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Research Paper

On Solutions of Generalized Implicit Equilibrium Problems with Application in Game Theory

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ABSTRACT

In this paper, first a brief history of equilibrium problems(EP) and generalized implicit vector equilibrium problems(GIVEP) are given. Then some existence theorems for GIVEP are presented, also some suitable conditions in order the solution set of GIVEP is compact and convex for set-valued mappings whose are a subset of the cartesian product of Hausdorff topological vector space and their range is a subset of a topological space values (not necessarily locally convex or a topological vector space). In almost all of published results for GIVEP the set-valued mappings are considered from a topological vector space (locally convex topological vector space) to a topological vector space while in this paper the range of the set-valued mappings are a subsets of a topological spaces. As applications of our results, we derive some suitable conditions for existing a normalized Nash equilibrium problem when the number of players are finite and the abstract case, that is infinite players. Finally, a numerical result, as an application of the main results, is given. The method used for proving the existence theorems is based on finite intersection theorems and Ky-Fan's theorema The results of this paper, can be considered as suitable generalizations of the published paper in this area.

1 Introduction

Game theory has been applied during the last two decades to an ever increasing number of important practical problems in economics, industrial organization, business strategy, finance, accounting, market design and marketing; including antitrust analyses, monetary policy, and firm restructuring. Game Theory is a method of modeling the interaction between two or more players in a situation with particular rules and expected outcomes. It is helpful in many fields, but mainly as a tool in economics. Game Theory helps with the fundamental analysis of industries and the interactions between two or more companies. Game Theory revolutionized economics and business analysis by addressing critical issues in the popular mathematical models. For example, neoclassical economists struggle to account for the concept of imperfect competition fully. Game Theory improves on that by switching the focus from constant equilibrium to analyzing the actual market process. An essential concept within Game Theory

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is the Nash Equilibrium, which represents a stable state in a game, also known as a ‘no regrets’ state. It is named after John Nash, who got the Nobel Prize for it in 1994. The concept represents an outcome within the game, at which point no player can increase payoff by changing their strategic decisions. Once we reach such a state, we usually don’t deviate from it. Unilateral moves no longer affect the situation, so it makes no sense to make them. And this is why we consider it a ‘no regrets’ state. A set of strategies is at a Nash Equilibrium if each is the best response to the others. If all players operate on a Nash Equilibrium strategy, they have no incentive to deviate, as discussed above. Each player has adopted a plan that’s the best course of action based on what the others are doing. The main theorem of this paper will apply to obtain some suitable conditions for existing a normalized Nash equilibrium problem when the number of players are finite and the abstract case, that is infinite players. The implicit vector equilibrium problem (IVEP) was introduced by Huang et al. [8] as follows:
Given a vector valued bifunction $f: K \times K \to Y$ and $g: K \to K$, find $x \in K$ such that

$$f(g(x), y) \not\in -intC, \quad \forall y \in K, \tag{1}$$

where X, Y are two topological vector spaces and K is a nonempty subset of X.denotes the space of all continuous linear operators from X to Y,
If $T: K \to L(X, Y), \theta: K \times K \to X$, and $g: K \to K$, then (IVEP) reduces to the generalized vector variational inequality (GWI) of finding $x \in K$ such that

$$\langle T(g(x)), \theta(y, g(x))\rangle \not\in -intC(x), \quad \forall y \in K, \tag{2}$$

where $L(X, Y)$ denotes the space of all continuous linear operators from X to Y, $\langle T(z), y\rangle$ denotes the evaluation of the linear operator $T(z)$ at y. The generalized vector equilibrium problem was first introduced in 1997 [1] as follows. Let K a nonempty, closed, and convex subset of topological vector space (tvs) X, C a closed and convex cone in Y with $intC \neq \emptyset$. Let $F: K \times K \to 2^Y$ be a set-valued mapping. The generalized vector equilibrium problem (GVEP) for F consists in finding $x \in K$ such that

$$F(x, y) \nsubseteq -intC, \quad \forall y \in K. \tag{3}$$

The authors of [16] considered the generalized implicit operator equilibrium problem (GIOEP) which consists of finding $f^* \in K$ such that

$$F(h(f^*), g) \nsubseteq -intC(f^*), \forall g \in K, \tag{4}$$

where $F: K \times K \longrightarrow 2^Y$ is a set-valued mapping, $h: K \longrightarrow K$ is a mapping, X and Y are two Hausdorff topological vector spaces, $L(X, Y)$ is the space of all continuous linear operators from X to Y, $K \subseteq L(X, Y)$ is a nonempty convex set, $C: K \longrightarrow 2^Y$ is a set-valued mapping such that for each $f \in K$, $C(f)$ is a closed and convex cone in Y with $intC(f) \neq \emptyset$($intC(f)$ is the interior of $C(f)$), 2^Y denotes the set of all non-empty subsets of Y. This paper is motivated and inspired by the recent paper [16] and its aim is to extend the results given in [16] to the setting of Hausdorff topological vector spaces with mild assumptions and relaxing some conditions. In the rest of this section we recall some definitions and results that we need in the next section. A subset C of Y is called a pointed and convex cone if and only if $C + C \subseteq C, tC \subseteq C, \forall t \geq 0$, and $C \cap -C = \{0_Y\}$ (see, for instance, [1, 3, 6, 8, 9, 10, 11, 13, 21]) The domain of a set-valued mapping $W: X \longrightarrow 2^Y$ is defined as

$$D(W) = \{x \in X: W(x) \neq \emptyset\} \tag{5}$$

and its graph is defined as

$$Gr(W) = \{(x, z) \in X \times Y: z \in W(x)\}. \tag{6}$$

Also W is said to be closed if its graph, that is, $Gr(W)$, is a closed subset of $X \times Y$.
A set-valued mapping $T: X \longrightarrow 2^Y$ is called upper semicontinuous (in short $u.s.c.$) at $x_0 \in X$ if for every open set $V \subseteq Y$ containing $T(x_0)$ there exists an open set $U \subseteq X$ containing x_0 such that $T(u) \subseteq V$, for all $u \in U$. The mapping T is said to be lower semicontinuous (in short $l.s.c.$) at $x_0 \in X$ if for every

open set $V \subseteq Y$ with $T(x_0) \cap V \neq \emptyset$ there exists an open set $U \subseteq X$ containing x_0 such that $T(u) \cap V \neq \emptyset, \forall u \in U$. The mapping T is continuous at x_0 if it is both $u.s.c.$ and $l.s.c.$ at x_0. Moreover, T is $u.s.c.$ $(l.s.c.)$ on X if T is $u.s.c.(l.s.c.)$ at each point of X. We need the following basic definitions and results in the sequel.

Lemma 1: [18] Let X and Y be two Hausdorff topological spaces and $T: X \longrightarrow 2^Y$ be a mapping. The following statements are true:

(a) For any given $x_0 \in X$ if T has compact value at x_0 (i.e., $T(x_0)$ is a compact), then T is $u.s.c.$ at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \longrightarrow x_0$ and for every $y_\alpha \in T(x_\alpha)$, there exist $y_0 \in T(x_0)$ and a subnet $\{y_{\alpha_\beta}\} \subseteq \{y_\alpha\}$ such that $y_{\alpha_\beta} \longrightarrow y_0$;

(b) T is $l.s.c.$ at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \longrightarrow x_0$ and for any $y_0 \in T(x_0)$, there exists $y_\alpha \in T(x_\alpha)$ such that $y_\alpha \longrightarrow y_0$.

Definition 1: [16] Let K be a non-empty subset of topological vector space X. A set-valued mapping $T: K \to 2^X$ is called a KKM mapping if for every finite subset $\{x_1, x_2, \dots, x_n\}$ of K, $Co\{x_1, x_2, \dots, x_n\}$ is contained in $\bigcup_{i=1}^{n} T(x_i)$, where Co denotes the convex hull.

The KKM -mappings were first considered by Knaster, Kuratowski and Mazurkiewicz (KKM) [19] in 1920, in order to guarantee the finite intersection property for values of the mapping.

Lemma 2: [5] Let K be a nonempty subset of a topological vector space X and $F: K \longrightarrow 2^X$ be a KKM-mapping with closed values in K. Assume that there exists a nonempty compact convex subset B of K such that $\cap_{x \in B} F(x)$ is compact. Then $\cap_{x \in K} F(x) \neq \emptyset$.
Remark that if $F: K \longrightarrow 2^X$ is a KKM-mapping with closed values in K, then the family $\{Gx: x \in X\}$ of sets has the finite intersection property.

Theorem 1: Let X and Y be two Hausdorff topological spaces and $T: X \longrightarrow 2^Y$ be a set-valued mapping with nonempty valued. Assume that T is closed valued and $u.s.c.$ mapping, then T has a closed graph.

Definition 2: Let X be a nonempty set and Y be a topological space. A set-valued mapping $T: X \to 2^Y$ is called a transfer closed mapping if $\cap_{x \in X} clT(x) = \cap_{x \in X} T(x)$.

2 Main Results

The results of this section theorem can be viewed as an extension, improvement and repairmen of the Theorem 3.1 given in [16] by relaxing and weakening some assumptions. Further, the main theorem of this section is implicit version of Corollary 2 in [1] from locally convex spaces to topological vector spaces and relaxing conditions (iv)-(vi) of it. Moreover, it is set-valued version of Theorem 3.1 and Theorem 3.2 in [18] with mild assumptions. Finally the range of the mappings are subsets of a topological spaces instead of locally convex topological vector spaces. In the rest of this section X is a Hausdorff topological vector space and Y is a Hausdorff topological space.

Theorem 2: Let K be a non-empty convex subset of X and $h: K \longrightarrow K$ be a mapping and $F: K \times K \longrightarrow 2^Y$ be a set-valued mapping. Suppose that the following assumptions hold:

(a) The set-valued mapping $x \longrightarrow F(h(x), y)$ is $u.s.c.$ with compact values, for all $y \in K$;
(b) The mapping $x \longrightarrow Y \setminus S(x)$ is $u.s.c.$;
(c) For each $y \in K$ the set $\{x \in D: F(h(x), y) \nsubseteq S(x)\}$ is finitely closed in K,
(i) $F(h(x), x) \nsubseteq S(x), \forall x \in K$;

(ii) $\{y \in K: F(h(x), y) \subseteq S(x)\}$ is convex, $\forall x \in K$.

Then the solution set of $GIVEP$ is nonempty, i.e. there exists $x^* \in K$ such that

$$F(h(x^*), y) \nsubseteq S(x^*), \forall y \in K.$$

Moreover, the solution set is compact if the following condition is satisfied:

(d) There exists a nonempty compact and convex subset B of K, such that for each $x \in K \backslash B$, there exists $y \in B$ such that $F(h(x), y) \subseteq S(x)$.

Proof. Let D be an arbitrary compact and convex subset of K.

Define $T: D \longrightarrow 2^D$ by

$$T(y) = \{x \in D: F(h(x), y) \nsubseteq S(x)\}, \forall y \in D. \tag{7}$$

We show that T satisfies all the assumptions of Lemma 1.3. We first prove that $T(y)$ is closed, for all $y \in K$. To see, let $\{x_\alpha\}$ be a net in $T(y)$ such that $x_\alpha \longrightarrow x^*$. Define the mapping $H_y: D \longrightarrow 2^Y$ by

$$H_y(x) = F(h(x), y). \tag{8}$$

It follows from $x_\alpha \in T(y)$ that $H_y(x_\alpha) \nsubseteq S(x_\alpha)$. Hence, for each α,

$$\exists z_\alpha \in H_y(x_\alpha) \quad s.t. \quad z_\alpha \in Y \backslash -intC(x_\alpha) = W(x_\alpha), \tag{9}$$

by (a) there exist $z \in H_y(x^*)$ and a subnet $\{z_{\alpha_\beta}\}$ such that $z_{\alpha_\beta} \longrightarrow z$. Also $(x_{\alpha_\beta}, z_{\alpha_\beta}) \longrightarrow (x^*, z)$ and $(x_{\alpha_\beta}, z_{\alpha_\beta}) \in G_r H_y$. By Theorem 1,

$$(x^*, z) \in G_r H_y \text{ and } z \in F(h(x^*), y). \tag{10}$$

On the other hand $z_{\alpha_\beta} \in W(x_{\alpha_\beta})$ and $(x_{\alpha_\beta}, z_{\alpha_\beta}) \in G_r W$. Since $W(x_{\alpha_\beta})$ is closed, by (b) and Theorem 1, we conclude that $(x^*, z) \in G_r W$, thus

$$z \in W(x^*) = Y \backslash S(x^*). \tag{11}$$

From (10), (11) we have

$$F(h(x^*), y) \nsubseteq S(x^*) \Longrightarrow x^* \in T(y). \tag{12}$$

Hence $T(y)$ is closed, for all $y \in K$.

Now we prove that the mapping $y \longrightarrow T(y)$ is a $KKM -$ mapping. Suppose to the contrary that there exists a finite subset $\{y_1, y_2, \dots, y_n\}$ of D such that $Co\{y_1, y_2, \dots, y_n\} \nsubseteq \cup_{i=1}^n T(y_i)$. Hence there exists $z \in Co\{y_1, y_2, \dots, y_n\}$ such that

$$z = \sum_{i=1}^n \lambda_i y_i, \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0, z \notin T(y_i), \forall i = 1,2, \dots, n. \tag{13}$$

Therefore

$$F(h(z), y_i) \subseteq S(z). \tag{14}$$

therefore by assumption (ii), we get $F(h(z), z) \subseteq S(z)$ which contradicts (i). Hence T is a KKM −mapping. Since D is compact and $T(y)$ is a closed subset of D, and T is a KKM −mapping. Hence by Lemma 2, we have

$$\cap_{y \in D} T(y) \neq \emptyset. \tag{15}$$

Now, we show that

$$\cap_{y \in K} T(y) \neq \emptyset. \tag{16}$$

Otherwise

$$\cap_{y \in K} T(y) = (\cap_{y \in D} T(y)) \cap (\cap_{y \in K \backslash D} T(y)) = \emptyset. \tag{17}$$

Thus $\cap_{y \in D} T(y) \subseteq \cup_{y \in K \backslash D} (T(y))^c$. Also, it is obvious that $\cap_{y \in D} T(y) \subseteq D$ and so $\cap_{y \in D} T(y)$ is compact.

(Note that $T(y)$ is closed for each $y \in D$ and D is compact).

Hence there exist $y_1', y_2', \dots, y_n' \in K \backslash D$ such that $\cap_{y \in D} T(y) \subseteq \cup_{i=1}^n (T(y_i'))^c$, which gives that

$$(\cap_{y\in D} T(y)) \cap (\cap_{i=1}^{n} (T(y_i'))) = \cap_{y\in D\cup\{y_1',y_2',\dots,y_n'\}} T(y) = \emptyset. \quad (19)$$

Thus

$$\cap_{y\in Co(D\cup\{y_1',y_2',\dots,y_n'\})} T(y) \subseteq \cap_{y\in D\cup\{y_1',y_2',\dots,y_n'\}} T(y). \quad (20)$$

Now, if we consider $B = Co(D \cup \{y_1', y_2', \dots, y_n'\})$, B is compact and convex and the mapping $T: B \longrightarrow 2^B$ is a KKM −mapping. Hence by Lemma1.3, $\cap_{y\in B} T(y) \neq \emptyset$.

By (20), we get

$$\emptyset \neq \cap_{y\in B} T(y) \subseteq \cap_{y\in D\cup\{y_1',y_2',\dots,y_n'\}} T(y) = \emptyset, \quad (21)$$

which is a contraction. Hence $\cap_{y\in K} T(y) \neq \emptyset$.

Thus there exists $x^* \in K$ such that

$$F(h(x^*), y) \nsubseteq -intC(x^*), \forall y \in K. \quad (22)$$

Now, we show that the solution set of $GIVEP$, which is equals to the set $\cap_{y\in k} T(y)$, is compact. If the condition (d) is satisfied, to see this, we show that $\cap_{y\in k} T(y) \subseteq B$. Otherwise, there exists $x_0 \in \cap_{y\in k} T(y)$ such that $x_0 \in K\backslash B$. By condition (d) $\exists y_0 \in B$ such that $F(h(x_0), y_0) \subseteq S(x_0)$. Thus $x_0 \notin T(y_0)$, which is a contradiction. Thus $\cap_{y\in K} T(y)$ is compact, This completes the proof.

The following theorem is a direct consequence of Theorem 2 which improves Theorem 4.1 in [8] by relaxing conditions (2), (3) of it and the locally convex space Y to topological space Y and replacing the family $\{C(x)\}_{x\in X}$ of convex cones to an arbitrary family of subsets of Y.

Theorem 3: Let K be a nonempty, closed, convex subset of X, $g: K \to K$ and $f: K \times K \to Y$ be a bi-function. Suppose that the following assumptions hold:

(a) $f(g(x), x) \notin S(x),\ \forall x \in K$;

(b) $x \to f(x, y)$ and g are continuous $\forall x \in K$;

(c) for each $x \in K$, the set $\{y \in K: f(x, y) \in S(x)\}$ is convex;

(d) the set-valued mapping $W: K \to 2^Y$ defined by $W(z) = Y\backslash S(z)$, for all $z \in K$, is closed;

(e) there exists a nonempty compact and convex subset D of K, such that for all $x \in K\backslash D, \exists y \in D$ such that $f(g(x), y) \in S(x)$. Then the set $\{x \in K: f(g(x), y) \notin S(x), \forall y \in K\}$ is nonempty and compact.

Proof. Let D be an arbitrary compact and convex subset of K.

Define $T: D \longrightarrow 2^D$ by

$$T(y) = \{x \in D: f(g(x), y) \notin S(x)\}, \forall y \in D. \quad (23)$$

It is easy to verify by (b) and (d) that for each $y \in D$, $T(y)$ is closed.

It is clear that the mapping $y \longrightarrow T(y)$ is a $KKM -$ mapping. Because otherwise there exists a finite subset $\{y_1, y_2, \dots, y_n\}$ of D such that

$$Co\{y_1, y_2, \dots, y_n\} \nsubseteq \cup_{i=1}^{n} T(y_i). \quad (24)$$

Hence there exists $z \in Co\{y_1, y_2, \dots, y_n\}$ such that

$$z = \sum_{i=1}^{n} \lambda_i y_i, \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, z \notin T(y_i), \forall i = 1,2, \dots, n. \quad (25)$$

Therefore

$$f(g(z), y_i) \in S(z) \quad (26)$$

and so (14), we get $F(h(z), z) \subseteq S(z)$ which contradicts (a). Hence T is a KKM −mapping.

Hence the set-valued mapping satisfies all the hypothesis of Lemma 1, and so

$$\cap_{y\in D} T(y) \neq \emptyset. \quad (27)$$

Now, we claim that

$$\cap_{y\in K} T(y) \neq \emptyset. \tag{28}$$

Otherwise

$$\cap_{y\in K} T(y) = (\cap_{y\in D} T(y)) \cap (\cap_{y\in K\setminus D} T(y)) = \emptyset. \tag{29}$$

Thus $\cap_{y\in D} T(y) \subseteq \cup_{y\in K\setminus D} (T(y))^c$. Also, it is obvious that $\cap_{y\in D} T(y) \subseteq D$ and so $\cap_{y\in D} T(y)$ is compact.

(Note that $T(y)$ is closed for each $y \in D$ and D is compact).

Hence there exist $y'_1, y'_2, \dots, y'_n \in K\setminus D$ such hat

$$\cap_{y\in D} T(y) \subseteq \cup_{i=1}^{n} (T(y'_i))^c, \tag{30}$$

which gives that

$$(\cap_{y\in D} T(y)) \cap (\cap_{i=1}^{n} (T(y'_i))) = \cap_{y\in D\cup\{y'_1,y'_2,\dots,y'_n\}} T(y) = \emptyset. \tag{31}$$

Thus

$$\cap_{y\in Co(D\cup\{y'_1,y'_2,\dots,y'_n\})} T(y) \subseteq \cap_{y\in D\cup\{y'_1,y'_2,\dots,y'_n\}} T(y). \tag{32}$$

Now, if we consider $B = Co(D \cup \{y'_1, y'_2, \dots, y'_n\})$, B is compact and convex and the mapping $T: B \longrightarrow 2^B$ is a KKM −mapping. Hence by Lemma 1, $\cap_{y\in B} T(y) \neq \emptyset$.

By (32), we get

$$\emptyset \neq \cap_{y\in B} T(y) \subseteq \cap_{y\in D\cup\{y'_1,y'_2,\dots,y'_n\}} T(y) = \emptyset, \tag{33}$$

which is a contraction. Hence $\cap_{y\in K} T(y) \neq \emptyset$.

Thus there exists $x^* \in K$ such that

$$f(g(x^*), y) \nsubseteq S(x^*), \forall y \in K. \tag{34}$$

Hence the set

$$\{x \in K: f(g(x), y) \nsubseteq S(x), \forall y \in K\} \tag{35}$$

s nonempty and the compactness of the set directly follows from (32) and the equality

$$\{x \in K: f(g(x), y) \nsubseteq S(x), \forall y \in K\} = \cap_{y\in k} T(y). \tag{36}$$

This completes the proof.

The following theorem is a generalization of Theorem 3 in [1] from locally convex spaces to topological spaces and moreover extending the domain of the set-valued mapping F from compact convex to convex and deleting conditions (ii), (iv)-(vii) of Theorem 3 in [1]. Further, it is implicit version of it.

Theorem 4: Let K be a nonempty convex subset of Hausdorff topological vector space X and $S: K \to 2^Y\setminus\emptyset$, where Y is a topological space. The set-valued mapping $F: K \times K \to 2^Y$, and single-valued mapping $g: K \to K$ satisfying in the following conditions.

(a) $F(g(x), x) \cap S(g(x)) \neq \emptyset, \forall x \in K$,

(b) $\{y \in K: F(x, y) \cap S(x) = \emptyset\}$ is convex, $\forall x \in K$,

(c) $\{x \in K: F(g(x), y) \cap S(g(x)) \neq \emptyset\}$ is closed, $\forall x \in K$,

(d) there exist compact convex set D and compact set M of K such that

$$\forall x \in K\setminus M, \exists y \in D, F(x, y) \cap S(x) = \emptyset. \tag{37}$$

Then there exists $x \in K$ such that the set

$$\{x \in K: F(x, y) \cap S(x) \neq \emptyset, \forall y \in K\}, \tag{38}$$

is nonempty and compact.

Proof. Assume that H is an arbitrary convex subset of K. Define $G: H \to 2^K$ by

$$G(x) = \{x \in K: F(x, y) \cap S(x) \neq \emptyset\}, \ \forall x \in K. \tag{39}$$

We prove that the mapping G is a $KKM -$ mapping. Suppose to the contrary there exists a finite subset

$\{y_1, y_2, \ldots, y_n\}$ of H such that

$$Co\{y_1, y_2, \ldots, y_n\} \nsubseteq \cup_{i=1}^{n} G(y_i). \tag{40}$$

Hence there exists $z \in Co\{y_1, y_2, \ldots, y_n\}$ such that

$$z = \sum_{i=1}^{n} \lambda_i y_i, \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, z \notin G(y_i), \forall i = 1,2, \ldots, n. \tag{41}$$

Therefore

$$F(g(z), y_i) \cap S(g(z)) = \emptyset. \tag{42}$$

Thus by assumption (b),we get $F(g(z), z) \cap S(g(z)) = \emptyset$, which is contracted by (a). Hence G is a KKM mapping and so the family $\{G(x)\}_{x \in H}$ has the finite intersection property. It follows from condition (d) that $\bigcap_{x \in D} G(x)$ is a closed subset of the compact set M. Consequently, $\bigcap_{x \in D} G(x) \neq \emptyset$. Now we claim that $\bigcap_{x \in K} G(x) \neq \emptyset$.

Otherwise $\bigcap_{x \in K} G(x) = (\bigcap_{x \in D} G(x)) \cap (\bigcap_{x \in K \setminus D} G(x)) = \emptyset$.

Hence $\bigcap_{x \in D} G(x) \subseteq \bigcup_{x \in K \setminus D} G^c(x)$ and since $\bigcap_{x \in D} G(x)$ is compact then there exist $x_1, \ldots, x_n$ of $K \setminus D$ such that $\bigcap_{x \in D} G(x) \subseteq \bigcup_{i=1}^{n} G^c(x_i)$. This mean that $\bigcap_{x \in D \cup \{x_1, x_2, \ldots, x_n\}} G(x) = \emptyset$. Hence

$$\bigcap_{x \in H = Co(D \cup \{x_1, x_2, \ldots, x_n\})} G(x) \subseteq \bigcap_{x \in D \cup \{x_1, x_2, \ldots, x_n\}} G(x) = \emptyset \tag{43}$$

which is a contradiction with being KKM of G on $H = Co(D \cup \{x_1, x_2, \ldots, x_n\})$. Hence there exists $z \in K$ such that $z \in \bigcap_{x \in K} G(x) = (\bigcap_{x \in D} G(x))$, and so

$$F(z, y) \cap C(z) \neq \emptyset, \forall y \in K. \tag{44}$$

The compactness of $\{z \in K: F(z, y) \cap S(z) \neq \emptyset, \forall y \in K\}$ directly follows from (d). This completes the proof.

Remark 1: It is easy to check that the result of Theorem 3 will be correct if one replaces the closedness of the set $\{x \in K: F(g(x), y) \cap C(g(x)) \neq \emptyset\}$, in condition (b) by the transfer closed (that is, if $z \notin G(y) = \{x \in K: F(g(x), y) \cap C(g(x)) \neq \emptyset\}$ then there exists $w \in K$ such that $z \notin \overline{G(w)}$, the closure of $G(w)$) of it.

The next result is a direct consequence of Theorem 4 which is an improvement version of Corollary 2 in [1].

Corollary 1: Let K be a nonempty convex subset of Hausdorff topological vector space X and P is a nonempty subset of the topological space Y. If The mappings $F: K \times K \to Y$ and $g: K \to K$ satisfy the following conditions conditions.

(a) $F(g(x), x) \in P, \quad \forall x \in K,$

(b) $\{y \in K: F(x, y) \notin P\}$ is convex, $\forall x \in K$,

(c) $\{x \in K: F(g(x), y) \in P\}$ is closed, $\forall x \in K$,

(d) there exist compact convex set D and compact set M of K such that

$$\forall x \in K \setminus M, \exists y \in D, F(x, y) \notin P, \tag{45}$$

Then there exists $x \in K$ such that the set $\{x \in K: F(x, y) \in P, \forall y \in K\}$, is nonempty and compact.

3 Applications

Definition 3(Game Theory Definitions): Any time we have a situation with two or more players that involve known payouts or quantifiable consequences, we can use game theory to help determine the most likely outcomes. Let's start by defining a few terms commonly used in the study of game theory:

- **Game**: Any set of circumstances that has a result dependent on the actions of two or more decision-makers (players)
- **Players**: A strategic decision-maker within the context of the game

- **Strategy**: A complete plan of action a player will take given the set of circumstances that might arise within the game
- **Payoff**: *T*he payout a player receives from arriving at a particular outcome (The payout can be in any quantifiable form, from dollars to utility.)
- **Information set**: The information available at a given point in the game (The term information set is most usually applied when the game has a sequential component.)
- **Equilibrium**: The point in a game where both players have made their decisions and an outcome is reached

Definition 4 (The Nash Equilibrium): Nash equilibrium is an outcome reached that, once achieved, means no player can increase payoff by changing decisions unilaterally. It can also be thought of as "no regrets," in the sense that once a decision is made, the player will have no regrets concerning decisions considering the consequences. The Nash equilibrium is reached over time, in most cases. However, once the Nash equilibrium is reached, it will not be deviated from. After we learn how to find the Nash equilibrium, take a look at how a unilateral move would affect the situation. Does it make any sense? It shouldn't, and that's why the Nash equilibrium is described as "no regrets." Generally, there can be more than one equilibrium in a game. However, this usually occurs in games with more complex elements than two choices by two players. In simultaneous games that are repeated over time, one of these multiple equilibria is reached after some trial and error. This scenario of different choices overtime before reaching equilibrium is the most often played out in the business world when two firms are determining prices for highly interchangeable products, such as airfare or soft drinks.

Let us recall the definition of the Nash equilibrium problem (NEP). There are N players, each player $v \in \{1,\ldots,\mathrm{N}\}$ controls the variables $x^v \in R^{n_v}$. All players' strategies are collectively denoted by a vector $x=(x^1,x^2,\ldots,x^N)^T \in R^n$, where $\mathrm{n}=n_1+\cdots+n_N$. To emphasize the v th player's variables within the vector x, we sometimes write $x=(\mathrm{x}^\nu,\mathrm{x}^{-\nu})^T$, where $x^{-v}\in R^{n_{-v}}$ subsumes all the other players' variables.

Let $\theta_v: R^n \to \mathrm{R}$ be the v th player's payoff (or loss or utility) function, and let $X^v \subseteq R^{n_v}$ be the strategy set of player v. Then, $x^*=(x^{*,1},x^{*,2},\ldots,x^{*,N})^T \in R^n$ is called a Nash equilibrium, or a solution of the NEP, if each block component $x^{*,v}$ is a solution of the optimization problem

$$\min \theta_v(\mathrm{x}^\nu,\mathrm{x}^{*,-\nu}) \text{ subject to } \mathrm{x}^\nu \in X^v. \tag{46}$$

A generalized Nash equilibrium problem (GNEP) consists of p players. Each player v controls the decision variable $\mathrm{x}^\nu \in \mathrm{C}_v$, where C_v is a non-empty convex and closed subset of $\mathrm{R}^{\mathrm{n}\nu}$. We denote by $\mathrm{x}=(x_1,\ldots,x_\mathrm{p}) \in \prod_{v=1}^p \mathrm{C}_v = C$ the vector formed by all these decision variables and by $\mathrm{x}^{-\nu}$, we denote the strategy vector of all the players different from player ν. The set of all such vectors will be denoted by $\mathrm{C}^{-\nu}$. We sometimes write $(\mathrm{x}^\nu,\mathrm{x}^{-\nu})$ instead of x in order to emphasize the ν-th player's variables within x. Note that this is still the vector $\mathrm{x}=(x^1,\ldots,x^v,\ldots,x^p)$, and the notation $(\mathrm{x}^\nu,\mathrm{x}^{-\nu})$ does not mean that the block components of x are reordered in such a way that x^v becomes the first block. Each player v has an objective function $\theta_v: \mathrm{C} \to \mathrm{R}$ that depends on all player's strategies. Each player's strategy must belong to a set identified by the set-valued map $\mathrm{K}_v: C^{-v} \rightrightarrows C_v$ in the sense that the strategy space of player v is $\mathrm{K}_v(x^{-v})$, which depends on the rival player's strategies x^{-v}. Given the strategy x^{-v}, player v chooses a strategy x^v such that it solves the following optimization problem

$$\min \theta_v(\mathrm{x}^\nu,\mathrm{x}^{-\nu}) \text{ subject to } \mathrm{x}^\nu \in \mathrm{K}_v(x^{-v}), \tag{47}$$

for any given strategy vector x^{-v} of the rival players. The solution set of problem (47) is denoted by $\mathrm{Sol}_\nu(x^{-v})$. Thus, a generalized Nash equilibrium is a vector $x^\wedge$ such that $x^{\wedge v} \in \mathrm{Sol}_\nu(x^{\wedge -v})$, for any ν. Associated to a GNEP, there is a function $f^{NI}: \mathrm{R}^n \times \mathrm{R}^n \to R$, defined by

$$f^{NI}(x,y) := \sum_{v=1}^{P} \{\theta_v(\mathrm{y}^\nu,\mathrm{x}^{-\nu}) - \theta_v(\mathrm{x}^\nu,\mathrm{x}^{-\nu})\}, \tag{48}$$

which is called Nikaido-Isoda function and was introduced in [14]. Additionally, we define the set-valued map $\mathrm{K}: \mathrm{C} \rightrightarrows \mathrm{C}$ by

$$K(x) \coloneqq \prod_{\upsilon=1}^{p} \mathrm{K}_{\upsilon}\ (\mathrm{x}^{-\nu}). \quad (49)$$

Definition 5: x^*, is a normalized Nash equilibrium of the GNEP, if $max_y f^{NI}(x^*, y) = 0$ holds, where $f^{NI}(x, y)$ denotes the Nikaido-Isoda function defined as (48).
Remark that the generalized Nash equilibrium problem is a generalization of the standard Nash equilibrium problem, in which both the utility function and the strategy space of each player may depend on the strategies chosen by all other players. This problem has been used to model various problems in applications.
The following theorem is a direct consequence of Corollary 1, which gives conditions of existence a normalized Nash equilibrium of the GNEP.

Theorem 4: Let K be a nonempty convex subset of Hausdorff topological vector space X and $P = (-\infty, 0]$ is a nonempty subset of the topological space Y. If The mappings $F: K \times K \to Y$ and $g: K \to K$ defined by

$$F(x, y) = f^{NI}(x, y) \coloneqq \sum_{\upsilon=1}^{P} \{\theta_{\upsilon}(\mathrm{y}^{\nu}, \mathrm{x}^{-\nu}) - \theta_{\upsilon}(\mathrm{x}^{\nu}, \mathrm{x}^{-\nu})\}, \quad g(x) = x \quad (50)$$

and satisfy the following conditions.

(a) $\{y \in K: F(x, y) = f^{NI}(x, y) \not\subseteq P\}$ is convex, $\forall x \in K$,
(b) $\{x \in K: F(g(x), y) = f^{NI}(g(x), y) \in P\}$ is closed, $\forall x \in K$,
(c) there exist compact convex set D and compact set M of K such that

$$\forall x \in K \backslash M, \exists y \in D, F(x, y) = f^{NI}(x, y) \not\subseteq P, \quad (51)$$

Then there exists a normalized Nash equilibrium of the GNEP.
Proof. It is easy to verify that the function F satisfies all the assumptions of Corollary 1. Hence the set of normalized Nash equilibrium for the GNEP is nonempty. This completes the proof.
The following example, taken from the reference [2], will be re-evaluated using the our main results.

Example 1 (Numerical Result): Let us consider two firms and two demand markets. Let $\underline{x}, \overline{x} \in L^2([0, T], \mathrm{R}^4)$ be the capacity constraints such that, a.e. in [0,1],

$$\underline{x}(\mathrm{t}) = \begin{pmatrix} 0 & 2t \\ 0 & 2t \end{pmatrix}, \quad \overline{x}(\mathrm{t}) = \begin{pmatrix} 100t & 200t \\ 100t & 200t \end{pmatrix},$$

and $\mathrm{p}, \mathrm{q} \in L^2([0, T], \mathrm{R}^4)$ be the production and demand functions such that, a.e. in [0,1],

$$p(\mathrm{t}) = \begin{pmatrix} 250t \\ 500t \end{pmatrix}, \quad q(\mathrm{t}) = \begin{pmatrix} 400t \\ 500t \end{pmatrix},$$

As a consequence, the feasible set is

$$K = \{x \in L^2([0, T], \mathrm{R}^4_+): \underline{x}_{ij}(t) \le x_{ij}(t)) \le \overline{x}_{ij}(t), \ \forall i = 1,2, \ \forall j = 1,2, a.e.in\ [0,1],$$
$$\sum_{j=1}^{2} x_{ij}(t) \le p_i(\mathrm{t}), i = 1,2, a.e.in\ [0,1], \ \sum_{i=1}^{2} x_{ij}(t) \le q_j(\mathrm{t}), j = 1,2, a.e.in\ [0,1]\}, \quad (52)$$

The set of feasible states is

$$\Omega = \{\omega \in L^2([0, T], \mathrm{R}^4_+): \underline{x}_{ij}(t) \le \omega_{ij}(t)) \le \overline{x}_{ij}(t), \ \forall i = 1,2, \ \forall j = 1,2, a.e.in\ [0, T]\}. \quad (53)$$

It is obvious Ω convex and weakly compact and $L^2([0, T])$ is a Hilbert space, hence we can take $K = \Omega$ in Theorem 2 and $L^2([0, T], \mathrm{R}^4)$.
Let us consider the profit function $v \in L^2([0,1] \times L^2([0,1], \mathrm{R}^4), \mathrm{R}^2)$ defined by

$$v_1(t, x(t)) = 6x_{11}^2(t) + 2x_{12}^2(t) + 2\alpha(t)x_{12}(t) - 2x_{11}(t)x_{12}(t) - 4x_{21}(t)x_{22}(t) - 2h_{11}(t)x_{11}(t) - 2h_{12}(t)x_{12}(t),$$
$$v_2(t, x(t)) = 6x_{21}^2(t) + 2x_{22}^2(t) + 2\beta(t)x_{22}(t) - 4x_{21}(t)x_{22}(t) - 2x_{11}(t)x_{12}(t) - 4h_{21}(t)x_{21}(t) - 4h_{22}(t)x_{22}(t),$$

where α, β are suitable functions depending on time and belonging to $L^2([0,1])$. Then, the operator

$$\nabla_D v\big(t, x(t)\big) = \left(\frac{\partial v_j\big(t,x(t)\big)}{\partial x_{ij}}\right)_{i=1,2} \in L^2([0,1] \times L^2([0,1],\mathrm{R}^4),\mathrm{R}^4) \tag{54}$$

is given by

$$\nabla_D v\big(t, x(t)\big) = \begin{pmatrix} 12x_{11}(t) - 2x_{12}(t) - 2h_{11}(t) & 4x_{12}(t) - 2x_{11}(t) - 2h_{12}(t) + 2\alpha(t) \\ 12x_{21}(t) - 4x_{22}(t) - 4h_{21}(t) & 4x_{22}(t) - 4x_{21}(t) - 4h_{22}(t) + 2\beta(t) \end{pmatrix}.$$

The dynamic oligopolistic market equilibrium distribution in presence of excesses is the solution to the evolutionary variational inequality:

$$\int_0^{\gamma} \sum_{i=1}^{2} \sum_{j=1}^{2} \left(-\frac{\partial v_i(t,x^*(t))}{\partial x_{ij}}\right) \left(x_{ij}(t) - x_{ij}{}^*(t)\right) dt \geq 0, \forall x \in \mathrm{K} \tag{55}$$

The inequality (55) has a solution, because in Theorem 2, it is enough, we take $X = L^2[0,1], \mathrm{g(x)} = \mathrm{x}$,

$$f(x,y) = \int_0^{\gamma} \sum_{i=1}^{2} \sum_{j=1}^{2} \left(-\frac{\partial v_i(t,x^*(t))}{\partial x_{ij}}\right) \left(x_{ij}(t) - x_{ij}{}^*(t)\right) dt \geq 0, \tag{56}$$

Where $x, y \in L^2[0,1], Y = R, S(x) = [0, +\infty)$.

In order to compute the solution to (55),we consider the following system

$$\begin{cases} -12x_{11}^*(t) + 2x_{12}^*(t) + 2h_{11}(t) = 0, \\ 2x_{11}^*(t) - 4x_{12}^*(t) + 2h_{12}(t) - 2\alpha(t) = 0, \\ -12x_{21}^*(t) + 4x_{22}^*(t) + 4h_{21}(t) = 0, \\ 4x_{21}^*(t) - 4x_{22}^*(t) + 4h_{22}(t) - 2\beta(t) = 0, \\ \qquad x^* \in \mathrm{K} \end{cases}$$

and we get the following solution, a.e.in [0,1],

$$x^*\big(t, h(t)\big) = \begin{pmatrix} \dfrac{2h_{11}(t) + h_{12}(t) - \alpha(t)}{11} & \dfrac{h_{11}(t) + 6h_{12}(t) - 6\alpha(t)}{11} \\ \dfrac{2h_{21}(t) + 2h_{22}(t) - \beta(t)}{4} & \dfrac{2h_{21}(t) + 6h_{22}(t) - 3\beta(t)}{4} \end{pmatrix}$$

In order to study the policy-maker's point of view, we have to solve the following inverse variational inequality

$$\int_0^{\gamma} \left(\sum_{i=1}^{2} \sum_{j=1}^{2} (\omega_{ij}{}^*(t) - x_{ij}(t, h^*(t))) \left(h_{ij}(t) - h_{ij}{}^*(t)\right) - \sum_{i=1}^{2} \sum_{j=1}^{2} h_{ij}{}^*(t) \left(\omega_{ij}(t) - \omega_{ij}{}^*(t)\right) \right) dt \geq 0,$$

$$\forall (h, \omega) \in \mathrm{L}^2\left([0,1], \mathrm{R}^4\right) \times \Omega. \tag{57}$$

Let us assume that $\omega_{ij}\ (t) = \omega_{ij}{}^*(t), \forall i = 1,2,\ \forall j = 1,2, a.e. in\ [0,1]$, in(55). As a consequence, we can consider the following system

$$\begin{cases} 2h_{11}^*(t) + h_{12}^*(t) - \alpha(t) - 11\omega_{11}{}^*(t) = 0, \\ h_{11}^*(t) + 6h_{12}^*(t) - 6\alpha(t) - 11\omega_{12}{}^*(t) = 0, \\ 2h_{21}^*(t) + 2h_{22}^*(t) - \beta(t) - 4\omega_{21}{}^*(t) = 0, \\ 2h_{21}^*(t) + 6h_{22}^*(t) - 3\beta(t) - 4\omega_{22}{}^*(t) = 0, \end{cases}$$

and we obtain the following solution, a. e. in [0,1],

$$h^*(t) = \begin{pmatrix} 6\omega_{11}{}^*(t) - \omega_{12}{}^*(t) & -\omega_{11}{}^*(t) + 2\omega_{12}{}^*(t) + \alpha(t) \\ 3\omega_{21}{}^*(t) - \omega_{22}{}^*(t) & -\omega_{21}{}^*(t) + \omega_{22}{}^*(t) + \dfrac{1}{2}\beta(t) \end{pmatrix}$$

Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 100t & 2t \\ 100t & 200t \end{pmatrix}.$$

Taking into account the direct method, it must be $h_{11}^*(t) > 0, h_{12}^*(t) < 0, h_{21}^*(t) > 0,\ h_{22}^*(t) > 0$. These conditions are true if and only if $\alpha(t) < 96t$ and $\beta(t) > -200t$. In this case the optimal regulatory tax and the optimal commodity distribution are, respectively

$$h^*(t) = \begin{pmatrix} 598t & \alpha(t) - 96t \\ 100t & \frac{1}{2}\beta(t) + 100t \end{pmatrix}, \quad x^*(t) = \begin{pmatrix} 100t & 2t \\ 100t & 200t \end{pmatrix},$$

which belongs to K. The production and demand excesses are $\varepsilon(t) = \begin{pmatrix} 148t \\ 200t \end{pmatrix}, \delta(t) = \begin{pmatrix} 200t148t \\ 298t \end{pmatrix}$, respectively. In particular, if $\alpha(t) = 50t - 50, \beta(t) = 200t + 50$, by the algorithm described in the previous section, we obtain the solution shown in Figure 1(see page 14). It represents the numerical approximation of the following exact optimal regulatory tax:

$$h^*(t) = \begin{pmatrix} 598t & -46t - 50 \\ 100t & 200t + 25 \end{pmatrix}.$$

It is possible to consider other 11 cases in which $\omega_{ij}{}^*(t)$ assumes minimal or maximal value and, like in the previous case, taking into account the direct method, it is possible, under appropriate conditions on the functions α and β, to compute the optimal regulatory tax, the optimal commodity distribution and the production and demand excesses. Let us underline that assuming $\omega_{11}{}^*(t), \omega_{12}{}^*(t)$both maximal, the previous procedure is not allowed since the correspondent commodity shipment $x^*(t)$ does not belong to the constraint set K because $x_{11}^*(t) + x_{12}^*(t) > 250t$. For this reason, let us consider the set

$$K^{\sim} = \{x \in L^2([0,T],\mathrm{R}^4{}_+): \underline{x}_{ij}(t) \le x_{ij}(t)) \le \overline{x}_{ij}(t),\ \ \forall i = 1,2,\ \forall j = 1,2, a.e.\ in\ [0,1],$$
$$x_{11}(t) + x_{12}(t) = p_1(t),\ \ x_{21}(t) + x_{22}(t) \le p_{21}(t), \textstyle\sum_{i=1}^{2} x_{ij}(t) \le q_j(\mathrm{t}), j = 1,2, a.e.\ in\ [0,1]\}. \quad (58)$$

In order to compute the solution to (57) we make use again of the direct method. We consider the following

$$\begin{cases} x_{11}^*(t) + x_{12}^*(t) - 25t = 0, \\ 14x_{11}^*(t) - 6x_{12}^*(t) - 2h_{11}(t) + 2h_{12}(t) - 2\alpha(t) = 0, \\ -12x_{21}^*(t) + 4x_{22}^*(t) + 4h_{21}(t) = 0, \\ 4x_{21}^*(t) - 4x_{22}^*(t) + 4h_{22}(t) - 2\beta(t) = 0, \\ \qquad x^* \in \mathrm{K} \end{cases}$$

and we obtain the following solution, a.e. in [0,1],

$$x^*(t, h(t)) = \begin{pmatrix} \frac{h_{11}(t) - h_{12}(t) + \alpha(t) + 750t}{11} & \frac{-h_{11}(t) + h_{12}(t) - \alpha(t) + 1750t}{11} \\ \frac{2h_{21}(t) + 2h_{22}(t) - \beta(t)}{4} & \frac{2h_{21}(t) + 6h_{22}(t) - 3\beta(t)}{4} \end{pmatrix}$$

In order to compute the solution to (57), we make use again of the direct method. For $\omega_{ij}\ \ (t) = \omega_{ij}{}^*(t), \forall i = 1,2,\ \forall j = 1,2, a.e.\ in$ [0,1] and the condition

$$\underline{x}_{1j}(t) \le x_{1j}(t)) \le \overline{x}_{1j}(t), \forall j = 1,2, a.e.\ in\ [0,1], \quad (59)$$

implies $h_{1j}^*(t) = 0, \forall j = 1,2, a.e.\ in$ [0,1], we can consider the following system

$$\begin{cases} h_{1j}^*(t) = 0, \forall j = 1,2, \\ 2h_{11}^*(t) + h_{12}^*(t) - \alpha(t) - 11\omega_{11}{}^*(t) = 0, \\ h_{11}^*(t) + 6h_{12}^*(t) - 6\alpha(t) - 11\omega_{12}{}^*(t) = 0, \\ 2h_{21}^*(t) + 2h_{22}^*(t) - \beta(t) - 4\omega_{21}{}^*(t) = 0, \\ 2h_{21}^*(t) + 6h_{22}^*(t) - 3\beta(t) - 4\omega_{22}{}^*(t) = 0, \end{cases}$$

and we get the following solutions a. e. in [0,1],

$$h^*(t) = \begin{pmatrix} 0 & 0 \\ 3\omega_{21}{}^*(t) - \omega_{22}{}^*(t) & -\omega_{21}{}^*(t) + \omega_{22}{}^*(t) + \frac{1}{2}\beta(t) \end{pmatrix}$$

and, moreover, $\omega_{11}{}^*(t) = \frac{\alpha(t)+750t}{10}, \omega_{12}{}^*(t) = \frac{-\alpha(t)+1750t}{10}$.

Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} \frac{\alpha(t) + 750t}{10} & \frac{-\alpha(t) + 1750t}{10} \\ 100t & 200t \end{pmatrix}.$$

As a consequence of the direct method, it must be $h_{11}^*(t) = 0, h_{12}^*(t) = 0, h_{21}^*(t) > 0, h_{22}^*(t) > 0,$ that are true if and only if $\beta(t) > -200t$. Hence, the optimal regulatory tax and the optimal commodity distribution are

$$h^*(t) = \begin{pmatrix} 0 & 0 \\ 100t & \frac{1}{2}\beta(t) + 100t \end{pmatrix}, x^*(t) = \begin{pmatrix} \frac{\alpha(t) + 750t}{10} & \frac{-\alpha(t) + 1750t}{10} \\ 100t & 200t \end{pmatrix},$$

which belongs to $K^{\sim}$ if and only if $-250t \leq \alpha(t) \leq 250t$. Hence, the production and demand excesses are

$$\varepsilon(t) = \begin{pmatrix} 0 \\ 200t \end{pmatrix}, \delta(t) = \begin{pmatrix} \frac{2250t - \alpha(t)}{10} \\ \frac{1250t + \alpha(t)}{10} \end{pmatrix},$$

In particular, if $\alpha(t) = 10t, \beta(t) = 100t + 100$ by the algorithm described in the previous section, we obtain the solution shown in Figure 2 (in the below). It represents the numerical approximation of the following exact optimal regulatory tax:

$$h^*(t) = \begin{pmatrix} 0 & 0 \\ 100t & 150t + 50 \end{pmatrix}$$

Finally, we remark that when $\omega_{11}{}^*(t), \omega_{12}{}^*(t)$ are both maximal, it is possible to consider other 3 cases in which $\omega_{ij}{}^*(t)(i = 2, j = 1,2)$ are minimal or maximal and, taking into account the direct method, like in last part, it is possible to compute the optimal regulatory tax, the optimal commodity distribution and the production and demand excesses under appropriate conditions on the functions α and β.

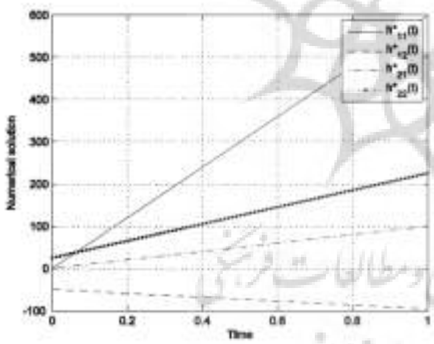

Fig. 1: Curves of optimal regulatory tax.

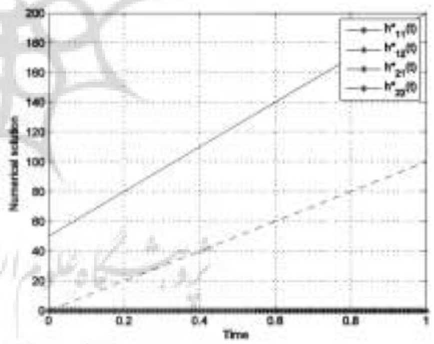

Fig. 2: Curves of optimal regulatory tax.

4 Conclusions

Existence theorem for a solution of game is designed for when the number of player is limited. Now the question arises that in a situation where the number of players is very large and even infinite, how to get existence theorems under appropriate assumptions. This article provides some answers to this question. A brief history of equilibrium problems and generalized implicit vector equilibrium problems(GIVEP) are stated. Then some existence theorems for GIVEP and some suitable conditions are presented, which under them the solution set of GIVEP is compact and convex for set-valued mappings whose are a subset of the cartesian product of Hausdorff topological vector space and their range is a subset of a topological space values (not necessarily locally convex or a topological vector space). Finally, the main theorem has been applied to obtain some suitable conditions for existing a normalized

Nash equilibrium problems when the number of players is finite and the abstract case, that is infinite players. Also, a numerical result is given.

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