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Introduction of New Risk Metric Using Kernel Density Estimation Via Linear Diffusion

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ABSTRACT

Any investor in stock markets around the world has a deep concern about the shortfalls of allocation wealth to any stock without accurate estimation of related risks. As we review the literature of risk management methods, one of the main pillars for the risk management framework in defining risk measurement approach using historical data is the estimation of the probability distribution function. In this paper, we propose a new measure by using kernel density estimation via diffusion as a nonparametric approach in probability distribution estimation to enhance the accuracy of estimation and consider some distribution characteristics, investor risk aversion and target return which will make it more accurate, comprehensive and consistent with stock historical performance and investor concerns.

1 Introduction

Investors in the stock market face so many issues for asset allocation. They want to gain maximum return on their investments but price volatilities may hurt investments and decrease their value. The other issue makes this decision a more complicated relationship between stock price movements in investment portfolio selection. Therefore, the calculation of price risk plays a key role for individual and corporate investors in the stock market. As investment managers and individuals try to optimize their stock portfolio, they should determine the model for risk measurement for each stock and they should determine the model of risk measurement for each stock and overall portfolio. One of the famous papers in the area published by Markowitz in 1952 [1] employing standard deviation as risk metric and stock expected return as return metric. So many other researches and academics have been trying to enhance the accuracy and quality of calculations to optimize portfolios ever since and have proposed various models and metrics. In different studies on stock price fluctuation risk, one of the most serious issues has been historical price movement analysis and the result applications. For this purpose, we can

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understand important characteristics by implying the probability distribution function in which so many papers have been discussed and benefit from them to build our statistical model. Some of the researchers assumed the historical daily price return of stocks follows Normal distribution or other parametric ones that many other papers have rejected [2] and this shows the importance of implying accurate probability nonparametric distribution. The other weakness in some studies for risk measurement is neglecting ivvsstor’s rikk vvrrsinn in mtt mmmttiaal aaluulatinn. Tee fcct tttt mmmm investors are more sensitive against risk in comparison to others is clear but the method to measure it in terms of risk aversion quantification is important and critical. On the other hand, the definition of downside for any investor depends on the expectations and his/her target, so it seems necessary to involve the ivvsstrr’s targtt return in risk measurement. For this purpose, researchers sometimes use decision making models to optimize their portfolio with predetermined goals [3, 4] or use target rate of return as a variable in statistical calculations. They also may use metaheuristic models for optimization to consider future estimations [5].

The main question here is what statistical distribution can be implied to consider stock price fluctuation aaarcctrristic,, ivvsstrr’s rikk vvrriinn ddd trrgtt rtturn in the same time for measuring the risk and optimize the stock portfolio? To answer this question, we set forming new risk metric using kind of statistical distribution considering broader variables to calculate risk of stock portfolio more accurately as main objective of this study.

2 Review of Related Literature

2.1 Research Background

For financial data analysis, PDF estimation may be one of the initial steps for academics and investors. So looking for the right probability distribution based on historical price movements seems important or even pivotal. As mentioned before, there can be segmentation in a class of distributions as parametric and nonparametric. If we assume the distribution estimation is based on one of a known parametric family of distributions, the distribution could then be estimated by finding the estimates of its parameters and substituting these estimates into distribution function. The assumption that the historical data follow parametric distributions such as Normal, Binomial and Poisson allows academics to run parametric tests for data analysis, but sometimes we conclude data does not follow these common distributions or it may be hard or impossible to assess assumptions, so we adopt another class of distributions named nonparametric.

Fix and Hodges, as two first academics, published a technical report in 1951 [6] about nonparametric models and they also first introduced two popular methods for nonparametric density estimation: the kernel density estimate and the nearest neighbour estimate [7]. One of the advantages of using nonparametric models refers to the time there is no reliable information about population parameters and the other one is regardless of historical data in regression, estimation of the function by parametric method would not be better than the best function with assumed parametric structure [8]. There have been many studies in the field of measuring investment risk, which is related to this study and will be discussed as fll lww... Ryy in 5952 bbblieeed a eeeer nmmdd ffff ety rrr tt ddd tee Hll digg ff Asstt”” add diccsssdd the importance of downside risk. He insisted on reasonability that an individual would seek to reduce the possibility of the chance of getting hurt from a catastrophic occurring as the Principle of Safety First [9]. Brian Rom and Kathleen Ferguson in 1994 explicitly discussed between bad or good variability with reminding that Markowitz and Sharp acknowledged drawbacks and limitations MPT theory. They

pointed out that Variance is a symmetric risk measure, which is counter-intuitive for investors in the real world. They also introduced the investment risk dependent on the ivvsstrr 's glll ddd any outcome more would not be recognized as a risk. They named their theory as Post-Modern Portfolio Theory [10]. Adams and Montesi in 1995 asserted that investment managers are more concerned about hedging downside risk and they assume lowering upside risk as a negative point [11]. Lien and Yiu TSE in 1998 constructed a bivariate APARCH-M mllll to rrr ive ivvsstmttt rikk's gggge ratiss in Nikkii cccck Average index.
They discussed using lower partial moments for calculation of hedge ratios and exhibited some advantages of this variable for risk measurement [12]. They also published another paper in 2000 discussing more LPM and insisted on the inefficiency of conventional minimum variance hedge when investors only care about hedging downside risk [13]. Chen et al in 2004, in their research for achieving accurate risk measurement and optimal hedge ratios, studied ratios based on different criteria such as expected utility, mean-Gini coefficient, generalized semi-variance and minimum variance under some normality and martingale conditions. They found out there would be convergence between these ratios if assumed conditions were satisfied [14].

2.2 Theoretical Fundamentals of Risk Measures

In this section, there will be a review of theoretical fundamentals about measurement approaches of ttkkk rr iee's vll tt ility rikk ddd rll atdd relll ts ff oteer tt ddiss. Bddd add aa thhll ())))) tt ddidd eemi-variance as an alternative for standard deviation in risk measurement. They mathematically discussed the properties of semi-variance under different conditions to compare the volatility risk measures to gauge their usefulness to practitioners. They mentioned a decisive factor in comparison between these measures in the conclusion of the paper, as when returns are symmetrically distributed, semi-variance will be inefficient [15].
Tee imrrr tatt draaaa kks aan ee seen in ooth maarrr es rr e nnfulfillmttt ff tkkigg ivvsstor's rikk vvrr-sion and target rate of return into account, which tells us the need to use more qualified measure. For this purpose, Fishborn [16], Harlow and Rao [17], Huang [18] and some other papers paid attention to wwwwr aa rtill mmmttt a as an alternative measure. For instance, Harlow and Rao tried to cover drawbacks by adding risk aversion and the target rate of return as new variables to their calculations [17]. On the other hand, some other papers (Porter and Gaumnitz [19]; Leibowitz and Langetieg [20]; Sortino and Forsey [21]) focused on the rr eferccce ff ss igg mnaan-Lower Partial mmmttt " pppraahh in eelcct-ing optimal portfolios for risk-averse investors.

2.3 Related Literatures to Kernel Estimation

Kernel density estimation method which also known as the Parzen-Rozenblatt window method (, after Emanuel Parzen and Murray Rozenblatt) is one of the famous nonparametric methods in estimation of the probability density for a random variable. This method is rooted in histogram methodology and it is naïve for uses the center of bins instead of bin edges for each point of estimation. Rozenblatt [21] defined the naïve kernel density estimator by using simply a bin centered at variable. Silverman [23] discussed more main methods available for univariate density estimation. Foster and Nelson [24] discussed kernel estimator by studying the point-wise asymptotic normality of the estimator and concluded some new results about optimal bandwidth and kernel functions. Botev et al. [25] studied mathematically kernel density estimators based on diffusion processes for multiple cases. They discussed the plug-in method for optimal bandwidth selection by considering this method would be adversely affected by normal reference rule [26, 27], so they proposed the method free from the arbitrary normal reference rules. Figueroa-Lopez and Li [28] discussed providing a formal justification of the optimal convergence rate of the kernel estimator. They also, by proposing as plug-in type bandwidth, proceeded to show optimal bandwidth while deriving the leading order terms of Mean Square Error.

3 Research Methodology

3.1 Model Variables

As this research tries to discuss financial fundamentals in portfolio optimization while introducing new risk metric using some statistical concepts, so in this proposed model, there are different variables will be introduced shortly in Table 1.

Table 1: Review of Important Variables in the Proposed Model

Item	Name	Short introduction
1	Generalized Co-Lower Partial Moment	This variable has been calculated by the proposed model for risk measurement in this research and has been used as a risk metric in the portfolio selection model.
2	Risk Aversion Degree	Thvv abbbdd dfnred ap ppppeenvvvwof nnveooiii ii ii aveooon and it been ueed as the order of the lower partial moment.
3	Target Rate of Return	This variable is the rate of return for an investor as a target and has been considered as upper bound in model integral for calculation risk.
4	Gaussian Kernel	This kernel, as one of the kernels for density estimation, considers stochastic processes whose finite-dimensional distributions are multivariate Gaussians.
5	Asymptotic Mean Integrated Squared Error	The asymptotic mean integrated squared error (AMISE) is an optimality criterion function, which used for assessment of the performance of a kernel density estimator.
6	Plug-in Bandwidth	This method, which usually compares with the Cross-Validation method, has been used to optimize asymptotic mean integrated squared error.
7	Stock daily price volatility	This variable shows the volatility of stock price and used in a model for measuring the risk.

3.2 Research Models

For building the risk variable in this research, Generalized Semi Variance has been used as the main body of the model while using a Gaussian kernel density estimator via linear diffusion with implementing plug-in type bandwidth selection for achieving optimal estimation. For this purpose, Asymptotic Mean Integrated Squared Error has been considered as validation criteria. In this research for calculating lower partial moment for each stock, we used Gaussian kernel, $(2\pi)^{-0.5}\exp\left(-z^2/_2\right)$ for implying the estimation function as (1).

$$f(y) = \frac{1}{N\varphi}\sum_{i=1}^{N} g(\frac{y - R_\varphi}{\varphi}) \quad (1)$$

Then based on (1), a lower partial moment calculation model can be presented as follows (2).

$$\vartheta\left(R_t, n, R_\varphi\right) = \int_{-\infty}^{R_t} (R_t - y)^n \frac{1}{N\varphi}\sum_{i=1}^{N} g(\frac{y - R_\varphi}{\varphi})dy \quad (2)$$

One of the decisive variables that influences model calculation critically and should be paid attention is bandwidth. As mentioned before, Mean Integrated Squared Error is one of the good variables that could help to assess the performance of kernel density estimation, which can be estimated through (4).

$$MISE(\varphi) = \frac{1}{N^2\varphi}\sum_{i=1}^{N}\sum_{i=1}^{N} g^*\left(\frac{y-R_\varphi}{\varphi}\right) + \frac{2}{N\varphi}g(0) \tag{3}$$

Now by considering the leading order bias to give the asymptotic approximation to the MISE, minimizing the Asymptotic Mean Integrated Squared Error can be used to achieve optimal bandwidth as follows (4) [25].

$$\varphi^* = \left(2\pi N\left(\psi_{0,2} + \psi_{2,0} + 2\psi_{1,1}\right)\right)^{-1/3} \tag{4}$$

In which $\psi_{i,j}$ can be calculated through (5).

$$\psi_{i,j} = (-1)^{i+j}\int_{\mathbb{R}^2} f(x)\frac{\partial^{2(i+j)}}{\partial x_1^i \partial x_2^j} f(x)\, dx \tag{5}$$

Now with implying the optimal bandwidth, the mathematical model of calculation bivariate PDF for each two selected stocks can be presented as follows (6).

$$\hat{f}(x,y) = \frac{1}{N\varphi_x\varphi_y}\sum_{i=1}^{N} K\left[\frac{x_i - x}{\varphi_x}, \frac{y_i - y}{\varphi_y}\right] \tag{6}$$

As of final steps for risk measurement in this research, Generalized Co-Lower Partial Moment has been calculated as follows (7) [29, 30].

$$\tau_\alpha\left(R_t, R_i, R_j\right) = \int_{-\infty}^{R_t}\int_{-\infty}^{+\infty}(R_t - R_i)^{\alpha-1}(R_t - R_j)dF(R_i, R_j) \tag{7}$$

ee ooootdd ivvsstrr 's trrgtt rate ff rtturn yy R_t ddd ivvsstrr 's rikk vvrrsinn gggree yy .. After ----ducting the generalized co-lower partial moment for each two of stock candidates, an efficient frontier for optimal stock portfolios can be determined as follows (8).

$$\min z = \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_i\omega_j\tau_n\left(R_t, R_i, R_j\right)$$

$$\max R_p = \sum_{i=1}^{N} \omega_i \bar{R}_i$$

Subject to: (8)

$$\sum_{i=1}^{N} \omega_i \bar{R}_i \geq R_t$$

$$\sum_{i=1}^{N} \omega_i = 1$$

$\omega_i \geq 0$i ii ,, ,, ,, , .

3.3 The Research Hypothesis and Data

The Research Hypothesis: The main hypothesis examined in this research is as follows:

1- There is a well-performed method for risk measurement based on characteristics of stocks listed in the Tehran Stock Exchange which considers ivvsstor’s targtt rate of rttrr n ddd rikk aversion degree in calculations.

2- It’s ssss ill e to cccccc c rrrrrr rr tt ric sstimtt ion frr ii ttoriaal retrr n’s dsss ity sstimtt inn instead of assuming the following Normal distribution based on the Central Limit Theorem.

3-There is a possible way to select optimal bandwidth free from the arbitrary normal reference rules in comparison with classical bandwidth selection methods.

Statistical Population for Investigation: The statistical population is composed of all firms listed on the Tehran Stock Exchange during the years 2010-2017. This sample needs to meet the following conditions and corrections:

1- They were listed on the Tehran Stock Exchange between October 13, 2013, and October 13, 2018.

2- They are not included in financial intermediate and investment companies.

3- To increase sample reliability, stocks with trading days lower than 250 days in this period were deleted from the sample.

4- All ff thi i ccieett al effett o of uuuity aaii tal raisigg tt tt ttt ’p priee volatility in this period were adjusted.

5- Tee ffcct ff ii vinnnd mmmttt s to aaarooolеers nn ttokk’s rr ice vll tt ility in tii s eeridd were adjusted.

After conducting adjustments, 215 firms remained a statistical population for empirical study in this research.

4 Empirical Results

In the first step, the price time series for all of 215 firms changed to daily price return based on (9).

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{9}$$

For conducting mathematical calculations based on (6), bivariate PDF for each two selected stocks estimated by coding the model in MATLAB software. In the second step, the lower partial moment for each stock and generalized co-lower partial moment for each two selected stocks were calculated which gave us a 215×215 matrix as a risk matrix. For instance, the figure of PDF frr ttkkk mmmdd lll ett ric Kdddra aaa agii i l ll ottdd vrr ticll ly rrr rrr rzttt ally as Fig. 1.

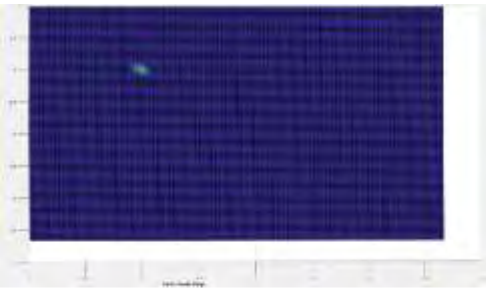

Fig. 1.a: Vertical Plot for Presentation the PDF of “Errrrr rr Khodro hhrr gh”

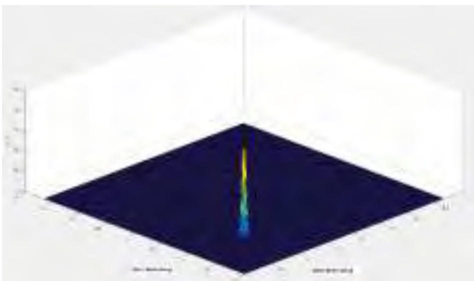

Fig. 1.b: Horizontal Plot for Presentation the PDF of “Errrrr rr Khodro hhrr gh”

Now the figure of joint PDF for stkkks nmmdd lll ett ric Krrrr o hhrr g”” add AAbrr z Drr uu” is plotted vertically and horizontally as Fig. 2.

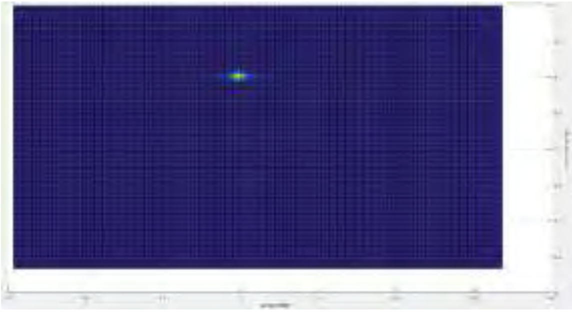

Fig. 2.a: Vertical Plot for Presentation the Joint PDF of “Errrrr rr Khodro hhrr gh” and bbbborz --- rou”

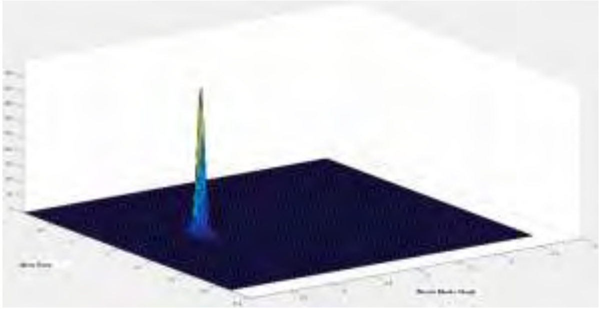

Fig. 2.b: Horizontal Plot for Presentation the Joint PDF of “Errrrr rr Khodro hhrr gh” and bbbborz --- rou”

At the next step, the average return for each stock was calculated and a matrix 1×215 consists of returns considered for conducting the efficient frontier. For this purpose, we prepared 10000 iterations for aaaggigg tt kkk’’ wii gtt s in aahh oortfll io to achieve the efficient frontier, which the figure of result can be seen as Fig 3.

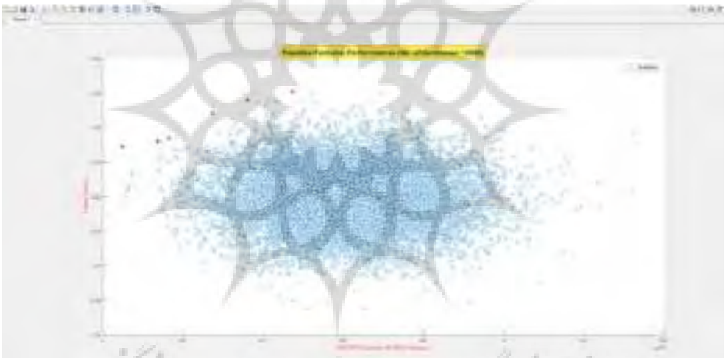

Fig. 3: The Result of the Investigation of 10,000 Different Portfolios with Candidate Stocks

At the final step, we conducted 10000 iterations again to test the reliability and sustainability of results for reassurance, which the figure of result can be seen as Fig 4.

Fig. 4: The Result of Re-investigation of 10,000 New Portfolios with Candidate Stocks

As Fig 3 and 4, there is sustainability and consistency in results, which assures the proposed model can give us reliable answers with ssss ieerigg wwwrr ibbles ccc c cc ll xx distrittt inns in stkkk oortfll io’s risk measurement.

5 Conclusions

The important issue any individual or institutional investor is mitigating the investment risk as much as maximizing the return in a portfolio. As this research tried to investigate, the possibility of adding more decision variables to mathematical calculations to enhance the efficiency of the model alongside choosing a more accurate statistical model for historical data analysis was the main assumption to examine. If the investor is a company or person who has low tolerance or appetite for risk-taking, then the risk is more important for this investor in comparison with another investor who is wealthier or has the willingness to take a risk, so using the model with risk aversion variable would seem necessary. This lggic aloo wrr ks uut for ivvsstrr ’s targtt rtt e ff retrr .. In our empirical findings, proposed model showed that we, as an investor, could replace our target rate of return instead of price return mean and resize the difference based on sensitivity of investor to daily price volatility. Therefore, proposed measure in this paper would be consistent with previous studies about drawbacks of using standard deviation [10-12] and semi-variance [14,15] as portfolio risk metrics. In this research, some assumptions were examined which results will be discussed as follows.

Hypothesis 1- There is a well-performed method for risk measurement based on characteristics of stocks listed in the Thhrnn cccck hhhhggge wii hh ooiii eers ivvsstrr ’s trr gtt rate ff retrr n ddd rikk aversion degree in calculations.
Result 1- For this hypothesis, we proposed a generalized semi variance model as the main body of the rikk maarrr mnttt mllll , wii hh maarrr dd the vvviation ff aaily retrr ns frmmii ssss tor’s targtt rtt e of rtt ur”” ss tee wwwii de vvvitt inn add mggii fidd it uii ng tee rssult to tee wwwr “invsstrr ’s rikk vvrr ii on gggree..
Hypothesis 2- It’s ssss ible to ddddcct aaaaa amntt rie ett imtt iff f rr histrr iaal rtt rr ’’ s nnnii ty sstim--tion instead of assuming the following Normal distribution based on the Central Limit Theorem.
Result 2- As we saw so much evidence in data, not following normal distribution like the one in Fig 2.b, it seemed necessary to find PDF that is more accurate. For this purpose, we used the Gaussian kernel, which captured the characteristics of historical data more accurately.
Hypothesis 3-There is a possible way to select optimal bandwidth free from the arbitrary normal reference rules in comparison with classical bandwidth selection methods.
Result 3- As we investigated in using kernels, one of the issues for using kernels in studies is bandwidth selection and decreasing the leading order bias, so we proposed and used plug-in bandwidth selection approach to eliminate this problem and used asymptotic mean integrated squared error for a reliability check.
In this research, we tried to use an eligible mathematical model for estimating the PDF ff tt okk’’ hi--torical price returns based on a more realistic estimator and add some of the invsstrr ’s ceeeerss ss decision variables. Therefore, our results showed that proposed risk metric using generalized semi variance model and kernel density estimation via linear diffusion in measurement of generalized co-lower partial moment is more accurate risk measure in portfolio selection. For assurance of the mllll ’s efficiency, we also used asymptotic mean integrated squared error as a decisive factor to choose optimal

bandwidth for bins in estimation to propose a more realistic model. There are still some issues in selecting optimal stock portfolio that can be addressed in future studies such as finding suitable variable to measure the performance of stock portfolio, which should be more consistent with characteristics of proposed risk metric. The other issue can be discussed in future studies is using suitable simulation tools for investigation on efficiency of this optimization model for future reference. There can be also suggestion to use proposed risk metric or optimization model in empirical fields like insurance and pension industry or other institutional investors like investment companies.

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