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Assessing Opportunities for Nocturnal Worship Based on Surah al-Muzzammil Using Data Envelopment Analysis (DEA)

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ABSTRACT:

Original Paper

This article aims to identify the most spiritually favorable nights of the year for believers and the optimal timing for nightly devotion to God by employing the Data Envelopment Analysis (DEA) technique. Given the emphasis on nocturnal worship in *Surah al-Muzzammil*, this study explores the relationship between the length of night and the opportunities it provides for worship. Using DEA, the nights of the year are evaluated based on the duration of night and environmental conditions in order to determine those nights that offer the greatest potential for devotion and supplication. In addition to temporal variables, nocturnal temperature is also considered as an influential factor in this evaluation. DEA serves here as an analytical tool for comparative performance assessment, facilitating the identification of nights with the most favorable conditions for nighttime worship. In the methodological section, two scenarios are implemented. The first scenario constructs a composite index based on the interval between the religious midnight (half of the canonical night) and *fajr* (dawn), as well as between *fajr* and sunrise. The second scenario applies the Fair Model (KA) and the AP Model with three indices: the interval from the religious midnight to *fajr*,

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from *fajr* to sunrise, and the temperature index. For more precise evaluation, the AP and KA models are also executed for the first scenario. The findings of this research can assist believers in planning their nocturnal worship during the most spiritually advantageous periods of the night.

KEYWORDS: The Qur'an, *Surah al-Muzzammil*, Data Envelopment Analysis (DEA), Nighttime worship, *al-layl*, Religious midnight.

1. Introduction

Religion, as a comprehensive system of beliefs, ethics, and regulations responsible for organizing human society and nurturing human beings (Javadi Amoli 1992), places special emphasis on worship as the means of attaining the highest spiritual and material ranks. From this perspective, the type and quality of worship deeply affect every individual and social matter. *Surah al-Muzzammil* emphasizes such concepts as nocturnal, Qur'anic recitation, patience, and social responsibility. Pedagogically, it embodies profound teachings that may serve as a framework for a spiritual-ethical lifestyle in the modern world. Scholars have defined worship as the utmost form of humility and obedience (al-Fayyūmī 1993, 2: 142; al-Zabīdī 1993, 2: 410), a deliberate act oriented toward divine proximity and the sanctification of God's majesty (Baḥrānī 1984, 2: 177).

Nightly worship (*al-tahajjud*) is among the most significant rituals to which followers of all revealed religions have been invited. It is a form of worship accompanied by seeking forgiveness and reciting the Qur'an (Ibn Manẓūr 1994, 3: 432). The Qur'an calls all believers to nocturnal devotion in nineteen verses (Q. 3:17, 113; 17:79; 25:64; 32:16–17; 20:130; 39:9; 51:17–18; 52:48; 73:2–4, 6, 20; 76:25–26; 89:2). The significance of the night is explicitly mentioned in over twenty Qur'anic verses, and those who spend the night in devotion are highly praised. God Almighty declares: "*And keep vigil for a part of the night, as a supererogatory [devotion] for you. It may be that your Lord will raise you to a praiseworthy station*" (Q. 17:79). Similarly, the verse Q. 25:64 refers to the servants who comprehend the sanctity of the night and dedicate themselves to nocturnal devotion (Kāshānī 1985, 6: 391): "*Those who spend the night for their Lord, prostrating and standing [in worship]*" (Q. 25:64). Moreover, in verse Q. 32:16-17, God praises the devout: "*Their sides vacate their beds to supplicate their Lord in fear and hope, and they spend out of what We have provided them. No one knows what has been kept hidden for them of comfort as a reward for what they used to do*". The night-watchers have been promised a joyful delight. From the commentators' perspective, this "joyful delight" denotes a form of blessing and exaltation that represents the fulfilment of human longing,

bringing great pleasure to the beholder (al-Ṭabrisī 1993, 8: 518).

God Almighty, in the verse Q. 76:26, commands: "*And worship Him for a watch of the night and glorify Him the night long*." Exegetes have understood this verse as a recommendation for nocturnal worship, especially during long nights. According to commentators, the description of *layl* (night) as *ṭawīl* (long) is explanatory, not restrictive, that is, it does not imply that short nights should be excluded from nocturnal glorification. The term *al-tasbīḥ* (glorification) here denotes the night prayer (*ṣalāt al-layl*) (Tabataba'i 1973, 20: 141). This notion is reaffirmed in Q. 51:17: "*They used to sleep a little during the night*." As al-Zamakhsharī (1994, 4: 116) observes, this verse employs multiple emphatic expressions: the very word *hujūʿ* implies avoidance of sleep, while the modifiers *qalīlan* (a little) and *mina al-layl* (of the night) both intensify the sense of minimal sleep. The addition of *mā* further amplifies the expression, depicting the believers as those who strive to remain awake for night prayer. Similarly, verse Q.52:49 commands: "*And also glorify Him during the night and at the receding of the stars*." Here again, *al-tasbīḥ* signifies nocturnal worship.

Exegetes agree that in the opening verses of *Surah al-Muzzammil*, the Prophet Muḥammad (PBUH) and, by extension, the early Muslim community, is commanded to engage in nightly devotion for a considerable portion of the night, accompanied by *tartīl* of the Qur'an, that is, recitation combined with contemplative reflection on its meanings. This injunction is understood as a formative command during the earliest phase of prophet hood, intended to prepare the Prophet spiritually for the reception of divine revelation (Tabataba'i 1973, 10: 575). *Surah al-Muzzammil* is among the gradually revealed chapters of the Qur'an, consisting of two distinct revelations: verses 1–19 form a cohesive unit, while verse 20 was revealed later. Evidence for this includes the linguistic and thematic coherence of verses 1–19, their consistent syntactic structure (connected pronouns, conjunctions, and emphatic particles such as *inna*) and unified focus on nocturnal devotion, in contrast to verse 20, which is syntactically and thematically independent. The latter begins with *inna*, a particle typically used to introduce independent sections or chapters (al-Darwazah 2004; Nekounam 2001). In verse Q. 73:20, God declares:

إِنَّ رَبَّكَ يَعْلَمُ أَنَّكَ تَقُومُ أَدْنَى مِنْ ثُلُثَيِ اللَّيْلِ وَنِصْفَهُ وَثُلُثَهُ وَطَائِفَةٌ مِنَ الَّذِينَ مَعَكَ وَاللَّهُ يُقَدِّرُ اللَّيْلَ وَالنَّهَارَ عَلِمَ أَنْ لَنْ تُحْصُوهُ فَتَابَ عَلَيْكُمْ فَاقْرَءُوا مَا تَيَسَّرَ مِنَ الْقُرْآنِ عَلِمَ أَنْ سَيَكُونُ مِنْكُمْ مَرْضَى وَآخَرُونَ يَضْرِبُونَ فِي الْأَرْضِ يَبْتَغُونَ مِنْ فَضْلِ اللَّهِ وَآخَرُونَ يُقَاتِلُونَ فِي سَبِيلِ اللَّهِ فَاقْرَءُوا مَا تَيَسَّرَ مِنْهُ وَأَقِيمُوا الصَّلَاةَ وَآتُوا الزَّكَاةَ وَأَقْرِضُوا اللَّهَ قَرْضًا حَسَنًا وَمَا تُقَدِّمُوا لِأَنْفُسِكُمْ مِنْ خَيْرٍ تَجِدُوهُ عِنْدَ اللَّهِ هُوَ خَيْرًا وَأَعْظَمَ أَجْرًا وَاسْتَغْفِرُوا اللَّهَ

إِنَّ اللَّهَ غَفُورٌ رَحِيمٌ (المزمل/20)

Indeed your Lord knows that you stand vigil nearly two thirds of the night or [at times] a half or a third of it along with a group of those who are with you. Allah measures the night and the day. He knows that you cannot calculate it [exactly], and so He was lenient toward you. So recite as much of the Qur'an as is feasible. He knows that some of you will be sick, while others will travel in the land seeking Allah's grace, and yet others will fight in the way of Allah. So recite as much of it as is feasible, and maintain the prayer and pay the zakat and lend Allah a good loan. Whatever good you send ahead for your souls you will find it with Allah [in a form] that is better and greater with respect to reward. And plead to Allah for forgiveness; indeed Allah is all-forgiving, all-merciful (Q.73:20).

According to exegetes, the particle *faʾ* in the phrase *fa-qraʾū* indicates that this mitigation (*al-takhfīf*) arises directly from God's awareness of human difficulty in calculating the night's precise divisions. The easing of the command broadens the scope of participation in nocturnal worship, allowing each believer to worship according to personal capacity. It does not, however, abrogate the earlier prescription of spending one-third, half or less than two-third of the night in prayer for those capable of such endurance, since "*Allah does not task any soul beyond its capacity*" (Q. 2:286). Moreover, in the verse the Prophet confirmed himself and a group of believers who used to keep vigil for one-third, half, or less than two-thirds of the night and were able to maintain these measures. Therefore, the impossibility of observing this degree of night-worship is attributed to the general public, including those who did not engage in night devotion, and those who did. On this basis the obligation was mitigated in such a way that the duty itself remained for those who were able to perform it, while a more lenient level was rendered practicable for everyone. Hence the injunction, "*recite as much of the Qur'an as is feasible*" (Q. 73:20) and the practice of night-worship, whether at its maximum degree or at its minimum (reciting an attainable portion of the Qur'an), is recommended for all believers (Tabataba'i 1973, 20: 75).

In this way, the exhortation to nocturnal devotion signifies the importance of spiritual discipline, avoidance of indulgence, and ritual orderliness. Moreover, the reference in the same verse to *"maintain the prayer and pay the zakat*" and *"others will fight in the way of Allah"* reveals that individual worship is incomplete without social engagement. The gradual revelation of the surah demonstrates the Qur'anic principle of progressive spiritual training. The surah begins with personal spiritual cultivation through nocturnal devotion and later extends to social responsibilities such as *al-jihād* and *al-zakāt*. This progression mirrors the divine pedagogy that aligns religious duty with human capacity. By

invoking the principle of *al-taysīr* (facilitation), the Qur'an presents a pedagogical model harmonious with human nature and capability. God, acknowledging human limitations such as fatigue, worldly duties, and differing capacities, modulated the command of nightly worship to a level attainable by all. The adjustment from "the entire night" to "whatever is easy" prevents spiritual burden and fosters sustainable devotion.

From the authors' perspective, applying the principle of *al-taysīr* (ease and flexibility in divine injunctions) to the identification of realistic opportunities for nightly worship is of great significance. By moderating this command, the Qur'an teaches that worship should not be conceived as an isolated act detached from daily life, but rather as a harmonized component of one's regular schedule. *Surah al-Muzzammil*, through its intelligent application of the principle of *al-taysīr*, illustrates that spiritual training must be realistic, proportionate to temporal and spatial capacities, and adaptable to various life circumstances. This approach transforms night worship into a sustainable opportunity for self-cultivation while shielding the individual from psychological strain.

There are significant traditions emphasizing the performance of *ṣalāt al-layl* (night prayer) during the last third of the night. Abī Baṣīr narrates from Imām al-Ṣādiq that when he asked about night and day worship, the Imam said: "*At dawn, perform eight units (al-rakʿah) of prayer, followed by the al-witr prayer, which consists of three al-rakʿahs—two together and one separately. The best time for the night prayer is at the end of the night*" (Khoei n.d.). Likewise, Ismāʿīl ibn Saʿīd al-Ashʿarī reports that when he asked Imām al-Riḍā about the *al-witr* prayer, the Imam replied, "*The most beloved time to me is the first dawn (al-fajr al-awwal).*" When asked about the best hours of the night, he said, "*The last third of the night*" (Khoei n.d.). Regarding the time of the night prayer, this time extends from midnight until the second dawn (*ṭulūʿ al-fajr al-ṣādiq*), and the most excellent period is at the final third of the night (*al-saḥar*), particularly close to dawn (Ṭabāṭabāʾī Yazdī 1989, 1: 525). Most classical and contemporary jurists affirm this ruling and consider *al-saḥar*, the last part of the night, the most meritorious time for *ṣalāt al-layl* (Ḥakīm n.d., 5: 113; al-Najafī 1983, 7: 312; al-Ṣadūq 1997). The Qur'anic phrase *"And Allah measures the night and the day"* (Q. 73:20) indicates the precise proportioning of day and night, showing that the timing of worship is naturally dependent on these variations. The expression "*recite as much of the Qur'an as is feasible*" (Q. 73:20) reflects the principle of *al-taysīr*, ease and flexibility, in structuring nightly devotions according to one's actual capacity and opportunity.

This study employs Data Envelopment Analysis (DEA) to examine the

varying conditions of nights for optimizing worship performance. The aim is to assist believers in effectively planning their nightly devotions based on two fundamental components:

1. Qur'anic and exegetical foundation: Verse Q.73:20 explicitly refers to the change in the duration of night worship according to the measurement of night and day and individual conditions. Thus, identifying the periods of the year with the longest nights and the most favorable conditions for devotion, constitutes a practical application of this verse and facilitates a more effective realization of *al-tahajjud*.
2. Scientific and empirical approach: Given the natural variations in night length throughout the year, measurable criteria, such as temperature, the interval between religious midnight and dawn, and between dawn and sunrise, must be quantitatively analyzed. The DEA method enables ranking of nights based on multiple indicators, thereby identifying those with the highest worship potential capacity.

Hence, a direct relationship exists between the content of verse Q. 73:20, emphasizing divine proportioning of night and day, and the principle of *al-taysīr* in worship, as well as the scientific necessity of this study. The objective is to integrate interpretive understanding with advanced analytical tools to propose a concrete and operational model for maximizing the spiritual opportunities of nightly worship, without departing from the Qur'anic spirit of balance and mercy. This study can therefore be classified as an interdisciplinary research (Brun et al. 2005; Klein & Newell 1997). Islamic education demands practical paradigms that embody religious teachings through lived behavior. Thus, *Surah al-Muzzammil* begins by addressing the Prophet, whose own nightly devotions, critical reflection, avoidance of superficiality, perseverance, and social responsibility serve as a living paradigm for applied spiritual training.

2. Literature Review

Data Envelopment Analysis (DEA), first introduced by Farrell in 1957, is a non-parametric method used to evaluate the efficiency of decision-making units (DMUs). DEA models can be categorized into constant returns to scale and variable returns to scale types. The constant returns to scale models, initially developed by Charnes, Cooper, and Rhodes in 1978, is known as the CCR model (Charnes et al. 1978). Later, the variable returns to scale models were introduced by Banker, Charnes, and Cooper in 1984, and is known as the BCC model. Various extensions of DEA have been proposed depending on data characteristics, such as fuzzy, interval, and stochastic approaches. Furthermore, based on network structures, several

models have been developed, including network DEA and dynamic DEA models.

Other notable methods and studies in this field include: multi-objective data envelopment analysis (Chen et al. 2009); congestion measurement in DEA models using weight restrictions and output exchanges (Jahanshahloo et al. 2011); a modified common-weight set approach for completing the ranking of decision-making units (Payan et al. 2011); fair allocation of shared fixed costs using the DEA method (Khodabakhshi & Aryavash 2013); ranking units using fuzzy DEA (Khodabakhshi & Aryavash 2014); a review of methods for estimating input congestion in DEA (Khodabakhshi et al. 2014); an optimistic–pessimistic structure in cross-efficiency (Khodabakhshi & Aryavash 2017); fair ranking of decision-making units using optimistic–pessimistic weights (Jahanshahloo et al. 2017); and pattern detection in DEA using a value-efficiency approach (Nasrabadi 2019). Additionally, some applied studies in the field of data envelopment analysis include Khodabakhshi and Mirkoushesh (2024) and Besharati and Khodabakhshi (2025). Among interdisciplinary studies linking religious studies and mathematics, Khodabakhshi et al. (2025) developed a self-assessment system based on the *Khuṭbah al-Muttaqīn* narrative and examined the relationships between its sections.

3. Methodology

In this study, Data Envelopment Analysis (DEA) is employed to identify the most suitable nights of the year for nocturnal worship from a believer's perspective. DEA serves as a mathematical technique for evaluating the efficiency of different entities using available indicators and data. Two distinct scenarios are considered. In the first scenario, two temporal indices are selected, based on the Qur'anic recommendations in *Surah al-Muzzammil*, which emphasize the regulation of nightly devotion and worship. In the second scenario, in addition to the two temporal indices, a temperature index is also included. For computational purposes, the first and second indices are analyzed under weight restrictions (Wong & Beasley 1990) in the first scenario, while the first, second, and third indices (including temperature) are used in the second scenario. The models used in this study include the Andersen–Petersen (AP) model (Andersen & Petersen 1993) and the KA fairness model (Khodabakhshi & Aryavash 2012). It should be noted that the prayer-time interval indices (www.praytimes.org) and the temperature data (www.visualcrossing.com) pertain to the city of Tehran.

Index 1: The interval between Islamic midnight and the dawn call to

prayer. This index represents the duration of the night suitable for nocturnal devotion and prayer. The time span between midnight and *al-fajr* reflects the opportunity believers have for spiritual engagement and supplication. In Islamic culture, longer nights provide greater capacity for worship and self-reflection, while shorter nights limit this opportunity. Thus, this index indirectly evaluates the quantity and quality of potential time for night worship.

Index 2: The interval between dawn and sunrise. This index complements the first one, covering the period between the dawn call to prayer and sunrise. During this interval, believers can engage in various devotional acts, such as post-prayer supplications, supererogatory morning prayers, recitation of Qur'an, and *al-dhikr* (remembrance of God). Therefore, this interval also contributes meaningfully to the overall assessment of optimal times for spiritual engagement during the night and early morning hours (Figure 1).

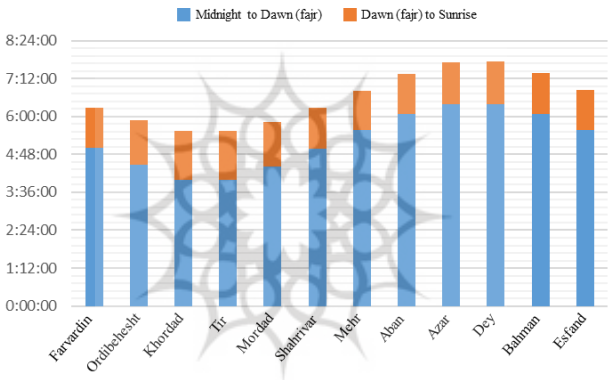

Figure 1. Bar chart of the total intervals between Islamic midnight and dawn and between dawn and sunrise for each month

Index 3: Air temperature. This index during different nights of the year plays a significant role in determining individual comfort and ease of nocturnal worship. Using this index, along with the temporal indicators, various nights of the year can be ranked from the believers' perspective to analyze the quality and suitability of night hours for devotion. To assess the temperature index for each month, a five-point Likert scale (ranging from 5=very good to 1=very poor) was defined as follows:

- Score 5 (Very Good): The average night temperature is between 20–23°C. This condition is ideal, requiring no heating or cooling systems. It represents the most favorable situation for nightly worship and should be assigned the highest output value in DEA.

- Score 4 (Good): The average night temperature is between 17–19°C and 24–25°C. Slightly less ideal but still comfortable, with only minor need for thermal adjustment.
- Score 3 (Moderate): The average night temperature is between 14–16°C and 26–27°C. Noticeable need for heating or cooling systems; comfort is reduced.
- Score 2 (Poor): The average night temperature is between 10–13°C and 28–30°C. Discomfort is evident; significant reliance on thermal regulation is required.
- Score 1 (Very Poor): The average night temperature is below 10°C or above 30°C. Conditions are extremely unfavorable for worship, with low tolerance and continuous dependence on temperature control systems.

For quantitative evaluation in the DEA framework, the Likert scale scores were converted to corresponding output index values as follows: (1 = 0.25, 2 = 0.50, 3 = 0.75, 4 ≈ 0.90, 5 = 1.00).

Table 1. Temperature Index Values Based on the Five-Point Likert Scale

Month	Mean Night Temperature (°C)	Likert Score	Temperature Index
Farvardin (March–Apr.)	16.7	3	0.75
Ordibehesht (Apr.–May)	20.67	5	1.00
Khordad (May–June)	27.78	3	0.75
Tir (June–July)	31.17	1	0.25
Mordad (July–Aug.)	32.36	1	0.25
Shahrivar (Aug.–Sep.)	28.58	2	0.50
Mehr (Sep.–Oct.)	22.73	5	1.00
Aban (Oct.–Nov.)	13.66	2	0.50
Azar (Nov.–Dec.)	7.09	1	0.25
Dey (Dec.–Jan.)	6.04	1	0.25
Bahman (Jan.–Feb.)	8.20	1	0.25
Esfand (Feb.–Mar.)	10.13	2	0.50

Accordingly, Ordibehesht (Apr.–May) and Mehr (Sep.–Oct.) have the most favorable nocturnal temperature conditions (index=1.00). Farvardin (March–Apr.) and Khordad (May–June) display moderately favorable conditions (index=0.75), while winter months and the hottest summer months receive the lowest comfort scores (index=0.25) (Table 1).

4. Analysis and Discussion

In this section, we analyze the results derived from constructing a

composite index as an independent method, as well as from implementing DEA models, and compare the two sets of findings. Using the common weighted-sum method, a composite index was created by assigning a weight of 0.7 to the indicator "interval between Islamic midnight and dawn" and a weight of 0.3 to the indicator "interval between dawn and sunrise." The ranking of this indicator shows the months in the following order: Azar and Dey; Bahman and Aban; Esfand and Mehr; Farvardin and Shahrivar; Ordibehesht and Mordad; and finally Tir and Khordad. In each pair, the two months are placed at the same position with only a very small difference between them (Table 2 - Figure 2).

Table 2. Composite Index Derived from Two Temporal Indicators (interval between Midnight–Dawn and Dawn–Sunrise)

Month	Composite Index (hh:mm:ss)
Farvardin	03:53:22
Ordibehesht	03:33:36
Khordad	03:15:58
Tir	03:15:57
Mordad	03:30:32
Shahrivar	03:52:50
Mehr	04:16:34
Aban	04:38:09
Azar	04:52:26
Dey	04:52:42
Bahman	04:38:31
Esfand	04:16:48

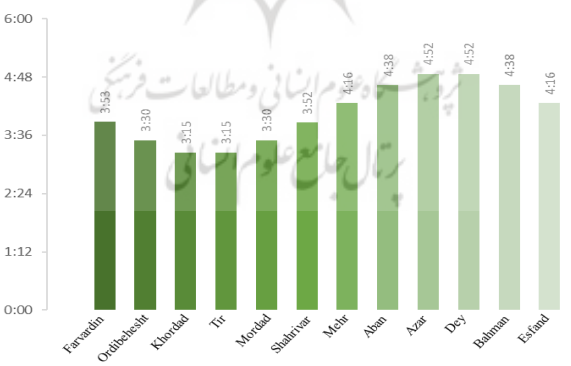

Figure 2. Bar chart of the composite index derived from the prayer-time intervals: from Islamic midnight to the Fajr (dawn), and from the Fajr to sunrise.

In the output-oriented AP model, lower super-efficiency values correspond to higher efficiency rankings. Accordingly, the ranking order is: Dey, Azar, Bahman, Aban, Mehr, Esfand, Farvardin, Shahrivar,

Ordibehesht, Mordad, Khordad, and Tir (Table 3).

Table 3. Month Ranking Based on the AP Model (Scenario 1)

Month	Super-Efficiency	Rank
Farvardin	1.253871	7
Ordibehesht	1.377272	9
Khordad	1.492076	11
Tir	1.492678	12
Mordad	1.389261	10
Shahrivar	1.256556	8
Mehr	1.140666	5
Aban	1.054851	4
Azar	1.000522	2
Dey	0.999284	1
Bahman	1.050914	3
Esfand	1.139693	6

To determine the most suitable months for nightly worship, three DEA models (CCR, AP, and KA) were applied using two indicators, the interval between Islamic midnight and dawn, and the interval between dawn and sunrise. A weight restriction was imposed to reflect the greater importance of the first indicator: $u_1 = 2.3 \times u_2$

Thus, the midnight-to-dawn interval carries 2.3 times more weight than the dawn-to-sunrise interval. This decision aligns with Qur'anic and hadith-based emphases on the *final third of the night* as the most virtuous time for night prayer.

Table 4. Fairness-Based KA Model Results (Scenario 1)

No.	Month	Pessimistic Efficiency	Optimistic Efficiency	Fair Efficiency	Rank
1	Farvardin	0.079474	0.079489	0.079481	8
2	Ordibehesht	0.071180	0.082881	0.077030	12
3	Khordad	0.063608	0.095273	0.079440	10
4	Tir	0.063582	0.095479	0.079530	7
5	Mordad	0.069986	0.087355	0.078671	11
6	Shahrivar	0.079262	0.079695	0.079478	9
7	Mehr	0.077427	0.088472	0.082949	5
8	Aban	0.079447	0.096411	0.087929	4
9	Azar	0.082922	0.101439	0.092181	2
10	Dey	0.083159	0.101492	0.092326	1
11	Bahman	0.079654	0.096517	0.088085	3
12	Esfand	0.077220	0.088578	0.082899	6

Based on Table 4, the KA model ranking identifies Dey, Azar, and Bahman as the top three months, followed by Aban, Mehr, and Esfand (with nearly equal efficiency values), and finally Tir, Farvardin, Shahrivar,

Khordad, Mordad, and Ordibehesht. These results indicate that winter and late autumn months show higher efficiency in terms of the studied indicators. The KA model integrates pessimistic, optimistic, and convex-combined (fair) efficiency values, providing a balanced assessment (Figure 3). Dey ranks highest in all three perspectives, confirming its overall superiority (Figure 4).

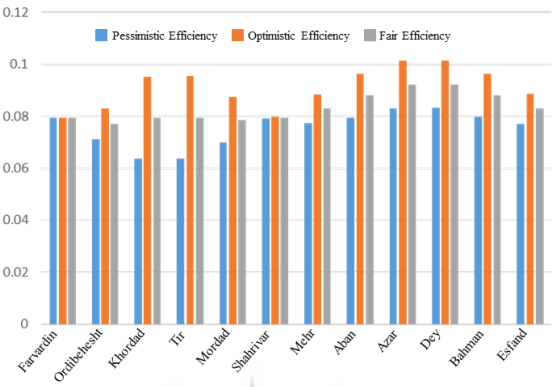

Figure 3. Bar chart of efficiency results for different months based on the fair KA model in three modes of pessimistic, optimistic, and fair efficiency in terms of two indicators.

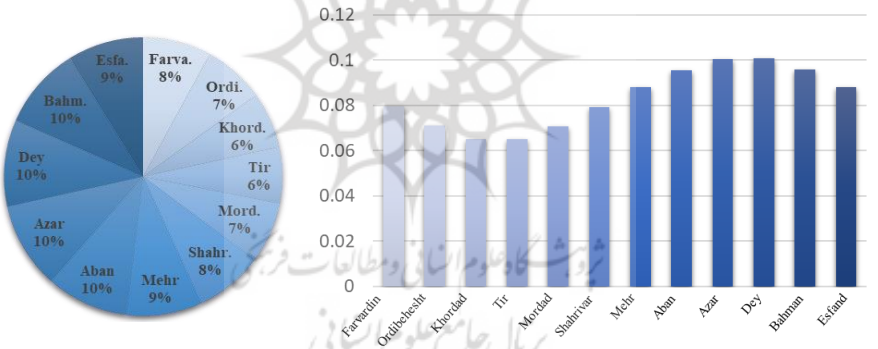

Figure 4. Pie chart and Bar chart of the efficiency of the fair model for different months in terms of two indicators

A more detailed CCR model analysis was then performed for Azar and Dey, which had the best performance across all models. The results show that seven nights in late Azar and eight nights in early Dey achieved full efficiency (score=1). These nights can thus be identified as the optimal nights of the year for worship, supplication, and night vigil, as they maximize the utilization of available devotional time according to Qur'anic guidance (Figure 5 & 6). Given Dey's top ranking in the KA model as well, a separate DEA run confirmed that the early nights of Dey represent the most favorable conditions for nocturnal devotion.

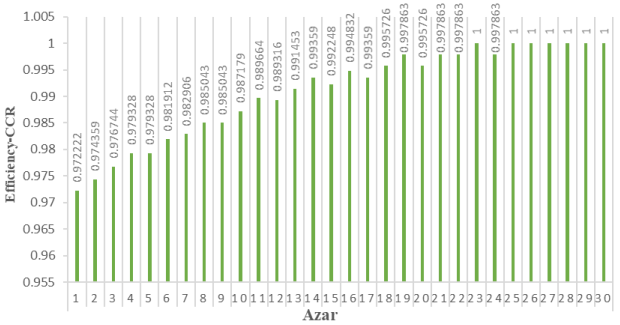

Figure 5. Efficiency of Nights in Azar Based on the CCR Model

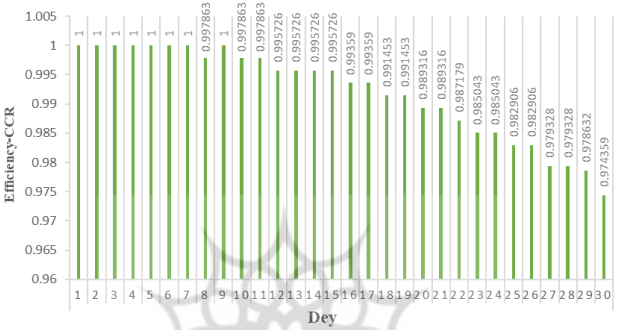

Figure 6. Efficiency of Nights in Dey Based on the CCR Model

To further refine the results, the temperature index was incorporated as a third indicator alongside the two temporal indices in the AP model (Scenario 2). The resulting month rankings are as follows: Mehr, Dey, Ordibehesht, Azar, Aban, Bahman, Esfand, Farvardin, Shahrivar, Khordad, Mordad, and Tir (Table 5).

Table 5. AP Model Results Incorporating Three Indicators (Scenario 2)

Month	Super-Efficiency
Farvardin	1.13059
Ordibehesht	1.00000
Khordad	1.31200
Tir	1.46420
Mordad	1.36823
Shahrivar	1.18572
Mehr	0.87506
Aban	1.00959
Azar	1.00050
Dey	0.99928
Bahman	1.04881
Esfand	1.08489

With the inclusion of air temperature, the analysis confirms that Mehr and Dey retain the highest efficiency levels, indicating optimal overall conditions (temporal and environmental) for nightly worship during these months.

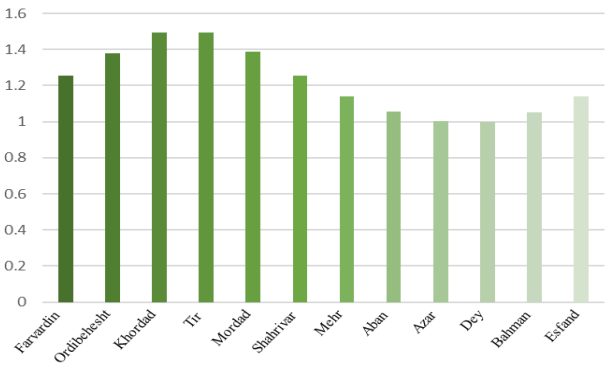

Figure 7. Bar chart illustrating super-efficiency results of the AP model based on two indicators

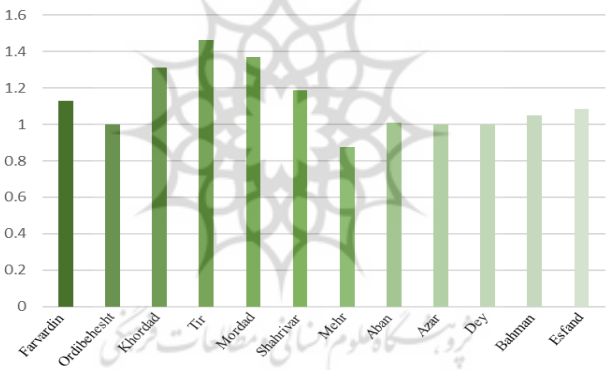

Figure 8. Bar chart illustrating super-efficiency results of the AP model based on three indicators

Figures 7 and 8 depict the bar graphs for super-efficiency values under the two- and three-indicator scenarios, respectively. In the super-efficiency model, lower numerical values correspond to higher efficiency ranks.

When the temperature indicator was incorporated, the results of the KA (fairness-based) model were obtained as shown in Table 6. The ranking order of the months in this case is as follows: Mehr, Ordibehesht, Farvardin, Khordad, Aban, Esfand, Shahrivar, Dey, Azar, Bahman, Tir, and Mordad (Figure 9).

Table 6. KA Model Results Based on Three Indicators

Month	Pessimistic Efficiency	Optimistic Efficiency	Fair Efficiency	Rank
Farvardin	0.079474	0.120000	0.098517	3
Ordibehesht	0.071180	0.160000	0.112917	2
Khordad	0.063608	0.120000	0.090107	4
Tir	0.040000	0.095479	0.066070	11
Mordad	0.040000	0.087355	0.062252	12
Shahrivar	0.079262	0.080000	0.079609	7
Mehr	0.077427	0.160000	0.116228	1
Aban	0.079447	0.096411	0.087419	5
Azar	0.040000	0.101439	0.068871	9
Dey	0.040000	0.101492	0.068896	8
Bahman	0.040000	0.096517	0.066558	10
Esfand	0.077220	0.088578	0.082557	6

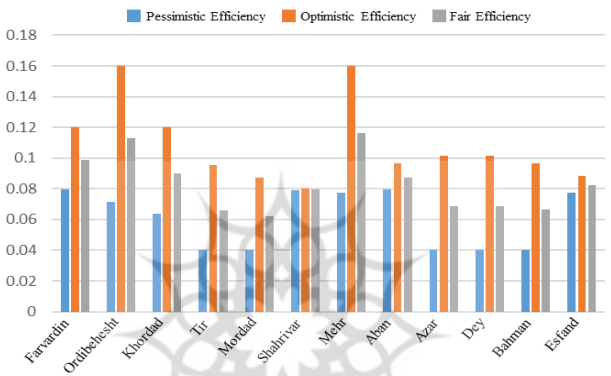

Figure 9. Bar chart illustrating pessimistic, optimistic, and fair efficiencies of the months according to the KA model under the three-indicator scenario.

Table 7 presents a comparison of the ranking of months across the year based on the AP and KA models for each of the two scenarios.

Table 7. Comparison of monthly rankings in the AP and KA models for the two scenarios

Month	AP (rank) Scenario 1	AP (rank) Scenario 2	KA (rank) Scenario 1	KA (rank) Scenario 2
Farvardin	7	8	8	3
Ordibehesht	9	3	12	2
Khordad	11	10	10	4
Tir	12	12	7	11
Mordad	10	11	11	12
Shahrivar	8	9	9	7
Mehr	5	1	5	1
Aban	4	5	4	5
Azar	2	4	2	9
Dey	1	2	1	8
Bahman	3	6	3	10
Esfand	6	7	6	6

The observed discrepancies between the AP model and the composite index reflect the additional influence of the temperature variable in the multi-indicator DEA framework (Table 7). The interpretive implications can be summarized as follows:

Mehr: Ranked *first* in the AP model due to the inclusion of the temperature factor, highlighting its moderate climate as a favorable condition for nocturnal worship. However, it ranks *sixth* in the composite index, which is purely based on prayer-time intervals. This demonstrates that environmental comfort significantly impacts its prioritization when temperature is considered.

Dey: Retains a consistently high ranking (*second in AP, first in the composite index*), indicating that time-based indices alone are sufficient to confirm Dey's exceptional suitability for night worship. The minor difference implies that the cold temperature does not diminish its devotional potential.

Ordibehesht: Shows a six-rank improvement in the AP model, underscoring the positive impact of mild temperatures on efficiency during this month. Time intervals alone placed it lower in the composite index, suggesting that thermal comfort substantially enhances its devotional feasibility.

Azar and Aban: Both experience slightly lower rankings in the AP model compared to the composite index. This suggests that temperature plays a lesser role during these cooler months, with time intervals being the more decisive factor in their ranking. In essence, their temporal efficiency already approaches the optimum, leaving limited room for environmental improvement.

Bahman and Esfand: These two months show minimal discrepancies between the AP model and the Composite Index (a maximum ranking difference of 3). In these months, the temperature index in the AP model may exert greater influence compared to the prayer-time intervals, yet the winter-adjacent months generally achieve higher ranks in the Composite Index.

Farvardin and Tir: Interestingly, Farvardin ranks eighth in both models, indicating a nearly identical evaluation. This similarity suggests that temperature effects during this month are less pronounced, while the timing of prayer intervals has a greater impact on the rankings. Tir also maintains the same twelfth position across both models.

Shahrivar: ranks ninth in the AP model and seventh in the Composite Index. This implies that in the AP model, temperature may have reduced its

rank, whereas in the Composite Index, the weighting of prayer-time intervals slightly improved its position.

Mordad and Khordad: Finally, Mordad and Khordad differ by only one rank between the two models, indicating that temperature effects were similar for these months.

This analysis demonstrates that the temperature index in the AP model significantly influences the rankings compared to the Composite Index. Overall, incorporating temperature into the AP model causes noticeable changes in certain months, particularly those with extreme thermal conditions. Comparative analysis reveals that the inclusion of temperature as a third variable leads to greater overall change in the KA model than in the AP model. In general, it can be stated that in the absence of sufficient heating or cooling systems, the three-index model yields more accurate results for planning night worship, while the two-index model is more suitable when such systems are available. It should be noted that since both prayer-time intervals and temperature indices are based on data from Tehran, the findings are context-specific and not directly generalizable to all cities.

5. Conclusion

The objective of this study was to produce a comprehensive ranking of nights according to their relative suitability for night worship, rather than merely identifying the longest night of the year. Using quantitative analysis based on Data Envelopment Analysis (DEA), the ranking is precise and reproducible. The analysis revealed a cyclic proportionality among prayer-time intervals across months, reflecting the natural alternation of night and day. Two analytical scenarios were considered. In Scenario 1, two indices were employed: the intervals midnight–Fajr and Fajr–sunrise. In Scenario 2, in addition to these two temporal indices, air temperature was included as a third indicator. All indices were computed for the city of Tehran. Scenario 1 demonstrated similar efficiency values for paired months (for example, Farvardin–Shahrivar, Ordibehesht–Mordad, Khordad–Tir, Mehr–Esfand, Aban–Bahman, and Azar–Dey). Another notable result was the continuity between the last ten days of Azar and the first eleven days of Dey, which exhibited nearly identical efficiency values and indicate a seamless transition suitable for structured planning of nocturnal worship. The final monthly rankings under the AP model in the first scenario are as follows: Dey, Azar, Bahman, Aban, Mehr, Esfand, Farvardin, Shahrivar, Ordibehesht, Mordad, Khordad, and Tir, while in the second scenario (which adds air temperature), the order is: Mehr, Dey, Ordibehesht, Azar,

Aban, Bahman, Esfand, Farvardin, Shahrivar, Khordad, Mordad, and Tir. In summary, the three-index model is preferable in contexts lacking adequate climate-control systems, whereas the two-index model is better suited to environments that are equipped with heating and cooling.

5.1. Recommendations

Future studies are encouraged to incorporate additional variables such as daily workload, the nature and intensity of hardships, social and religious conditions, and individual spiritual inclination. By quantifying these factors, it would be possible to design a personalized algorithm to determine the most effective nighttime periods for worship tailored to each individual. It is also recommended that subsequent research systematically account for seasonal variations in human activity patterns.

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