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Research Paper

An anticipating Class of Fuzzy Stochastic Differential Equations

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ABSTRACT

In this paper, we consider an anticipating stochastic differential equation in which the integrands are not adapted to the filtration generated by a Wiener process in the stochastic integrals. By leveraging the correspondence between the Skorohod integral and the Itô-Skorohod integral, we propose solving these equations using standard iterative techniques. Subsequently, we discuss the existence and uniqueness of strong solutions to these equations. The incorporation of non-adapted, fuzzy, and random processes in such equations makes them applicable in financial models.

1 Introduction

Stochastic differential equations (SDEs) are widely used in various real-world systems, including economics and finance. Fuzzy stochastic differential equations (FSDEs) are particularly useful in addressing problems involving dual uncertainties of fuzziness and randomness concurrently (see [2, 8, 9, 21]). Several papers have explored FSDEs using different approaches. In this study, we focus on the application of fuzzy stochastic integrals of both Itô and Lebesgue-Aumann types.
The definition of the fuzzy Itô integral was introduced by Kim [10], while Malinowski presented novel approaches to define the fuzzy stochastic Itô integral in [13, 15, 16]. The method involves transforming the fuzzy Itô integral into a fuzzy set space, resembling classical d-dimensional Itô integrals. The fuzzy stochastic integral in [13, 15, 16] is driven by a combination of the Wiener process and fuzzy non-anticipating stochastic processes. To provide an overview of different approaches to studying FSDEs, Michta [20] presents three methods. There are also papers similar to the setup in [17], considering a class of FSDEs driven by a continuous local martingale. Fei in [6] introduces the fuzzy stochastic integral for the continuous local martingale class, incorporating non-Lipschitzian conditions to establish the existence and uniqueness of solutions. In [7], a class of FSDEs with a Lipschitzian condition driven by a continuous local martingale is explored. Additionally, [8] examines FSDEs driven by fractional Brownian motion. Recent progress has been made in SDEs with anticipating integrands, which find applications in finance. The Malliavin calculus plays a pivotal role in the analysis of stochastic differential equations with non-adapted

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processes and in sensitivity analysis of price functions in finance. The Skorohod integral, or anticipating integral, extends the Itô integral to accommodate non-adapted integrands. However, the existence and uniqueness of solutions for anticipating SDEs are still unknown. In the Skorohod integral, the boundedness of the Malliavin derivative of the process is required, but closed-form formulas are generally not available, and the conventional Picard iteration method cannot be applied. Some results exist only for specific cases of crisp SDEs (see [4]). In [9], the Gaussian Malliavin derivative in fuzzy space was defined to investigate the existence and uniqueness of solutions to linear Skorohod fuzzy stochastic differential equations involving non-adapted fuzzy processes.

This paper proposes a class of FSDEs with non-adapted process integrands. By utilizing the correspondence between Skorohod integrals and Itô-Skorohod integrals established in [26], we introduce a class of anticipating equations that can be solved using standard iterative techniques. The Black-Scholes model, commonly employed in option pricing (see [3]), assumes the absence of arbitrage opportunities in the market and describes the price movements of financial instruments through geometric Brownian motion. However, the model parameters may be imprecisely estimated as fuzzy numbers. The structure of the paper is as follows: Section 2 provides preliminaries and definitions of fuzzy random variables, measurable multi-functions, anticipating fuzzy stochastic integrals, fuzzy stochastic processes, and Gaussian Malliavin calculus results. In Section 3, we propose a class of Itô-Skorohod FSDEs and present their solutions using iterative methods, along with an example in finance. The conclusion is presented in the final section.

2 Preliminaries

2.1 Gaussian Malliavin Calculus

The purpose of this section is to recall some definitions and results on Malliavin calculus operators defined on Gaussian space. We can also refer the reader to [21] for more details. Consider a centered Gaussian process W on $[0,T]$ defined on its canonical complete probability space (Ω, A, P). Denote by S a family of smooth functionals

$$F = f\big(W_{t_1}, \dots, W_{t_n}\big), \quad t_1, \dots, t_n \in [0,T], \tag{1}$$

where $n \geq 1$, and the function f is an infinitely differentiable on $\mathbb{R}^n$ such that all its partial derivatives have polynomial growth property. The Malliavin derivative DF that belongs to $L^2([0,T] \times \Omega)$ is defined by

$$D_t F = \sum_{i=1}^{n} \partial_i f(W_{t_1}, \dots, W_{t_n}) 1_{[0,t_i]}(t). \tag{2}$$

The Malliavin derivative is a dense and closed operator in $L^2(\Omega)$, that its domain $Dom\, D$ is the closure of smooth random variables with respect to the following norm

$$\mathbb{E}|F|^2 + \mathbb{E}\int_0^T |D_t F|^2 dt. \tag{3}$$

The iterated derivative $D^k F$ belongs to the space $L^2\big(T^{\times k} \times \Omega\big)$, for every $k \geq 0$. For every $p \geq 1$ and $k \geq 0$, $\mathbb{D}^{k,p}$ denotes the closure of S with respect to the norm

$$\|F\|_{k,p}^{p} = \|F\|_{L^2(\Omega)}^{p} + \sum_{j=1}^{k} \left\|D^j F\right\|_{L^2([0,T]^k)}^{p}. \tag{4}$$

The adjoint of the operator D is denoted by δ, and is called the Skorohod integral. The domain of δ, denoted by Domδ, is the set of square-integrable random variables $u \in L^2([0,T] \times \Omega)$ such that

$$\left|\mathbb{E} \int_0^T u(t) D_t F dt\right| \leq C\|F\|_{L^2(\Omega)}.$$

Then, the duality relationship is

$$\mathbb{E}\big(F\delta(u)\big) = \mathbb{E} \int_0^T u_t D_t F dt, u \in Dom\delta, F \in S. \tag{5}$$

The variance of the Skorohod integral is

$$\mathbb{E}(\delta^2(u)) = \mathbb{E} \int_0^T u_t^2 dt + \mathbb{E} \int_0^T \int_0^T D_s u_t D_t u_s dt ds. \tag{6}$$

The space $\mathbb{L}^{k.p} := \mathbb{D}^{k.p}\big(L^2([0,T])\big)$, coincides with the class of processes $u \in L^2([0,T] \times \Omega)$ such that $u_t \in \mathbb{D}^{k.p}$ for almost all t.

Let $u \in Dom\delta$ and $F \in \mathbb{D}^{1.2}$ such that $\mathbb{E}\left(F^2 \int_0^T u_t^2 dt\right) < \infty$, then

$$\int_0^T F u_t dW_t = F \int_0^T u_t dW_t dt + \int_0^T (D_t F) u_t dt, \tag{7}$$

such that $Fu \in Dom\delta$ if and only if the right-hand side of (7) is square-integrable.

We can consider the following processes

$$\int_0^t u(s) dW(s),$$
$$\int_0^t \mathbb{E}\big(v(s)\big|A_{[s,t]^c}\big) dW(s),$$

where $u, v \in \mathbb{L}^{k.p}$, for $k \geq 1$, $p \geq 2$; as Skorohod and Itô-Skorohod integral processes respectively (see [26]). It can be shown that (see [27]) the two classes coincide for regular integrands.

2.2 Fuzzy Background

This section includes some preliminaries and definitions on fuzzy numbers and fuzzy stochastic integrals which are taken from [15], [17], [12], [18], [24] and references therein. Let us denote by $K(\mathbb{R})$ the family of all convex, compact and nonempty subsets of $\mathbb{R}$. We define the Hausdorff metric d_H by

$$d_H(A,B) = max\left\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\right\}.$$

The metric space $K(\mathbb{R})$ with respect to d_H is separable and complete. If $A, B, C \in K(\mathbb{R})$, then $d_H(A + C, B + C) = d_H(A,B)$.

Definition 2.1 *Consider the probability space* (Ω, A, P). *The mapping* $F: \Omega \to K(\mathbb{R})$ *is said to be* $A-$ *measurable if it satisfies:*

$$\{\omega \in \Omega: F(\omega) \cap C \neq \phi\} \in A,$$

for any closed set $C \subset \mathbb{R}$.

Denote by M, a family of $A-$ measurable multifunctions with values in $K(\mathbb{R})$.

Definition 2.2 *A multifunction $F \in M$ is $L^p -$ integrably bounded, for $p \geq 1$, if there is $h \in L^p(\Omega, A, P, \mathbb{R}_+)$ such that $|||F||| \leq h \ a.s.$, and*

$$|||F||| = d_H(F, \{0\}) = \sup_{f\in F}\|f\|, \quad for\ F \in K(\mathbb{R}), \mathbb{R}_+ = [0, \infty).$$

$F \in M$ is L^p $-$integrably bounded if and only if $|||F||| \in L^p(\Omega, A, P, \mathbb{R}_+)$. Define

$$L^p(\Omega, A, P; K(\mathbb{R})) = \{F \in M: |||F||| \in L^p(\Omega, A, P, \mathbb{R}_+)\}.$$

A fuzzy set $u \in \mathbb{R}$ is determined by its membership function, $u: \mathbb{R} \to [0,1]$ and $u(x)$, for $x \in \mathbb{R}$. is the membership degree of x in fuzzy set u. Let us denote by $F(\mathbb{R})$ the fuzzy sets $u: \mathbb{R} \to [0,1]$ such that $[u]^\alpha \in K(\mathbb{R})$ for every $\alpha \in [0.1]$, where $[u]^\alpha = \{x \in \mathbb{R}: u(x) \geq \alpha\}$.

For scalar multiplication and addition in fuzzy set space $F(\mathbb{R})$ one can write

$$[\lambda u]^\alpha = \lambda[u]^\alpha,$$

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha,$$

where $u, v \in F(\mathbb{R})$, $\lambda \in \mathbb{R}$, and $\alpha \in [0,1]$. The metric $d_\infty: F(\mathbb{R}) \times F(\mathbb{R}) \to [0, \infty)$ is defined by

$$d_\infty(u, v) = \sup_{\alpha\in[0,1]} d_H([u]^\alpha, [v]^\alpha),$$

then it is famous that $(F(\mathbb{R}), d_\infty)$ is a complete metric space with metric d_∞ in $F(\mathbb{R})$. For every fuzzy elements $u, v, w, z \in F(\mathbb{R})$, $\lambda \in \mathbb{R}$, we have the following properties (see [17] and [22]):

- $d_\infty(u + w, v + w) = d_\infty(u, v)$,
- $d_\infty(u + w, v + w) = d_\infty(u, w) + d_\infty(v, z)$,
- $d_\infty(\lambda u, \lambda v) = |\lambda| d_\infty(u, v)$.

Define $\langle\cdot\rangle: \mathbb{R} \to F(\mathbb{R})$ as an embedding of $\mathbb{R}$ into $F(\mathbb{R})$,

$$\langle r\rangle(a) = \begin{cases} 1 & for\ a = r, \\ 0 & for\ a \in R \setminus\{r\} \end{cases}.$$

Definition 2.3 *A function $X: \Omega \to F(\mathbb{R})$ on the probability space (Ω, A, P) is a fuzzy random variable if for all $\alpha \in [0,1]$ the mapping $[X]^\alpha: \Omega \to K(\mathbb{R})$ is A $-$measurable multifunction.*

Consider a metric ρ in the set $F(\mathbb{R})$, and σ $-$algebra B_ρ that is derived by ρ. A fuzzy random variable is defined as a mapping between two spaces (Ω, A) and $(F(\mathbb{R}), B_\rho)$. That is, X is $A|B_\rho$ $-$measurable. The following metric is also used

$$d_s(u.v) := \inf_{\lambda\in\Lambda} max\left\{\sup_{t\in[0,1]} |\lambda(t) - t|, \sup_{t\in[0,1]} d_H(X_u(t), X_u(\lambda(t))\right\}.$$

where Λ is the set of strictly increasing continuous functions $\lambda: [0,1] \to [0,1]$ such that $\lambda(0) = 0$, $\lambda(1) = 1$, and $X_u, X_v: [0,1] \to F(\mathbb{R})$ are cadlag representations for the fuzzy sets $u, v \in F(\mathbb{R})$ (see [5]).

Proposition 2.4 *[14] Consider $X: \Omega \to F(\mathbb{R})$ on the probability space (Ω, A, P), then*

- X is a fuzzy random variable if and only if X is measurable with respect to $A|B_{d_s}$.
- If X is measurable with respect to $A|B_{d_s}$, then it is a fuzzy random variable; the opposite implication is not true.

Therefore, the measurability on $F(\mathbb{R})$ with respect to the metric d_∞ is not equivalent to the measurability of the $\alpha-$level mappings.

Definition 2.5 *A fuzzy random variable X, is L^p-integrably bounded, for $p \geq 1$, if $[X]^\alpha \in L^p(\Omega, A, P, K(\mathbb{R}))$, for every $\alpha \in [0,1]$.*

Let us denote by $L^p(\Omega, A, P; F(\mathbb{R}))$, the set of all L^p-integrably bounded fuzzy random variables. The random variables $X, Y \in L^p(\Omega, A, P; F(\mathbb{R}))$ are identical if $P(d_\infty(X,Y) = 0) = 1$.

It is easy to see that for $X: \Omega \to F(\mathbb{R})$ being a fuzzy random variable and $p \geq 1$, the following conditions are equivalent:

a) $X \in L^p(\Omega, A, P; F(\mathbb{R}))$,

b) $[X]^0 \in L^p(\Omega, A, P, K(\mathbb{R}))$,

c) $\||[X]^0|\| \in L^p(\Omega, A, P, \mathbb{R}_+)$.

Let $I := [0, T]$, and (Ω, A, P) be complete with a filtration $\{A_t\}_{t\in I}$ satisfying an increasing and right continuous family of sub $\sigma-$algebras of A, and contains null sets.

Definition 2.6 *If $X(t): \Omega \to F_c^b(\mathbb{R})$, for every $t \in I$, is a fuzzy random variable, then $X: I \times \Omega \to F_c^b(\mathbb{R})$ is a fuzzy stochastic process.*

Definition 2.7 *If almost all trajectories of a fuzzy process $X(\cdot, \omega): I \times \Omega \to F(\mathbb{R})$ are $d_\infty-$continuous then the process is $d_\infty-$continuous.*

Definition 2.8 *If $[X]^\alpha: I \times \Omega \to K(\mathbb{R})$ is $B(I) \otimes A-$ measurable function for all $\alpha \in [0,1]$, where $B(I)$ is Borel $\sigma-$algebra of subsets of I. Then, the process X is measurable.*

Definition 2.9 *A fuzzy process X is L^p-integrably bounded $(p \geq 1)$, if there exists a real-valued process $h \in L^p(I \times \Omega, B(I) \otimes A; \mathbb{R}_+)$ such that*

$$\||[X(t, \omega)]^0|\| \leq h(t, \omega),$$

for almost all $(t, \omega) \in I \times \Omega$.

Denote by $L^p(I \times \Omega, B(I) \otimes A; F(\mathbb{R}))$ the set of L^p-integrably bounded fuzzy stochastic processes. Consider $X \in L^p(I \times \Omega, B(I) \otimes A; F(\mathbb{R}))$, according to the Fubini's theorem, the fuzzy integral is defined by

$$\int_0^T X(s, \omega) ds.$$

where $\omega \in \Omega \backslash N$, $N \in A$ and $P(N) = 0$. The level sets of this fuzzy integral are the set-valued Aumann integrals of level sets of $X(\cdot,\omega)$. For every $\alpha \in [0,1]$, and every $\omega \in \Omega \backslash N$, the Aumann integral $\int_0^T [X(s,\omega)]^\alpha ds$ belongs to $K(\mathbb{R})$ (see [11]), so we have a fuzzy random variable $\int_0^T X(s,\omega)ds \in F(\mathbb{R})$ for every $\omega \in \Omega \backslash N$ (see [16]).

Definition 2.10 *The fuzzy stochastic Lebesgue-Aumann integral of $X \in L^1(I \times \Omega, B(I) \otimes A; F(\mathbb{R}))$ is defined as:*

$$L_x(t.\omega) = \begin{cases} \int_0^T 1_{[0,t]}(s)X(s,\omega)ds & for\ every\ \omega \in \Omega \setminus N \\ \langle 0 \rangle & for\ every\ \omega \in N \end{cases}. \tag{8}$$

Proposition 2.11 *[14] For the stochastic integral L_x, we have the following properties:*

1) For $p \geq 1$, if $X \in L^p\big(I \times \Omega, B(I) \otimes A; F(\mathbb{R})\big)$, then $L_x(\cdot,\cdot) \in L^p(I \times \Omega, B(I) \otimes A; F(\mathbb{R}))$.

2) If $X \in L^1\big(I \times \Omega, B(I) \otimes A; F(\mathbb{R})\big)$, then $\{L_x(t)\}_{t\in I}$ is d_∞ −continuous.

3) Let $X, Y \in L^p\big(I \times \Omega, B(I) \otimes A; F(\mathbb{R})\big)$, for $p \geq 1$, then
$$\sup_{u\in[0,t]} d_\infty^p\left(L_x(u), L_y(u)\right) \leq t^{p-1} \int_0^t d_\infty^p\big(X(s), Y(s)\big)ds \quad a \cdot e \cdot$$

Let us denote by $\langle\cdot\rangle: \mathbb{R} \to F(\mathbb{R})$ an embedding of $\mathbb{R}$ into $F(\mathbb{R})$ i.e. for $r \in \mathbb{R}$,

$$\langle r \rangle(a) = \begin{cases} 1 & for\ a = r, \\ 0 & for\ a \in R \setminus \{r\} \end{cases}.$$

For a random variable $X: \Omega \to \mathbb{R}$ on the probability space (Ω, A, P), the embedding $\langle X \rangle: \Omega \to F(\mathbb{R})$ is a fuzzy random variable.

We consider fuzzy stochastic Itô integral by the fuzzy random variable as $\left\langle \int_0^T X(s)dW(s) \right\rangle$, where W is a Wiener process. The following properties will be useful.

Proposition 2.12 *(Ref. 15) Let the process $X \in L^2(I \times \Omega, B(I) \otimes A; \mathbb{R})$. then $\left\{\left\langle \int_0^t X(s)dW(s) \right\rangle\right\}_{t\in I}$ is a fuzzy stochastic process and $\left\langle \int_0^t X(s)dW(s) \right\rangle$ belongs to $L^2(I \times \Omega, B(I) \otimes A; F(\mathbb{R}))$.*

If the process $X \in L^2(I \times \Omega, B(I) \otimes A; \mathbb{R})$, then we can easily show that $\left\{\left\langle \int_0^t X(s)dW(s) \right\rangle\right\}_{t\in I}$ is d_∞ −continuous.

3 Itô-Skorohod Fuzzy Stochastic Differential Equations

In this paper, we consider the following fuzzy stochastic differential equation

$$\begin{cases} dX(t) = f\big(t, X(t)\big)dt + g\big(t, X(t)\big)dW(t), \\ X(0) = x_0, \end{cases}$$

on a complete probability space (Ω, A, P) with filtration $(A(t))_{t\geq 0}$, where $f: [0,\infty) \times F(\mathbb{R}) \to F(\mathbb{R})$

and $g:[0,\infty)\times F(\mathbb{R})\to F(\mathbb{R})$. Here $F(\mathbb{R})$ is the family of all fuzzy sets of which level sets are nonempty closed convex subsets of , the set of all real numbers $\mathbb{R}$, and $(W(t))_{t\geq 0}$ is a 1- dimennsional Brownian motion. The solution of fuzzy stochastic differential equation is satisfying in

$$X(t)=x+\int_0^t f\big(s,X(s)\big)ds+\int_0^t g\big(s,X(s)\big)dW(s),\ \text{a.s.}$$

In this section, we consider an example of the fuzzy stochastic differential equations under the Lebesgue-Aumann integral, random coefficients, and Brownian motion. Using the Picard's iterations method, we state a theorem of existence and uniqueness for a class of anticipating stochastic differential equations. We know that this iteration method cannot be implemented in anticipating stochastic calculus since the mean square of the Skorohod integral formula includes the Malliavin derivative and a closed form formula cannot be found. Here, we consider a class of anticipating FSDEs that can be solved by the iteration method.

3.1 Picard Iteration Method

The Picard's iteration method is famous in SDEs and the differential equations theory. One can construct a random processes sequence which converges to a solution of the equation (see [1]). Define $X^n(t)$ by the equation

$$X^{n+1}(t)=X_0+\int_0^t f\big(s,X^n(s)\big)ds+\int_0^t g\big(s,X^n(s)\big)dW(s),X_0=X(0), \tag{9}$$

where $\{W(t)\}_{t\geq 0}$ be a standard Brownian motion on a probability space (Ω,A,P), and functions f and g satisfy these conditions for some constant $k\geq 0$

C1) $|f(t,x)-f(t,y)|^2+|g(t,x)-g(t,y)|^2\leq k|x-y|^2,\ x,y\in\mathbb{R},\ t\geq 0$

C2) $|f(t,x)|^2+|g(t,x)|^2\leq k(1+x^2),\ x\in\mathbb{R},\ t\geq 0.$

All the processes $X^n(t)$ are well-defined with continuous paths. By induction and showing that the limit process is a solution to the SDE and the sequence $X^n(t)$ uniformly converges on compact time intervals, the existence of solutions for the equation (9) is proved. The sequence of random variables $X^n(t)$ for each $t\geq 0$ converges in L^2 to a random variable $X(t)$. The first term of the sequence X_0 and then for t in any bounded interval $[0,T]$, $X^1(t)$ are bounded uniformly in L^2. Then there exists $C=C_T<\infty$ for each $T<\infty$ such that for all $t\leq T$ we have $\mathbb{E}(X^1(t)-X^0(t))^2\leq C$. By hypotheses C1 and C2, the Itô isometry property and application of the triangle inequality, one can obtain $\mathbb{E}(X^n(t)-X^{n-1}(t))^2\leq\infty$,. for all $n\in\mathbb{N}$. Hence, the sequence of Picard's iterations $X^n(t)$ has a limit $X(t)$ in L^2 at all $t\geq 0$. For the uniqueness of the solution, consider that there exist two continuous solutions for some initial value x as follows:

$$X(t)=x+\int_0^t f\big(s,X(s)\big)ds+\int_0^t g\big(s,X(s)\big)dW(s),$$

and

$$Y(t)=x+\int_0^t f(s,Y(s))ds+\int_0^t g(s,Y(s))dW(s).$$

Then the difference is

$$Y(t)-X(t)=\int_0^t (f(s,X(s))-f(s,Y(s)))ds+\int_0^t (g(s,X(s))-g(s,Y(s)))dW(s). \tag{10}$$

Although the second moment in (10) can be bounded, its second integral cannot be bounded pathwise.

From the isometry property, we have

$$\mathbb{E}\left\{\int_0^t (g(s,X(s))-g(s,Y(s)))dW(s)\right\}^2 \le k^2\int_0^t \mathbb{E}(X(s)-Y(s))^2 ds,$$

where k is a constant. Then

$$\mathbb{E}(X(t)-Y(t))^2 \le 2k^2(1+T)\int_0^t \mathbb{E}(X(s)-Y(s))^2 ds.$$

One can suppose the finite and integrable $J(t)=\mathbb{E}(X(t)-Y(t))^2$ on compact time intervals, from the Gronwall inequality we have $J(t)=0$ for all $t\in I$, then the uniqueness would be proved.

3.2 Itô-Skorohod Model

Consider the following Skorohod fuzzy stochastic differential equation

$$X(t)=X_0+\int_0^t f(s,X(s))ds+\left\langle\int_0^t g\left(s,\mathbb{E}\left(X(s)\middle|A_{[s,t]^c}\right)\right)dW(s)\right\rangle, X_0=X(0)\quad a.s.,$$

with coefficients $f: I\times\Omega\times F(\mathbb{R})\to F(\mathbb{R})$, $g: I\times\Omega\times F(\mathbb{R})\to F(\mathbb{R})$, and the fuzzy random variable $X_0:\Omega\to F(\mathbb{R})$.

Definition 3.1 *A strong solution to (11) is a fuzzy process* X *such that*

- $X\in L^2\big(I\times\Omega, B(I)\otimes A; F(\mathbb{R})\big)$,
- X is a continuous fuzzy process with respect to d_∞,
- A strong solution X is known to be strongly unique, if

$$d_\infty\big(X(t,\omega),Y(t,\omega)\big)=0$$

where $Y: I\times\Omega\to F(\mathbb{R})$ is any strong solution of (11).

Assumptions 3.2 *Now we consider assumptions on the equation coefficients:*

A1) The mapping $f: I\times\Omega\times F(\mathbb{R})\to F(\mathbb{R})$ is $B(I)\otimes A\otimes B_{d_s}|B_{d_s}$ −measurable. and $g: I\times\Omega\times F(\mathbb{R})\to\mathbb{R}$ is $B(I)\otimes A\otimes B_{d_s}|B(\mathbb{R})$ −measurable.

A2) For every $t\in I$, and every $u,v\in F(\mathbb{R})$ there exists a constant $L>0$,

$$max\{d_\infty^2\big(f(t,\omega,u),f(t,\omega,v)\big),|g(t,\omega,u)-g(t,\omega,v)|^2\}\le L d_\infty^2(u,v), P-a\cdot e\cdot$$

A3) For every $t\in I$ and every $u\in F(\mathbb{R})$ there exists a constant $C>0$ such that

$$max\{d_\infty^2(f(t,\omega,u),\langle 0\rangle),|g(t,\omega,u)|^2\}\le C\big(1+d_\infty^2(u,\langle 0\rangle)\big), P-a\cdot e\cdot$$

Proposition 3.3 *Let* $X,Y\in L^2(I\times\Omega,B(I)\otimes A;\mathbb{R})$, *then for every* $t\in I$

$$\mathbb{E}\sup_{u\in[0.t]} d_\infty^2\left(\left\langle\int_0^u \mathbb{E}(X(s)|A_{[s,t]^c})dW(s)\right\rangle,\left\langle\int_0^u \mathbb{E}(Y(s)|A_{[s,t]^c})dW(s)\right\rangle\right)$$
$$\le 4\mathbb{E}\int_0^t d_\infty^2\left(\left\langle\mathbb{E}\left(X(s)\middle|A_{[s,t]^c}\right)\right\rangle,\left\langle\mathbb{E}(Y(s)|A_{[s,t]^c})\right\rangle\right)ds. \quad (12)$$

Proof: Let us consider the process $Z(t)=\int_0^t \mathbb{E}\left(X(s)\middle|A_{[s,t]^c}\right)dW(s)$, then $\mathbb{E}(Z(t)-Z(s)|A_{[s,t]^c})=0$, which implies that the projection of $Z(t)$ on the Brownian filtration is a martingale. From (6), we have an isometry property

$$\mathbb{E}(\int_0^t \mathbb{E}(X(s)|A_{[s,t]^c})dW(s))^2 = \mathbb{E}\int_0^t (\mathbb{E}(X(s)|A_{[s,t]^c}))^2 ds.$$

Now due to the Doob inequality. isometry property, and Itô integral we have

$$\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\left(\left\langle\int_0^u \mathbb{E}(X(s)|A_{[s,t]^c})dW(s)\right\rangle, \left\langle\int_0^u \mathbb{E}(Y(s)|A_{[s,t]^c})dW(s)\right\rangle\right)$$
$$= \mathbb{E}\sup_{u\in[0,t]} d_H^2\left(\left\{\int_0^u \mathbb{E}(X(s)|A_{[s,t]^c})dW(s)\right\}, \left\{\int_0^u g(s, \mathbb{E}(Y(s)|A_{[s,t]^c}))dW(s)\right\}\right)$$
$$= \mathbb{E}\sup_{u\in[0,t]} \left\|\int_0^u \left(\mathbb{E}(X(s)|A_{[s,t]^c}) - \mathbb{E}(Y(s)|A_{[s,]^c})\right)dW(s)\right\|^2$$
$$\leq 4\mathbb{E}\left\|\int_0^u \left(\mathbb{E}(X(s)|A_{[s,t]^c}) - \mathbb{E}(Y(s)|A_{[s,t]^c})\right)dW(s)\right\|^2$$
$$= 4\mathbb{E}\int_0^u \left\|\mathbb{E}(X(s)|A_{[s,t]^c}) - \mathbb{E}(Y(s)|A_{[s,t]^c})\right\|^2 ds$$
$$= 4\mathbb{E}\int_0^t d_\infty^2(\langle\mathbb{E}(X(s)|A_{[s,t]^c})\rangle, \langle\mathbb{E}(Y(s)|A_{[s,t]^c})\rangle)ds.$$

Theorem 3.4 *Let $X_0 \in L^2(\Omega, A_0, P; F(\mathbb{R}))$. Suppose that $f: I \times \Omega \times F(\mathbb{R}) \to F(\mathbb{R})$ and $g: I \times \Omega \times F(\mathbb{R}) \to \mathbb{R}$, satisfy assumptions (A1)-(A3). Then Equation (11) has a unique strong solution.*

Proof: Consider the Picard iterations. $X^0(t) = X_0$, and for n = 1, 2, . . .

$$X^{n+1}(u) = X_0 + \int_0^u f(s, X^n(s))ds + \left\langle\int_0^u g\left(s, \mathbb{E}\left(X^n(s)\middle|A_{[s,t]^c}\right)\right)dW(s)\right\rangle, \quad a.s. \tag{13}$$

Denote

$$j_n(t) = \mathbb{E}\sup_{u\in[0.t]} d_\infty^2(X^n(u), X^{n-1}(u)),$$

for $n \in \mathbb{N}$ and $t \in I$. Then, using propositions 2.11, 3.3, and assumption A3, we can write

$$j_1(t) = \mathbb{E}\sup_{u\in[0,t]} d_\infty^2\left(\int_0^u f(s, X^0(s))ds + \left\langle\int_0^u g(s, \mathbb{E}(X^0(s)|A_{[s,t]^c})dW(s)\right\rangle, \langle 0\rangle\right)$$
$$\leq 2\left[\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\left(\int_0^u f(s, X^0(s))ds, \langle 0\rangle\right)\right.$$
$$\left.+\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\left(\left\langle\int_0^u g(s, \mathbb{E}(X^0(s)|A_{[s,t]^c})dW(s)\right\rangle, \langle 0\rangle\right)\right]$$
$$\leq 2\left[t\mathbb{E}\int_0^t d_\infty^2(f(s, X^0(s)), \langle 0\rangle)ds + 4\mathbb{E}\int_0^t d_\infty^2(\langle g(s, \mathbb{E}(X^0(s)|A_{[s,t]^c})\rangle, \langle 0\rangle)ds\right]$$
$$\leq K_1 t,$$

for every $t \in I$, where $K_1 = 4C^2(T+4)(1 + \mathbb{E}\||[X]^0|\|^2) < \infty$. Then, similarly

$$j_{n+1}(t) \leq 2(t+4)L\mathbb{E}\int_0^t d_\infty^2(X^n(s), X^{n-1}(s))ds$$
$$\leq 2(t+4)L\int_0^t \mathbb{E}\sup_{u\in[0,s]} d_\infty^2(X^n(u), X^{n-1}(u))ds$$
$$\leq 2(T+4)L\int_0^t j_n(s)ds.$$

Hence

$$j_n(t) \leq 2^{n+1}C(T+4)^n(1 + \mathbb{E}\||[X]^0|\|^2)L^{n-1}\frac{t^n}{n!}, t \in I, n \in \mathbb{N}.$$

From the Chebyshev's inequality it follows that

$$P(\sup_{u\in I} d_\infty^2(X^n(u), X^{n-1}(u)) > \frac{1}{2^n}) \leq 2^n j_n(T).$$

The series $\sum_{n=1}^{\infty} 2^n j_n(T)$ is convergent. From Borel-Cantelli lemma, we can write

$$P\left(\sup_{u\in I} d_\infty(X^n(u),X^{n-1}(u)) > \frac{1}{(\sqrt{2})^n} \quad infinitely \quad often\right) = 0.$$

For almost all $\omega \in \Omega$, there exists $n_0(\omega)$ such that

$$\sup_{u\in I} d_\infty\big(X^n(u),X^{n-1}(u)\big) \le \frac{1}{(\sqrt{2})^n}, \quad if \quad n \ge n_0.$$

The sequence $X^n(\cdot,\omega)$ is uniformly convergent to a d_∞-continuous function $\tilde{X}^n(\cdot,\omega)$ for every $\omega \in \Omega_c$, where $\Omega_c \in A$ and $P(\Omega_c) = 1$. For the mapping $X: I \times \Omega \to F(\mathbb{R})$, we can define $X^n(\cdot,\omega) = \tilde{X}^n(\cdot,\omega)$ if $\omega \in \Omega_c$ and $X^n(\cdot,\omega)$ as freely chosen fuzzy function in the case $\omega \in \Omega \setminus \Omega_c$. For every $\alpha \in [0,1]$ and every $t \in I$ with a.s. we have

$$d_H([X^n(t)]^\alpha,[X(t)]^\alpha) \to 0 \quad as \quad n \to \infty.$$

Therefore, X is a continuous fuzzy stochastic process. Then from $X^n \in L^2(I \times \Omega, B(I) \otimes A; F(\mathbb{R}))$ we get $X \in L^2(I \times \Omega. B(I) \otimes A; F(\mathbb{R}))$. Hence. we can verify

$$\mathbb{E}\sup_{t\in I}\left[d_\infty(X^n(t),X(t)) + d_\infty\left(X^n(t),X_0 + \int_0^t f(s,X(s))ds + \left\langle\int_0^t g(s,\mathbb{E}(X(s)|A_{[s,t]^c}))dW(s)\right\rangle\right)\right]^2$$

tends to zero as n goes to infinity. Then

$$\mathbb{E}\sup_{t\in I} d_\infty^2\left[\left(X(t),X_0 + \int_0^t f(s,X(s))ds + \left\langle\int_0^t g(s,\mathbb{E}(X(s)|A_{[s,t]^c}))dW(s)\right\rangle\right)\right] = 0.$$

Therefore

$$\sup_{t\in I} d_\infty\left[\left(X(t),X_0 + \int_0^t f(s,X(s))ds + \left\langle\int_0^t g(s,\mathbb{E}(X(s)|A_{[s,t]^c}))dW(s)\right\rangle\right)\right] = 0, \quad a.s.,$$

which shows the existence of the strong solution.

Let us now assume that $X, Y: I \times \Omega \to F(\mathbb{R})$ are strong solutions. Consider

$$j(t) = \mathbb{E}\sup_{u\in[0,t]} d_\infty^2\big(X(u),Y(u)\big),$$

then

$$j(t) \le (m+1)(t+4m)L\mathbb{E}\int_0^t d_\infty^2(X(s),Y(s))ds \le (m+1)(T+4m)L\int_0^t j(s)ds.$$

Applying the Gronwall inequality yields $j(t) = 0$ for $t \in I$. Hence

$$\sup_{t\in I} d_\infty\big(X(t),Y(t)\big) = 0, \quad a.s.,$$

which completes the proof of strongly uniqueness.

Remark 3.5 *Consider the following equation*

$$X(t) = \mathbb{E}(X(0)|A_{t^c}) + \int_0^t f(s,\mathbb{E}(X(s)|A_{[s.t]^c}))ds + \left\langle\int_0^t g(s,\mathbb{E}(X(s)|A_{[s,t]^c}))dW(s)\right\rangle. \quad (14)$$

Following the lines of the proof of Theorem 3.4. we can easily show that (14) admits a unique solution.

Corollary 3.6 *Consider the following stochastic fuzzy differential equation*

$$X(t) = \mathbb{E}(X(0)|A_{t^c}) + \int_0^t a\mathbb{E}\big(X(s)\big|A_{[s,t]^c}\big)ds + \left\langle\int_0^t \frac{b}{2}\mathbb{E}\left(\left(X_L^1(s) + X_U^1(s)\right)\middle|A_{[s,t]^c}\right)dW(s)\right\rangle, \quad (15)$$

where $X_0 \in L^2\big(\Omega, A_0, P; F(\mathbb{R})\big)$, $a, b \in \mathbb{R}$, and $X_L^1, X_U^1: \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that $[X(t)]^1 = [X_L^1(t), X_U^1(t)]$. Then, there exists an explicit solution.

Proof: The equation (15) satisfy assumptions of Theorem 3.4. In order to find a closed explicit form of solution to (15), for $a \ge 0$, we can write the following systems

$$\begin{cases} X_L^1(t) = \mathbb{E}(X_L^1(0)|A_{t^c}) + \int_0^t a\mathbb{E}(X_L^1(s)|A_{[s,t]^c})ds + \int_0^t \frac{b}{2}\mathbb{E}((X_L^1(s) + X_U^1(s))|A_{[s,t]^c})dW(s) \\ X_U^1(t) = \mathbb{E}(X_U^1(0)|A_{t^c}) + \int_0^t a\mathbb{E}(X_U^1(s)|A_{[s,t]^c})ds + \int_0^t \frac{b}{2}\mathbb{E}((X_L^1(s) + X_U^1(s))|A_{[s,t]^c})dW(s) \end{cases},$$

then

$$X_L^1(t) + X_U^1(t) = \mathbb{E}(X_L^1(0) + X_U^1(0)|A_{t^c}) + \int_0^t a\mathbb{E}(X_L^1(s) + X_U^1(t)|A_{[s,t]^c})ds$$
$$+ \int_0^t b\mathbb{E}\left(\left(X_L^1(s) + X_U^1(s)\right)\middle|A_{[s,t]^c}\right)dW(s),$$

Hence, it has a unique solution

$$X_L^1(t) + X_U^1(t) = \mathbb{E}(X_L^1(0) + X_U^1(0)|A_{t^c})exp\left\{bW(t) + (a - \frac{b^2}{2})t\right\}. \tag{16}$$

Now for every $\alpha \in [0,1]$, we apply similar procedure to obtain the following systems

$$\begin{cases} X_L^\alpha(t) = \mathbb{E}(X_L^\alpha(0)|A_{t^c}) + \int_0^t a\mathbb{E}(X_L^\alpha(s)|A_{[s,t]^c})ds + \int_0^t \frac{b}{2}\mathbb{E}((X_L^1(s) + X_U^1(s))|A_{[s,t]^c})dW(s) \\ X_U^\alpha(t) = \mathbb{E}(X_U^\alpha(0)|A_{t^c}) + \int_0^t a\mathbb{E}(X_U^\alpha(s)|A_{[s,t]^c})ds + \int_0^t \frac{b}{2}\mathbb{E}((X_L^1(s) + X_U^1(s))|A_{[s,t]^c})dW(s) \end{cases}.$$

We apply the solution (16) to get the following system

$$X_L^\alpha(t) = \mathbb{E}(X_L^\alpha(0)|A_{t^c}) + \int_0^t a\mathbb{E}(X_L^\alpha(s)|A_{[s,t]^c})ds$$
$$+ \int_0^t \frac{b}{2}\mathbb{E}(X_L^1(0) + X_U^1(0)|A_{t^c})exp\left\{bW(s) + (a - \frac{b^2}{2})s\right\}dW(s)$$
$$X_U^\alpha(t) = \mathbb{E}(X_U^\alpha(0)|A_{t^c}) + \int_0^t a\mathbb{E}(X_U^\alpha(s)|A_{[s.t]^c})ds$$
$$+ \int_0^t \frac{b}{2}\mathbb{E}(X_L^1(0) + X_U^1(0)|A_{t^c})exp\left\{bW(s) + (a - \frac{b^2}{2})s\right\}dW(s) \tag{17}$$

Hence. we obtain the unique solution as follows

$$\begin{cases} X_L^\alpha(t) = e^{at}\left[\mathbb{E}(X_L^\alpha(0)|A_{t^c}) + \frac{b}{2}\mathbb{E}(X_L^1(0) + X_U^1(0)|A_{t^c})\int_0^t exp\left\{bW(s) - \frac{b^2}{2}s\right\}dW(s)\right] \\ \\ X_U^\alpha(t) = e^{at}\left[\mathbb{E}(X_U^\alpha(0)|A_{t^c}) + \frac{b}{2}\mathbb{E}(X_L^1(0) + X_U^1(0)|A_{t^c})\int_0^t exp\left\{bW(s) - \frac{b^2}{2}s\right\}dW(s)\right] \end{cases}. \tag{18}$$

Then the fuzzy solution X for $a \geq 0$ is

$$X(t) = e^{at}\mathbb{E}(X(0)|A_{t^c}) + \left\langle\frac{b}{2}\mathbb{E}(X_L^1(0) + X_U^1(0)|A_{t^c})e^{at}\int_0^t exp\left\{bW(s) - \frac{b^2}{2}s\right\}dW(s)\right\rangle. \tag{19}$$

For $a < 0$, we can show that

$$X_L^\alpha(t) = \mathbb{E}(X_L^\alpha(0)|A_{t^c}) + \int_0^t a\mathbb{E}(X_U^\alpha(s)|A_{[s,t]^c})ds$$
$$+ \int_0^t \frac{b}{2}\mathbb{E}(X_L^1(0) + X_U^1(0)|A_{t^c})exp\left\{bW(s) + (a - \frac{b^2}{2})s\right\}dW(s)$$
$$X_U^\alpha(t) = \mathbb{E}(X_U^\alpha(0)|A_{t^c}) + \int_0^t a\mathbb{E}(X_L^\alpha(s)|A_{[s,t]^c})ds$$
$$+ \int_0^t \frac{b}{2}\mathbb{E}(X_L^1(0) + X_U^1(0)|A_{t^c})exp\left\{bW(s) + (a - \frac{b^2}{2})s\right\}dW(s). \tag{20}$$

then

$$X_L^\alpha(t) = \mathbb{E}(X_L^\alpha(0)|A_{t^c}) \cdot cosh(at) + \mathbb{E}(X_U^\alpha(0)|A_{t^c}) \cdot sinh(at)$$
$$+ \frac{b}{2}\mathbb{E}(X_L^1(0) + X_U^1(0)|A_{t^c})\int_0^t exp\left\{bW(s) - \frac{b^2}{2}s\right\}dW(s)$$
$$X_U^\alpha(t) = \mathbb{E}(X_L^\alpha(0)|A_{t^c}) \cdot sinh(at) + \mathbb{E}(X_U^\alpha(0)|A_{t^c}) \cdot cosh(at)$$

$$+\frac{b}{2}\mathbb{E}(X_L^1(0)+X_U^1(0)|A_{t^c})\int_0^t exp\left\{bW(s)-\frac{b^2}{2}s\right\}dW(s). \quad (21)$$

Therefore, in the case of $a<0$ we obtain

$$X(t)=cosh(at)\mathbb{E}(X(0)|A_{t^c})+sinh(at)\mathbb{E}(X(0)|A_{t^c})$$
$$+\left\langle\frac{b}{2}\mathbb{E}(X_L^1(0)+X_U^1(0)|A_{t^c})e^{at}\int_0^t e^{bW(s)-\frac{b^2}{2}s}dW(s)\right\rangle. \quad (22)$$

3.3 Application in Finance

In this section , we give an example in a market model with price dynamic with Itô-Skorohod SDE stated in previous section. On the probability space (Ω, A, P), we suppose two assets: the safe investment $B=(B_t)_{t\in[0,T]}$ that $B_t=1+r\int_0^t B_s ds$, and the risky asset $S=(S_t)_{t\in[0,T]}$ with the following price dynamic

$$S(t)=\mathbb{E}(S(0)|A_{t^c})+\int_0^t a\mathbb{E}(S(s)|A_{[s,t]^c})ds+\int_0^t b\mathbb{E}(S(s)|A_{[s,t]^c})dW(s). \quad (23)$$

The stochastic integral is Skorohod integral since the integrand is not adapted the filtration generated by a Wiener process or the initial condition is anticipating. Using the correspondence between the Skorohod integral and Ito-Skorohod integral, the equations can be solved by using standard iterative techniques.

In the real market, the data may not be precise. In a linguistic expression, around a value, leads to consider the fuzzy theory in our financial model. Hence, we can write

$$S(t)=\mathbb{E}(S(0)|A_{t^c})+\int_0^t a\mathbb{E}(S(s)|A_{[s,t]^c})ds+\left\langle\int_0^t \frac{b}{2}\mathbb{E}((S_L^1(s)+S_U^1(s))|A_{[s,t]^c})dW(s)\right\rangle. \quad (24)$$

From (18) we have

$$\begin{cases} S_L^\alpha(t)=e^{at}\left[\mathbb{E}(S_L^\alpha(0)|A_{t^c})+\frac{b}{2}\mathbb{E}(S_L^1(0)+S_U^1(0)|A_{t^c})\int_0^t exp\left\{bW(s)-\frac{b^2}{2}s\right\}dW(s)\right] \\ S_U^\alpha(t)=e^{at}\left[\mathbb{E}(S_U^\alpha(0)|A_{t^c})+\frac{b}{2}\mathbb{E}(S_L^1(0)+S_U^1(0)|A_{t^c})\int_0^t exp\left\{bW(s)-\frac{b^2}{2}s\right\}dW(s)\right] \end{cases}. \quad (25)$$

Then, from (19), for $a\geq 0$ the closed form solution of the equation (23) is

$$S(t)=e^{at}\mathbb{E}(S(0)|A_{t^c})+\left\langle\frac{b}{2}\mathbb{E}(S_L^1(0)+S_U^1(0)|A_{t^c})e^{at}\int_0^t exp\left\{bW(s)-\frac{b^2}{2}s\right\}dW(s)\right\rangle.$$

In the case of $a<0$, from (22) we obtain

$$S(t)=cosh(at)\mathbb{E}(S(0)|A_{t^c})+sinh(at)\mathbb{E}(S(0)|A_{t^c})$$
$$+\left\langle\frac{b}{2}\mathbb{E}(S_L^1(0)+S_U^1(0)|A_{t^c})e^{at}\int_0^t e^{bW(s)-\frac{b^2}{2}s}dW(s)\right\rangle.$$

4 Conclusion

In this paper, we presented an Itô-Skorohod FSDE with anticipating integrands. The Skorohod integral, which is the Itô integral expansion to non-adapted integrands are applied in financial models and option pricing. We used the Gaussian Malliavin calculus operators to introduce the fuzzy anticipating integrals. We applied the Picard's iteration procedure to show the existence and uniqueness of the solution for this class of anticipating SDEs. As an example, in financial models, we obtained the solution for an equation with linear coefficients.

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