
| 0.129 | 0.074 | -0.159 | 0.075 | 0.147 | -0.180 | 0.129 | -0.165 | -0.188 | 0.010 |
| 0.247 | -0.113 | -0.004 | 0.347 | -0.052 | -0.288 | -0.037 | 0.160 | -0.005 | -0.111 |
| 0.136 | 0.131 | 0.135 | -0.076 | 0.115 | 0.139 | 0.011 | -0.175 | -0.023 | -0.017 |
| 0.181 | 0.287 | 0.107 | -0.286 | 0.056 | 0.040 | -0.197 | 0.011 | -0.002 | -0.397 |
| 0.205 | 0.124 | -0.065 | 0.161 | 0.079 | -0.029 | -0.357 | -0.250 | 0.099 | -0.181 |
| 0.010 | 0.168 | 0.412 | -0.057 | 0.288 | -0.184 | -0.024 | 0.171 | 0.361 | 0.250 |
| 0.129 | 0.139 | 0.298 | -0.109 | -0.177 | 0.152 | 0.335 | -0.038 | -0.012 | -0.194 |
| 0.303 | -0.148 | -0.059 | -0.225 | -0.104 | -0.095 | 0.301 | -0.066 | 0.154 | 0.013 |
| 0.188 | -0.025 | -0.037 | 0.146 | 0.155 | -0.051 | -0.080 | -0.199 | 0.302 | -0.149 |
| 0.292 | -0.115 | -0.275 | 0.082 | 0.151 | -0.165 | -0.234 | 0.047 | -0.002 | 0.092 |
| 0.296 | 0.034 | 0.045 | -0.254 | -0.225 | 0.006 | -0.050 | 0.014 | -0.211 | 0.191 |
| 0.240 | -0.089 | -0.036 | 0.206 | -0.110 | 0.078 | 0.305 | -0.251 | 0.015 | -0.003 |
| 0.170 | 0.119 | 0.119 | 0.030 | 0.146 | -0.478 | 0.188 | -0.065 | -0.123 | 0.240 |
| 0.261 | 0.127 | -0.132 | -0.160 | -0.245 | 0.039 | 0.066 | 0.313 | 0.234 | 0.121 |
| 0.003 | 0.056 | 0.230 | 0.127 | 0.165 | -0.048 | 0.211 | 0.230 | -0.045 | -0.182 |
| 0.236 | -0.178 | -0.061 | 0.002 | -0.065 | -0.087 | 0.227 | 0.099 | 0.004 | -0.331 |
| 0.035 | 0.429 | -0.251 | 0.154 | 0.006 | 0.224 | 0.009 | 0.234 | -0.278 | 0.257 |
| 0.017 | 0.459 | -0.284 | -0.138 | 0.209 | -0.235 | 0.191 | 0.042 | -0.236 | -0.197 |
| 0.059 | 0.059 | 0.366 | 0.145 | -0.137 | -0.065 | -0.129 | 0.271 | -0.301 | -0.017 |
| 0.150 | -0.026 | 0.004 | -0.123 | 0.282 | 0.369 | -0.057 | -0.114 | -0.073 | 0.034 |

| $SV_1$ | $SV_2$ | $SV_3$ | $SV_4$ | $SV_5$ | $SV_6$ | $SV_7$ | $SV_8$ | $SV_9$ | $SV_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.003 | 0.410 | 0.110 | 0.247 | -0.275 | -0.132 | -0.030 | -0.250 | 0.275 | 0.073 |

Now, if the proposed GS algorithm is executed, the matrix will contain Schmidt vectors, as presented in Table 6.

*Table 6. The Matrix Containing Schmidt Vectors Through the Novel Approach*

| $SV_1$ | $SV_2$ | $SV_3$ | $SV_4$ | $SV_5$ | $SV_6$ | $SV_7$ | $SV_8$ | $SV_9$ | $SV_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.247 | 0.112 | -0.270 | -0.082 | 0.049 | -0.064 | -0.060 | 0.085 | 0.245 | -0.158 |
| 0.111 | 0.177 | -0.004 | -0.006 | 0.190 | 0.233 | -0.311 | -0.053 | -0.049 | 0.196 |
| 0.247 | -0.052 | 0.005 | 0.231 | 0.106 | -0.055 | -0.033 | -0.199 | -0.332 | -0.223 |
| 0.017 | 0.226 | 0.197 | -0.199 | 0.264 | 0.138 | 0.090 | 0.014 | -0.027 | -0.151 |
| 0.111 | 0.151 | 0.260 | -0.297 | 0.218 | -0.255 | 0.175 | -0.051 | -0.174 | 0.107 |
| 0.115 | 0.243 | -0.126 | -0.057 | -0.220 | -0.168 | 0.185 | 0.251 | -0.210 | 0.183 |
| 0.042 | 0.351 | 0.157 | 0.040 | -0.268 | 0.216 | -0.074 | -0.431 | -0.190 | -0.007 |
| 0.310 | 0.008 | -0.261 | -0.054 | 0.082 | -0.010 | 0.127 | -0.050 | -0.050 | 0.330 |
| 0.132 | 0.080 | -0.017 | 0.240 | 0.219 | 0.020 | 0.241 | -0.066 | -0.087 | -0.150 |
| 0.129 | -0.093 | 0.267 | -0.087 | -0.035 | -0.110 | 0.155 | -0.035 | 0.185 | 0.096 |
| 0.247 | -0.177 | 0.103 | 0.165 | -0.225 | 0.190 | 0.090 | -0.018 | -0.140 | 0.272 |
| 0.136 | 0.215 | 0.001 | -0.117 | 0.022 | -0.087 | 0.097 | 0.093 | 0.055 | -0.128 |
| 0.181 | 0.197 | -0.174 | 0.177 | -0.321 | -0.127 | 0.024 | 0.078 | 0.207 | -0.330 |
| 0.205 | 0.080 | -0.173 | -0.100 | -0.231 | 0.217 | 0.304 | -0.079 | 0.198 | -0.024 |
| 0.010 | 0.172 | 0.110 | 0.368 | 0.303 | 0.274 | 0.206 | 0.313 | -0.108 | -0.211 |
| 0.129 | 0.260 | 0.126 | -0.092 | -0.154 | -0.128 | -0.297 | 0.287 | -0.101 | -0.112 |
| 0.303 | -0.235 | 0.151 | -0.122 | 0.083 | 0.013 | -0.248 | 0.133 | -0.017 | -0.187 |
| 0.188 | -0.017 | 0.044 | -0.136 | -0.130 | 0.310 | 0.209 | -0.040 | -0.007 | -0.174 |
| 0.292 | -0.252 | -0.054 | 0.092 | 0.063 | 0.088 | 0.227 | -0.245 | 0.091 | 0.073 |
| 0.296 | -0.031 | -0.186 | 0.035 | 0.168 | -0.223 | -0.175 | 0.148 | 0.152 | 0.109 |
| 0.240 | 0.010 | 0.220 | -0.374 | -0.040 | 0.064 | -0.104 | 0.028 | -0.087 | 0.113 |
| 0.170 | -0.143 | 0.368 | 0.151 | 0.157 | -0.020 | 0.191 | 0.311 | 0.164 | 0.166 |
| 0.261 | 0.000 | -0.043 | 0.231 | 0.154 | 0.156 | -0.415 | -0.088 | 0.063 | -0.070 |
| 0.003 | 0.139 | 0.260 | 0.217 | -0.163 | -0.073 | 0.033 | 0.081 | -0.246 | -0.021 |
| 0.236 | -0.198 | 0.155 | -0.004 | -0.297 | -0.053 | -0.167 | -0.018 | -0.168 | -0.083 |

| $SV_1$ | $SV_2$ | $SV_3$ | $SV_4$ | $SV_5$ | $SV_6$ | $SV_7$ | $SV_8$ | $SV_9$ | $SV_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.035 | 0.353 | 0.107 | 0.175 | 0.186 | -0.156 | -0.105 | -0.368 | 0.240 | 0.300 |
| 0.017 | 0.036 | 0.402 | 0.216 | -0.192 | -0.246 | 0.016 | -0.123 | 0.460 | -0.094 |
| 0.059 | 0.163 | -0.129 | 0.307 | -0.125 | -0.158 | 0.023 | 0.208 | -0.164 | 0.338 |
| 0.150 | 0.203 | -0.094 | -0.171 | 0.147 | -0.221 | 0.167 | -0.163 | -0.066 | -0.215 |
| 0.003 | 0.264 | 0.058 | -0.089 | -0.068 | 0.468 | -0.065 | 0.236 | 0.294 | 0.166 |

The error rate of Schmidt vectors was reported as $7.84 \times 10^{-14}$ in the baseline approach, whereas it was reported as $3.86 \times 10^{-15}$ in the novel approach. Therefore, the novel approach yields a lower error rate. It can then be concluded that Schmidt vectors are more accurate in the novel approach.

**Numerical Example (3)**

In this section, 20 random examples are provided. Table 7 presents the relevant data.

***Table 7. 20 Random Examples***

| | X | | Error | |
|---|---|---|---|---|
| **Example** | **Number of Rows(m)** | **Number of Columns(n)** | **Old Method** | **New Method** |
| 1 | 8 | 4 | 3.580E-15 | 7.086E-16 |
| 2 | 8 | 6 | 3.793E-15 | 1.096E-15 |
| 3 | 9 | 8 | 2.563E-14 | 4.727E-15 |
| 4 | 5 | 5 | 1.441E-14 | 2.755E-15 |
| 5 | 13 | 9 | 1.258E-14 | 2.521E-15 |
| 6 | 9 | 5 | 2.489E-15 | 7.459E-16 |
| 7 | 4 | 4 | 1.998E-15 | 1.360E-15 |
| 8 | 4 | 3 | 7.633E-16 | 1.665E-16 |
| 9 | 10 | 10 | 6.442E-14 | 4.303E-15 |
| 10 | 5 | 5 | 4.316E-15 | 2.366E-15 |
| 11 | 6 | 4 | 1.166E-15 | 7.685E-16 |
| 12 | 8 | 3 | 7.355E-16 | 4.996E-16 |
| 13 | 8 | 5 | 5.676E-15 | 1.429E-15 |
| 14 | 11 | 8 | 9.963E-15 | 3.777E-15 |
| 15 | 10 | 5 | 3.779E-15 | 1.645E-15 |
| 16 | 13 | 8 | 1.434E-14 | 2.344E-15 |

| | | | | |
|---|---|---|---|---|
| 17 | 8 | 7 | 3.953E-14 | 3.556E-15 |
| 18 | 10 | 7 | 5.721E-15 | 2.221E-15 |
| 19 | 10 | 5 | 2.007E-15 | 1.135E-15 |
| 20 | 8 | 7 | 6.529E-15 | 2.592E-15 |
| | **Sum** | | **2.234E-13** | **4.072E-14** |

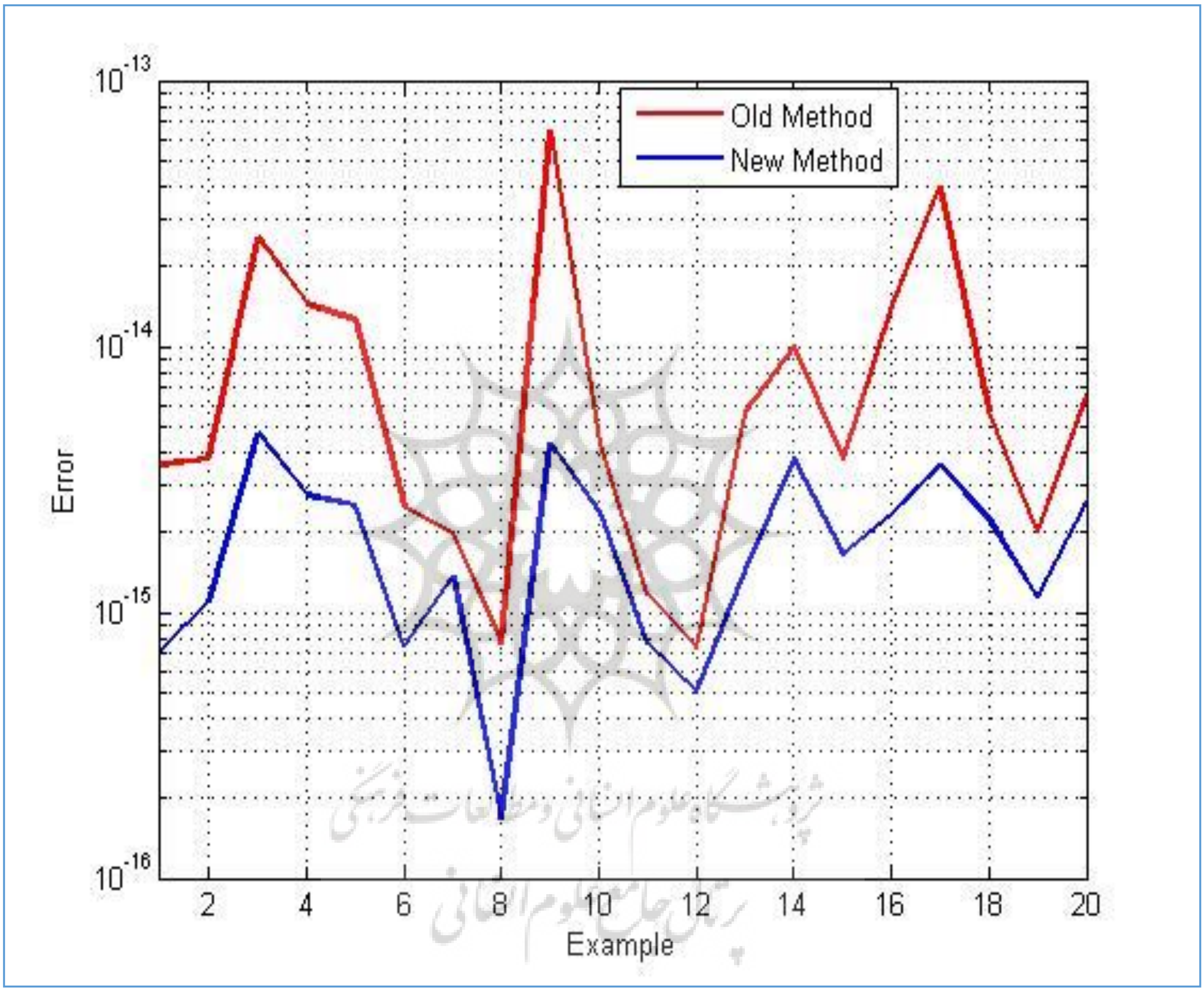


*Figure 2. Error Rates of Both Methods*

Figure 2 illustrates the error rates of both methods in a semi-logarithmic plot. Accordingly, the novel method has a lower error rate than the baseline method.

## 6. Case Study

Imam Khomeini Shazand Arak Oil Refining Company is an Iranian oil refining and petrochemical company, whose facilities are located in Markazi Province, 20 kilometers from Arak (Shahriari et al., 2019). This company is currently the largest producer of gasoline among Iranian oil refineries, producing 16.5 million liters of gasoline per day, and the second largest producer of gasoline in Iran after the Persian Gulf Star Condensate Refinery (Jafari et al., 2022). It is also the second largest producer of liquefied petroleum gas among Iranian oil refineries after Tabriz Refinery, producing 3.6 million liters of LPG per day (Hasani et al., 2024; Vaezi Hir et al., 2021). The company was established in 1992 by the National Iranian Oil Company and was titled as the Shazand Arak Oil Refining Company (Mortazavi & Saberinasab, 2016). In this example, the performance of thirteen units from the subunits of the Shazand Oil Refinery (SOR) is examined based on three different indicators (These indicators have been identified based on the opinions of the organization's expert specialists). Considering the confidentiality of the information of the organization under study, the names of the mentioned units have not been disclosed. The data related to the 13 units under study, based on three different criteria (pollution level ($C_1$), annual cost of raw materials ($C_2$), and the number of required workforce ($C_3$)), are presented in Table 8.

*Table 8. Data Related to the 13 Units Under Study*

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $Alternative_1$ | 5 | 285138 | 20 |
| $Alternative_2$ | 7 | 249582 | 15 |
| $Alternative_3$ | 5 | 670033420 | 20 |
| $Alternative_4$ | 5 | 554045180 | 20 |
| $Alternative_5$ | 3 | 13865985 | 16 |
| $Alternative_6$ | 1 | 1234129 | 20 |
| $Alternative_7$ | 3 | 556698 | 18 |
| $Alternative_8$ | 7 | 1167176925 | 15 |
| $Alternative_9$ | 9 | 1014356 | 36 |
| $Alternative_{10}$ | 9 | 973694 | 39 |
| $Alternative_{11}$ | 3 | 785438390 | 12 |
| $Alternative_{12}$ | 5 | 112252903 | 36 |
| $Alternative_{13}$ | 5 | 171915000 | 36 |

**Pollution Level:** This indicator signifies the pollution level of the unit under study and is classified as a cost type.

**Annual Cost of Raw Materials:** This indicator represents the annual expenditure on raw materials and is classified as a cost type.

**Number of Required Workforce:** This indicator shows the number of human resources needed and is classified as a cost type.

The correlation coefficient matrix, before implementing DGS, is presented in Table 9.

*Table 9. Correlation Coefficient Matrix, Before Implementing DGS*

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $C_1$ | 1.0000 | 0.0181 | 0.5133 |

| | | | |
|---|---|---|---|
| **$C_2$** | 0.0181 | 1.0000 | -0.4199 |
| **$C_3$** | 0.5133 | -0.4199 | 1.0000 |

For a better understanding, the absolute values of these numbers are illustrated in a bar chart in Figure 3.

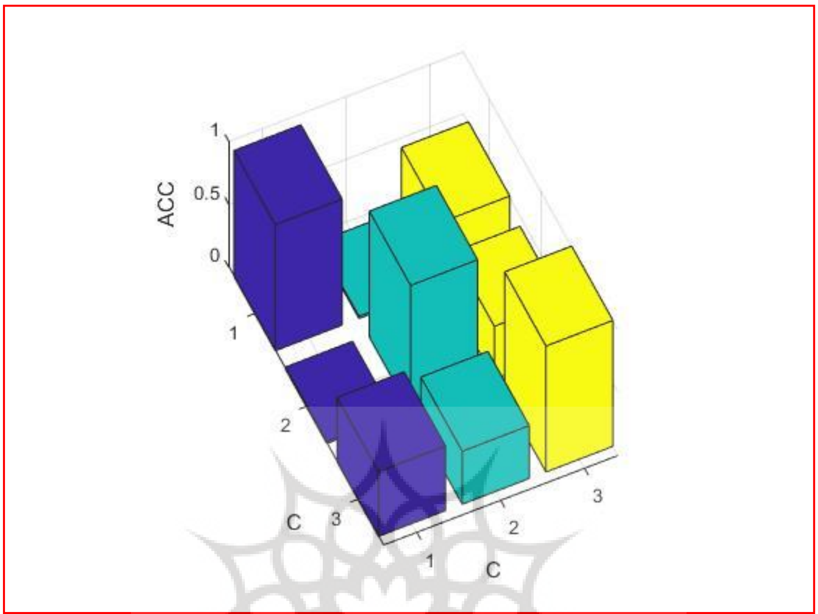


***Figure 3. Display of Absolute Correlation Coefficients (ACC) Before Implementing the Proposed Approach***

Subsequently, the Schmidt Vectors (SVs) obtained using the proposed approach (DGS) are presented in Table 10.

***Table 10. Matrix Containing Schmidt Vectors Obtained Using the Proposed Approach (DGS)***

| | **$SV_1$** | **$SV_2$** | **$SV_3$** |
|---|---|---|---|
| $Alternative_1$ | 0.2211 | -0.0995 | 0.1727 |
| $Alternative_2$ | 0.1658 | -0.0746 | 0.6050 |
| $Alternative_3$ | 0.2211 | 0.3413 | -0.0495 |
| $Alternative_4$ | 0.2211 | 0.2650 | -0.0110 |
| $Alternative_5$ | 0.1769 | -0.0707 | -0.0126 |
| $Alternative_6$ | 0.2211 | -0.0989 | -0.4128 |
| $Alternative_7$ | 0.1990 | -0.0894 | -0.0641 |
| $Alternative_8$ | 0.1658 | 0.6935 | 0.2179 |
| $Alternative_9$ | 0.3980 | -0.1788 | 0.3108 |
| $Alternative_{10}$ | 0.4311 | -0.1938 | 0.2270 |
| $Alternative_{11}$ | 0.1327 | 0.4572 | -0.1569 |

| | | | |
|---|---|---|---|
| $\text{Alternative}_{12}$ | 0.3980 | -0.1056 | -0.3114 |
| $\text{Alternative}_{13}$ | 0.3980 | -0.0663 | -0.3312 |

The correlation coefficient matrix after implementing DGS is presented in Table 11.

***Table 11. Correlation Coefficient Matrix, After Implementing DGS***

| | $SV_1$ | $SV_2$ | $SV_3$ |
|---|---|---|---|
| $SV_1$ | 1.0000 | -0.5557 | -0.1282 |
| $SV_2$ | -0.5557 | 1.0000 | -0.0113 |
| $SV_3$ | -0.1282 | -0.0113 | 1.0000 |

For a better understanding, the absolute values of these numbers are illustrated in a bar chart in Figure 4.

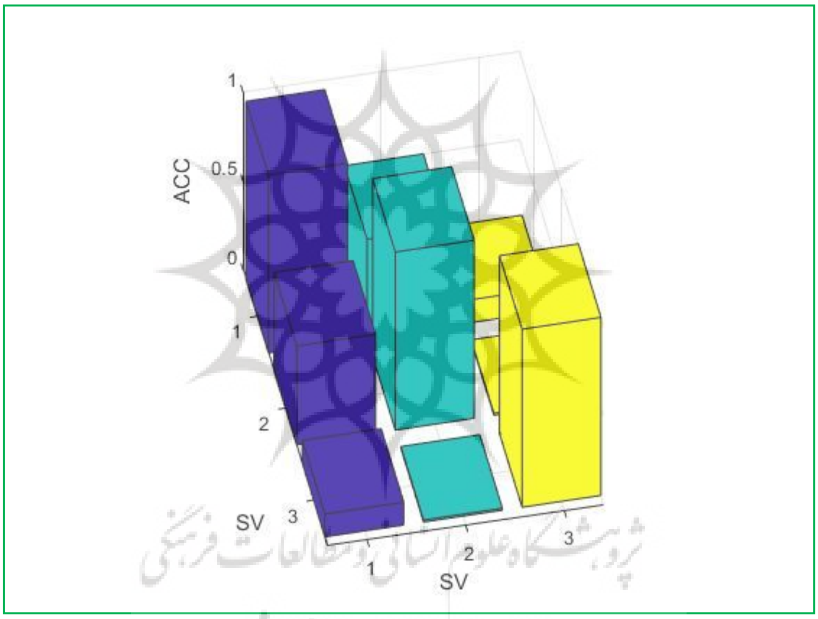


***Figure 4. Display of Absolute Correlation Coefficients (ACC) After Implementing the Proposed Approach***

The average absolute correlation coefficient before implementing the DGS method is 0.55, and the average absolute correlation coefficient after implementing the DGS method is 0.48. Therefore, as can be observed, the matrix containing Schmidt Vectors has less correlation compared to the initial decision matrix (i.e., Table 8). Thus, in the continuation of the discussion, with the help of the TOPSIS technique and this matrix, the performance of the 13 units of the Shazand Oil Refinery has been evaluated (The TOPSIS technique is one of the most well-known methods in MCDM) (Davoodi et al., 2022; Jafarzadeh Ghoushchi et al., 2019; Jokar et al., 2025; Karamian et al., 2025; Nouri, 2021; Mohseni Kiasari & Fartash, 2023).

The weight of the new criteria (The method obtained from the proposed Gram-Schmidt method) has also been calculated using the variance technique. The value of these weights is $W = (w_{C'_1} = 0.0114\,,\ w_{C'_2} = 0.0794\,,\ w_{C'_3} = 0.0831)$.

It should be noted that the error of the Schmidt Vectors in the basic method equals $1.179 \times 10^{-16}$, and the error of the Schmidt Vectors in the new method equals $3.001 \times 10^{-16}$.

Therefore, the new method has fewer errors, and it can be concluded that the computational accuracy of the Schmidt Vectors is higher in it. In Table 12, using the relation $d_{i,j} = sv_{i,j} + 1 - \underset{\forall k,l}{min}\{sv_{k,l}\}$, we have corrected the negative and zero data. This method is a common approach for generating positive decision data (Salehi et al., 2023).

***Table 12. Decision Matrix for Implementing the TOPSIS Method***

| | **C1'** | **C2'** | **C3'** |
|---|---|---|---|
| $Alternative_1$ | 1.6339 | 1.3133 | 1.5856 |
| $Alternative_2$ | 1.5786 | 1.3382 | 2.0179 |
| $Alternative_3$ | 1.6339 | 1.7541 | 1.3634 |
| $Alternative_4$ | 1.6339 | 1.6778 | 1.4018 |
| $Alternative_5$ | 1.5897 | 1.3422 | 1.4002 |
| $Alternative_6$ | 1.6339 | 1.3139 | 1.0000 |
| $Alternative_7$ | 1.6118 | 1.3234 | 1.3487 |
| $Alternative_8$ | 1.5786 | 2.1063 | 1.6307 |
| $Alternative_9$ | 1.8108 | 1.2340 | 1.7236 |
| $Alternative_{10}$ | 1.8440 | 1.2190 | 1.6398 |
| $Alternative_{11}$ | 1.5455 | 1.8700 | 1.2559 |
| $Alternative_{12}$ | 1.8108 | 1.3072 | 1.1014 |
| $Alternative_{13}$ | 1.8108 | 1.3465 | 1.0816 |

In Table 13, the scores obtained from the TOPSIS method and the final ranking of each of the studied units are presented.

***Table 13. Scores Obtained and Final Ranking After Implementing the TOPSIS Method***

| | **Score** | **Rank** |
|---|---|---|
| $Alternative_1$ | 0.5957 | 6 |
| $Alternative_2$ | 0.4178 | 12 |
| $Alternative_3$ | 0.5400 | 11 |
| $Alternative_4$ | 0.5545 | 8 |
| $Alternative_5$ | 0.6965 | 5 |
| $Alternative_6$ | 0.9327 | 1 |
| $Alternative_7$ | 0.7346 | 4 |
| $Alternative_8$ | 0.2690 | 13 |

| | | |
|---|---|---|
| $Alternative_9$ | 0.5496 | 9 |
| $Alternative_{10}$ | 0.5916 | 7 |
| $Alternative_{11}$ | 0.5419 | 10 |
| $Alternative_{12}$ | 0.8971 | 2 |
| $Alternative_{13}$ | 0.886887 | 3 |

In Figure 5, the scores obtained for each alternative are presented. It can be stated that the unit number 6 has had the best performance.

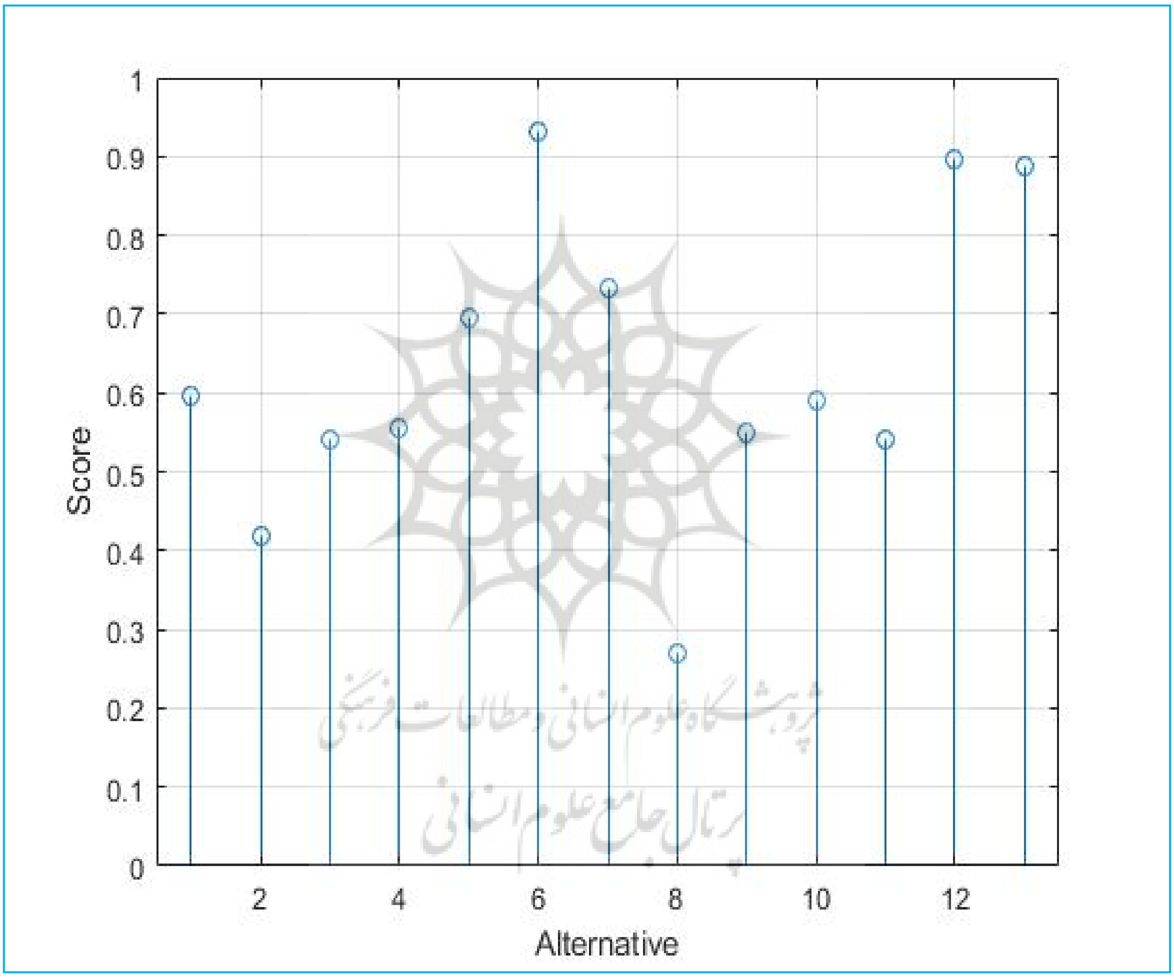


*Figure 5. The Scores Obtained for Each of the Alternatives*

## 7. Conclusion

This paper proposes a Developed Gram–Schmidt Algorithm (DGSA). In the baseline GS algorithm, vectors are selected randomly; however, this process is performed more sensitively in the proposed approach. A few numerical examples were employed to indicate the superiority of the proposed approach to the baseline method; therefore, researchers are recommended to make further use of the proposed approach in their studies. Researchers usually pay less attention to the interdependencies of criteria when using existing MCDM methods, and this is an important issue regarding these techniques that should be taken seriously. Additionally, to demonstrate the applicability of the proposed approach, a case study at the Shazand Oil Refinery was used. The results of the case study section confirmed that the developed Gram-Schmidt method was able to minimize variable correlation. In this section, one of the most reputable techniques available in MCDM, called TOPSIS, was employed. The criteria weights were also calculated using the variance method.

It is suggested that future researchers develop the DGS method under uncertainty. These researchers are also advised to use the Gram-Schmidt method in fuzzy and grey MCDM problems after its development.

### Data Availability Statement

Some or all data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request.

### Disclosure Statement

The authors do not have any conflict of interest to declare.

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