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Optimal Promotional Effort Policy for Innovation Diffusion Model in a Fuzzy Environment

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Abstract

In today's era when a substitute for almost every product is readily available, acceptance and adoption of a new product in a market requires substantial amount of promotion. Here we formulate and analyze policies for promoting sales of a product in a market through optimal control theory problems. The market is partitioned into various segments depending upon multifarious demands of customers and promotion of the product is done segment-wise. The aim is to maximize the profits keeping in mind the demand requirements and the available budget for promotion. In order to provide a realistic model, the total available budget is taken to be imprecise. The optimal control model with fuzzy parameter is converted into crisp form using necessity and possibility constraints, and thereafter solved by using Pontryagin Maximum principle. To illustrate this technique, a numerical example is also considered by discretizing the model. The analysis also gives a deep insight of how the promotional effort should be planned by the decision makers keeping in mind the financial constrains without hindering the promotional effort at the end of the planning period. This paper mirrors the real time situation that could be faced by any industry, including that of software development, where budgets may have variable components and promotion of products may vary according to different regions and markets. The experimental data reveals that profitability can still be maximized if real-life constraints are applied in promotional planning by any industry.

Keywords: Fuzzy parameter; Segmentation; Optimal control problem.

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Introduction

With markets being flooded with substitutes, there has been an increased need to promote products more aggressively and strategically. Manufacturers cannot take monopoly for granted anymore and they must compete with other substitutes to sell their product. In order to popularize its new products manufacturers continuously formulate newer advertising plans. The customers approached by a manufacturer via some promotional strategies may be of varied interests, age groups and purchasing power. Hence, there is a need to partition the market and promote the products in a segment-wise manner. In addition to promoting products to the entire population as a whole, it is vital to take into account diffusion of innovation in a target market to formulate promotional strategies. In this paper we consider the total promotional effort cost to be consisting of cost of segment-wise promotion and mass promotion. In real world problems, the budget to be utilized for promotion may not be known completely. The model thus proposed in this paper inter alia allows the available budget to be a fuzzy parameter while using optimal control theory to analyze diffusion of innovation.

The concept of fuzzy sets was pioneered by Zadeh (1965), who offered a practical way to model imprecise data by using fuzzy numbers and fuzzy intervals. Thereafter, the task of modelling real problems with vague parameters could be dealt in a more efficient manner.

Optimal control theory has proved to be a tool of great importance for analyzing models related to diffusion of innovation. Since the time when basic model was given by Bass (1969) in the field of marketing, many modifications and variations have been proposed to meet specific requirements. These variations have made a considerable effect on various parameters in the diffusion process. Authors Robinson and Lakhani (1975) have studied the impact of price. On the other hand, authors Horsky and Simmon (1983) have analyzed the effect of advertising on the diffusion of a new product in the market. The application of optimal control theory in advertising models for markets that are partitioned have been carried out by (Little and Lodish, 1969; Grosset and Viscolani, 2005). Later Jha et al. (2009) optimized the advertising efforts for promotion of a new product in a segmented market. In their paper, the authors considered the sales outcome under two situations. In one case, the effect on sales outcome was considered when advertising was done separately and independently in each segment. In the other case, the effect of advertising on the entire population as a whole was considered. Significant research has been made in application of optimal control theory in product promotion strategies in both partitioned and consolidated markets for several variants such as price, sales outcome etc. but not in respect of other important variants such as “budget”. Various other innovation-diffusion models for dynamic and segmented markets have been given by (Huang et al., 2012; Helmes et al., 2013). In all these models the parameters that have been considered are crisp in nature. In most cases, the available resources are fixed. But in real world problems, this may not be the case. In reality it may so happen that due to abrupt fluctuations in demand or price of a product, there could be a sudden need to increase the number of resources.

To incorporate the imprecise behavior, many fuzzy optimal control theory models have been proposed in the past. The concept of fuzzy set theory is applied to such problems where the resource parameters are assumed to be some specific kind of fuzzy numbers. A fuzzy optimal control problem, which is non-linear in nature with fuzzy objective function, has been considered by Filev and Angelov (1992). Maity and Maiti (2005) have discussed production inventory models in which, the production and holding costs are not known precisely. The authors have used weighted average technique to convert a multi-objective program to a single objective problem. The products considered in this paper are deteriorating in nature and the optimal control problem is solved for values of demand, production and inventory levels. A fuzzy optimal control problem in the field of portfolio optimization has been considered by Zhu (2009), where the risk assets are assumed to give fuzzy returns. For a multilevel system with fuzziness at each stage, a model to solve optimal control problem is discussed by Zhu (2011). Ramezanzadeh and Heydari (2011) have considered solving an optimal control problem with fuzzy parameters. Use of Pontryagin principle and K-T conditions have been made to solve the resultant crisp problem and “Chance constrained programming” technique has been employed to obtain the corresponding crisp constraints. Roul et al. (2019) has studied control theory problems with control variable as the rate of production. The objective was to minimize the production cost with fuzzy cost constraints. In addition to that, Optimal Control Theory problem in which the fuzziness in parameters is expressed as fuzzy intervals has been studied by Campos et.al. (2020).

A study of literature also provides us with numerous production and inventory models, which are formulated under a fuzzy environment and analyzed using the concept of optimal control theory. Maity and Maiti (2007) have applied the theory of optimal control to solve a production inventory problem with multiple products. The production process has been assumed to result in some defective products and thereby creating shortages. The space and the amount invested has been assumed to be fuzzy in nature and the corresponding contract is firstly converted to crisp ones and the optimal control model obtained is solved using Pontryagin’s principle, K-T conditions and Generalized reduced gradient technique. In a paper given by Mandal et. al. (2010), the authors have presented a model in the area of production and inventory for defective items; and the theory of optimal control has been used to achieve the minimum cost for each of the ‘n’ items produced. The demand rate has been taken to be dependent on the level of stock. This model provides a production inventory scheme with time period, unit production and holding costs taken to be fuzzy in nature. It may be gleaned from the above papers that optimal control theory and fuzzy concepts have been merged in various studies relating to inventory and production. A multi-product optimal control problem, which considers the production and inventory models, has been considered in the paper given by Roul et.al. (2015). The constraint which limits the cost of production, has been taken to be both crisp and imprecise in nature. The demand is assumed to be dependent on time and the defectiveness in the production process is reliability and time dependent.

While assuming some variants as fuzzy in nature, there have been studies in respect of innovation-diffusion models as well. In their paper, Cheng et.al. (2009) have discussed an innovation-diffusion model for Information and Communication Technology products on the Internet. The authors have developed a trend-weighted time-series model which is fuzzy in nature to predict more efficiently the process of innovation and diffusion of such products. Similarly, innovation diffusion models for new products has been studied by Aggarwal et. al. (2012). The model in their paper is assumed to have a potential market of dynamic nature. The co-efficient of innovation and the initial potential market size is taken to be imprecise. Representing these parameters as certain fuzzy numbers, the authors aim to calculate the economic order quantity. Chanda and Kumar (2016) have further developed an economic order quantity model for dynamic potential markets. The initial price, advertising and market size are assumed to be fuzzy in nature and are represented by fuzzy numbers. Mehta et.al. (2020) have considered an innovation diffusion model in order to find optimal promotion efforts to be made when a new product is introduced in the market. The authors have considered the market size to be dynamic in nature along with an additional assumption that the market is segmented. Policies to maximize profits have also been formulated.

To the best of our knowledge, the models in literature have considered various situations but, budget has not been a variant and various other parameters are also considered as fixed while developing promotional strategies. Fuzziness has also been introduced in various studies relating to inventory and production and innovation diffusion. However, in this paper we have discussed a model in which the process of innovation diffusion of a new product is studied in an environment where promotional efforts are distributed over various market segments. Moreover, the total available budget for promotion is one of the variants and is additionally considered to be imprecise in nature, like in most of the real-world problems. The use of fuzzy set theory is made to incorporate such fuzzy parameters in the model and the parameter is represented by a triangular fuzzy number. The imprecise budget constraint assumed to be of possibility or necessity type, or both, is defuzzified using the techniques given by (Dubois and Prade, 1988,1997; and Liu and Iwamura, 1998). The crisp model is then solved by using Pontryagin's maximum principle to maximize profits.

In any industry, including that of software development, it is necessary to promote the products across the world. A very straightforward way to plan promotion is to allot a budget, identify regions/markets and execute the promotion plan while exhausting the financial support sanctioned in a budget. However, real-time situations are not straightforward and seldom demonstrate a linear relationship as illustrated in the aforesaid simple example. In real-time situations, the budget may not be fixed and might laid out as a broad range with some component as variable. Similarly, markets may not be homogenous and the promotion may have to be executed in a segment-wise manner. Thus, by mirroring real-life situations, what we formulate here is a method to increase efficiency and optimize budget utilization for both homogenous and segmented markets while considering the budget to be fuzzy.

Preliminaries

Definitions

Definition 1 *Fuzzy Set*: It is a set $\tilde{X}$ in the universe of discourse which is characterized by the membership function $\mu_{\tilde{X}} \rightarrow [0, 1]$

Definition 2 *Fuzzy Number:* A fuzzy set $\tilde{X}$ in R is said to be a fuzzy number if $\exists$ real numbers 'a' and 'b', a $\leq$ b such that the membership function for $\tilde{X}$ is given by:

$$\mu_{\tilde{X}}(x) = \begin{cases} l(x), x < a \\ 1, a \leq x \leq b \\ r(x), \ x > b \end{cases}$$

where $l(x): (-\infty, a) \rightarrow [0,1]$ is a function which is increasing, continuous from the right and satisfying l(x)=0 whenever $x < l_1$ for some $l_1 < a$;

and $r(x): (b, \infty) \rightarrow [0,1]$ is a function which is decreasing, continuous from the left and satisfying r(x) = 0 whenever $x > r_1$ for some $r_1 > b$.

The figure below illustrates the membership function for the fuzzy number 3.

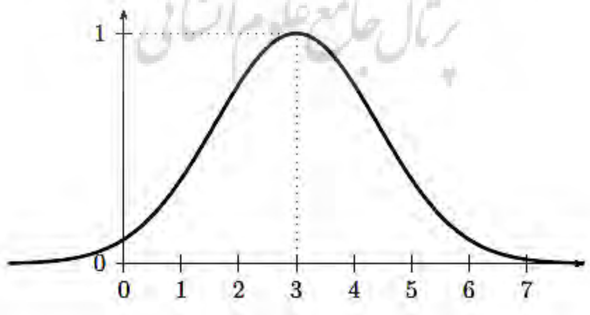

Figure 1: Membership function for the fuzzy number 3

Definition 3 *Triangular Fuzzy Number:* A fuzzy set $\tilde{X}$ in R is said to be a triangular fuzzy number if its membership function is given by:

$$\mu_{\tilde{X}}(x) = \begin{cases} 0, & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 < x \le a_3 \\ 0, & x > a_3 \end{cases}$$

The number $\tilde{X}$ is denoted by the triplet a_1, a_2, a_3) and it's membership function is in the form of a triangle, as depicted by the figure given below:

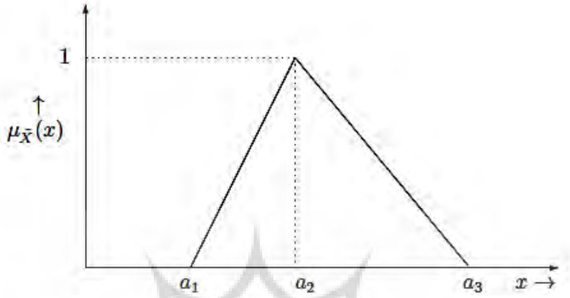

Figure 2. Membership function for the triangular fuzzy number

Operations on Fuzzy numbers

The operation * on fuzzy numbers $\tilde{X}$ and $\tilde{Y}$ in R gives a fuzzy set in R. It can be seen that $\tilde{X}$ *$\tilde{Y}$ thus obtained is a fuzzy number adeh((1555). Further using Zadeh's Extension principle, the fuzzy number $\tilde{X}$ $\tilde{Y}$ is given by the membership function:

$$\mu_{\tilde{X}*\tilde{Y}}(z) = \sup_{z=x*y;z\in R} \min\left(\mu_{\tilde{X}}(x), \mu_{\tilde{Y}}(y)\right)$$

More specifically, we have

$$\mu_{\tilde{X}(+)\tilde{Y}}(z) = \sup_{z=x+y} \min\left(\mu_{\tilde{X}}(x), \mu_{\tilde{Y}}(y)\right)$$

$$\mu_{\tilde{X}(-)\tilde{Y}}(z) = \sup_{z=x-y} \min\left(\mu_{\tilde{X}}(x), \mu_{\tilde{Y}}(y)\right)$$

$$\mu_{\tilde{X}(.)\tilde{Y}}(z) = \sup_{z=x.y} \min\left(\mu_{\tilde{X}}(x), \mu_{\tilde{Y}}(y)\right)$$

$$\mu_{\tilde{X}(\div)\tilde{Y}}(z) = \sup_{z=\frac{x}{y}} \min\left(\mu_{\tilde{X}}(x), \mu_{\tilde{Y}}(y)\right)$$

For triangular fuzzy numbers $\tilde{X}$ given by (a_1, a_2, a_3) and $\tilde{Y}$ given by (b_1, b_2, b_3), it can be seen that $\tilde{X}$ (+) $\tilde{Y}$, $\tilde{X}$ (-) $\tilde{Y}$, -$\tilde{X}$ and k$\tilde{X}$ are triangular fuzzy numbers with

$\tilde{X}$ (+) $\tilde{Y}$ given by ($a_1 + b_1$, $a_2 + b_2$, $a_3 + b_3$), -$\tilde{Y}$ given by (-b_3, -b_2, -b_1), k$\tilde{X}$ given by (ka_1, ka_2, ka_3), and $\tilde{X}$ (-) $\tilde{Y}$ given by ($a_1 - b_3$, $a_2 - b_2$, $a_3 - b_1$).

Possibility and Necessity on Fuzzy numbers

Let$\tilde{X}$X and $\tilde{Y}$ be two fuzzy numbers in R with membership functions $\mu_{\tilde{X}}(x)$ and $\mu_{\tilde{Y}}(y)$. As given by (Zadeh, 1999; Dubois and Prade, 1988,1997; Liu and Iwamura, 1998a, 1998b) the Possibility and Necessity of certain events in the fuzzy environment are given as follows:

Pos(X $\tilde{Y}$)= $sup\{min(\mu\ (x), \mu_{\tilde{Y}}(y)) : x, y \in R, x \quad y\}$

Pos($\tilde{}$ <<$\tilde{Y}$)= $sup\{min(\mu_{\tilde{X}}(x), \mu_{\tilde{Y}}(y)) : x, y \in R, x < y\}$

and Pos $\tilde{}$ ==$\tilde{Y}$)= $sup\{ min(\mu_{\tilde{X}}(x), \mu_{\tilde{Y}}(x)) : x \in R\}$

If $\tilde{X}$=c is crisp real number, then the above possibilities become:

Pos(c≤≤$\tilde{Y}$)= $sup\{min(\mu_{\tilde{Y}}(y)) : y \in R, c \leq y\}$

Pos(c < $\tilde{Y}$)= $sup\{min(\mu_{\tilde{Y}}(y)) : y \in R, c < y\}$

and Pos(c = $\tilde{Y}$)= $\mu_{\tilde{Y}}(c)$

In general, the Possibility of $\tilde{}$ $\tilde{Y}$ is given by:

Pos(($\tilde{X}$**$\tilde{Y}$) = $sup\{min(\mu_{\tilde{X}}(x), \mu_{\tilde{Y}}(y)) : x, y \in R, x * y\}$

where * represents any one of the relational operators <, ≤, =, >, ≥.

Further, the Necessity is defined as follows:

Nes(($\tilde{X}$**$\tilde{Y}$)= $inf\{max(\mu_{\tilde{X}}(x), \mu_{\tilde{Y}}(y)) : x, y \in R, x * y\}$

Considering the fact that necessary and possible events share a dual relationship, we get the following expression:

$$Nes(\tilde{X} * \tilde{Y}) = 1 - Pos(\overline{\tilde{X} * \tilde{Y}})$$

In particular for two triangular fuzzy numbers $\tilde{X}$ = (X_1, X_2, X_3) and $\tilde{Y}$ = (Y_1, Y_2, Y_3) the value of Pos(($\tilde{X}$≤≤$\tilde{Y}$) is given by the following cases:

Case 1: When $X_2 \quad Y_2$, Pos $\tilde{Y}$) =1

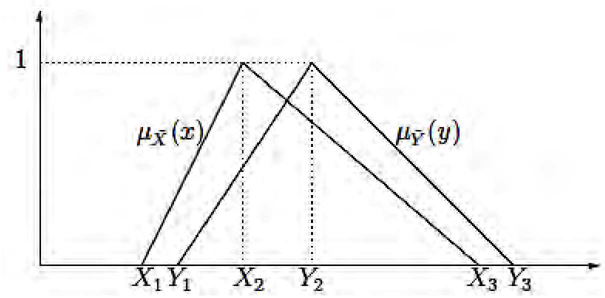

Figure 3. Membership function for two triangular fuzzy numbers with $X_2 \leq Y_2$

Case 2: When $X_2 > Y_2$ and $Y_3 > X_1$, Pos($\tilde{X} \leq \tilde{Y}$) = δ, where $\delta = \frac{Y_3 - X_1}{X_2 - X_1 + Y_3 - Y_2}$

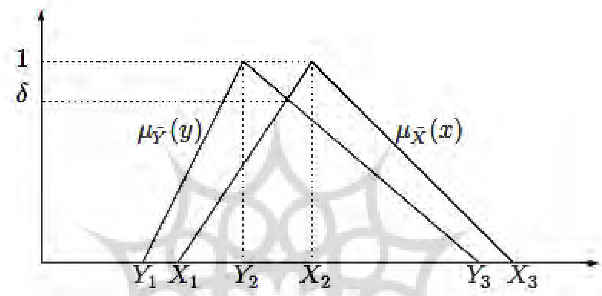

Figure 4. Membership function for two triangular fuzzy numbers with $X_2 > Y_2$ and $Y_3 > X_1$

Case 3: When $X_1 \geq Y_3$, Pos($\tilde{X} \leq \tilde{Y}$) = 0

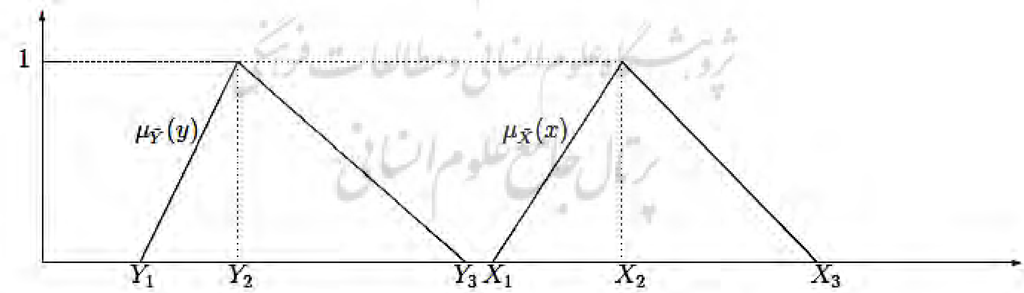

Figure 5: Membership function for two triangular fuzzy numbers with $X_1 \geq Y_3$

Hence Pos($\tilde{X} \leq \tilde{Y}$) can be given as:

$$\text{Pos}(\tilde{X} \leq \tilde{Y}) = \begin{cases} 1 & , X_2 \leq Y_2 \\ \delta = \frac{Y_3 - X_1}{X_2 - X_1 + Y_3 - Y_2} & , X_2 > Y_2 \text{ and } Y_3 > X_1 \\ 0 & , X_1 \geq Y_3 \end{cases} \quad (1)$$

Programming Problem in fuzzy environment

Let us consider the following programming problem with fuzzy parameters:

Maximize J

subject to $g \quad \tilde{Y}$ (FP) (1)

where J is the objective function and g is the constraint involving the fuzzy parameter $\tilde{Y}$.

To convert the constraints to their equivalent crisp versions, similar to the concepts given by Liu and Iwamura (1998a, 1998b), we can write the problem (FP) problem under necessity and possibility constraints as

Maximize J

subject to $Nes\{g \le \tilde{Y}\} \ge \eta_1$ and/or $Pos\{g \le \tilde{Y}\} \ge \eta_2$

where η_1 and η_2 are predetermined confidence levels for the fuzzy constraint.

Model Notations

To define the model, the following notations are adopted throughout this paper:

M: the total number of market segments and a discrete variable

$\overline{X}_i$: the total number of potential customers of the product in i^{th} segment

$x_i(t)$: the number of adopters for the i^{th} segment at time t

$u_i(t)$: the differentiated promotional effort rate for i^{th} segment at time t

$u(t)$: the mass market promotional effort rate at time t

α_i: the segment spectrum of mass promotion

p_i: the coefficients of external influence in i^{th} segment

q_i: the coefficients of internal influence in i^{th} segment

$\phi_i(u_i(t)) = \frac{\epsilon_i}{2} u_i^2(t)$: the differentiated market promotional effort cost

$\varphi(u(t)) = \frac{\epsilon}{2} u^2(t)$: the mass market promotional effort cost

P_i: the sales price per unit for i^{th} segment

C_i: the production cost for i^{th} segment

$\widetilde{W}$: fuzzy variable representing the total available budget

Model Development

In this section, we consider a monopolistic firm which produces a single product in market with M(>1) market segments. The firm simultaneously uses mass market promotion and differentiated market promotion to capture the potential market in each segment, respectively. The total available budget for promotion is taken to be imprecise. Under the influence of mass market and differentiated market promotion, the problem is formulated as follows:

$$Maximize\, J = \int_0^T \left(\sum_{i=1}^{M} \left[(P_i - C_i)\dot{x}_i(t) - \frac{\epsilon_i}{2} u_i^2(t) \right] - \frac{\epsilon}{2} u^2(t) \right) dt \tag{2}$$

subject to

$$\dot{x}_i(t) = \left(p_i + q_i \frac{x_i(t)}{\bar{X}_i} \right) \left(u_i(t) + \alpha_i u(t) \right) \left(\bar{X}_i - x_i(t) \right), \tag{3}$$

$$x_i(0) = x_{i0};\; u_i^l \le u_i(t) \le u_i^u \;\forall\, i = 1 \ldots M;\; u^l \le u(t) \le u^u \tag{4}$$

$$\int_0^T \left(\sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) \right) dt \le \widetilde{W} \tag{5}$$

Using properties of definite Integral, we get from equation (5)

$$\sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) \le \frac{\widetilde{W}}{T} \tag{6}$$

There are two different forms of the fuzzy constraint given by equation (6) depicting two different scenarios.

Scenario 1: $Nes\{\sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) < \frac{\widetilde{W}}{T}\} \ge \eta_1$

This can also be written as

$$Pos\left\{ \sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) \ge \frac{\widetilde{W}}{T} \right\} \le 1 - \eta_1 \tag{7}$$

Scenario 2:

$$Pos\left\{ \sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) \le \frac{\widetilde{W}}{T} \right\} \ge \eta_2 \tag{8}$$

Equivalent crisp representation of the proposed model

Let $\widetilde{W} = (W_1, W_2, W_3)$ be a triangular fuzzy number. Then $\frac{\widetilde{W}}{T} = (W_1/T, W_2/T, W_3/T)$ be given by the triangular fuzzy number $\left(W_1^{'}, W_2^{'}, W_3^{'}\right)$.

The problem represented by (2)-(6) reduces to the following:

$$Maximize\ J = \int_0^T \left(\sum_{i=1}^{M} \left[(P_i - C_i)\dot{x}_i(t) - \frac{\epsilon_i}{2} u_i^2(t) \right] - \frac{\epsilon}{2} u^2(t) \right) dt \tag{9}$$

subject to (3) and (4) for all scenarios and

for Scenario 1:

$$\frac{\sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) - W_1'}{W_2' - W_1'} \leq 1 - \eta_1 \tag{10}$$

for Scenario 2:

$$\frac{W_3' - \left(\sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) \right)}{W_3' - W_2'} \geq \eta_2 \tag{11}$$

Mathematical techniques to solve the above crisp model

For the problem (9)-(11), the corresponding Hamiltonian function is:

$$H = \sum_{i=1}^{M} \left[(P_i - C_i)\dot{x}_i(t) - \frac{\epsilon_i}{2} u_i^2(t) \right] - \frac{\epsilon}{2} u^2(t) + \sum_{i=1}^{M} \mu_i(t)\dot{x}_i(t) \tag{12}$$

The corresponding Lagrangian is

for Scenario 1

$$L = H + \lambda \left[(1 - \eta_1) W_2' + \eta_1 W_1' - \left(\sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) \right) \right] \tag{13}$$

and for Scenario 2

$$L = H + \lambda \left[(1 - \eta_2) W_3' + \eta_2 W_2' - \left(\sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) \right) \right] \tag{14}$$

In general for Scenario k=1,2, L may be given by

$$L = H + \lambda\left[(1-\eta_k)W'_{k+1} + \eta_k W'_k - \left(\sum_{i=1}^{M}\frac{\epsilon_i}{2}u_i^2(t) + \frac{\epsilon}{2}u^2(t)\right)\right] \tag{15}$$

K-T Conditions for the $k^{th}(k = 1,2)$ Scenario are

$$\lambda\left[(1-\eta_k)W'_{k+1} + \eta_k W'_k - \left(\sum_{i=1}^{M}\frac{\epsilon_i}{2}u_i^2(t) + \frac{\epsilon}{2}u^2(t)\right)\right] = 0 \tag{16}$$

The adjoint functions $\mu_i(t)$ are obtained by using the Maximum Principle as follows:

$$\frac{d}{dt}\mu_i(t) = -\frac{\partial L}{\partial x_i(t)} \tag{17}$$

$$\mu_i(T) = 0 \tag{18}$$

$$Now, \frac{\partial L}{\partial x_i(t)} = \left(P_i - C_i + \mu_i(t)\right)\frac{\partial \dot{x}_i(t)}{\partial x_i(t)} \tag{19}$$
$$= \left(P_i - C_i + \mu_i(t)\right)\left(u_i(t) + \alpha_i u(t)\right)\left(q_i - p_i - 2q_i\frac{x_i(t)}{\overline{X_i}}\right)$$

Hence from equation (17), $\mu_i(t)$ satisfies equation (18) and is given by

$$\frac{d}{dt}\mu_i(t) = -\left(P_i - C_i + \mu_i(t)\right)\left(u_i(t) + \alpha_i u(t)\right)\left(q_i - p_i - 2q_i\frac{x_i(t)}{\overline{X_i}}\right) \tag{20}$$

According to the Maximum principle, the Lagrangian is to be maximised with respect to the control variables $u_i(t)$, $\forall i = 1 \dots M$ & $u(t)$ at every instant of time. Using equation (15) we get:

$$\frac{\partial L}{\partial u_i(t)} = -\epsilon_i u_i(t) + \left(P_i - C_i + \mu_i(t)\right)\left(p_i + q_i\frac{x_i(t)}{\overline{X_i}}\right)\left(\overline{X_i} - x_i(t)\right) - \lambda\epsilon_i u_i(t) \tag{21}$$

and

$$\frac{\partial^2 L}{\partial u_i^2(t)} = -\epsilon_i(\lambda + 1) \tag{22}$$

Letting $A = \frac{\left(P_i - C_i + \mu_i(t)\right)\left(p_i + q_i\frac{x_i(t)}{\overline{X_i}}\right)\left(\overline{X_i} - x_i(t)\right)}{\epsilon_i(\lambda+1)}$, we can infer the following three cases:

Case 1: If $\frac{\partial L}{\partial u_i(t)} > 0$ then the Lagrangian L is an increasing function of $u_i(t)$ and $u_i(t) = minimum\{A, u_i^u\}$

Case 2: If $\frac{\partial L}{\partial u_i(t)} = 0$ then L is maximised for the value of $u_i(t)$ as $u_i^*(t) = A\ (for\ \lambda \geq 0)$

Case 3: If $\frac{\partial L}{\partial u_i(t)} < 0$ then L is a decreasing function of $u_i(t)$ and $u_i(t) = maximum\{A, u_i^l\}$

Similarly using equation (15) we get:

$$\frac{\partial L}{\partial u(t)} = -\epsilon u(t) + \sum_{i=1}^{M} \alpha_i(P_i - C_i + \mu_i(t))\left(p_i + q_i \frac{x_i(t)}{\bar{X}_i}\right)(\bar{X}_i - x_i(t)) - \lambda\epsilon u(t) \tag{23}$$

and

$$\frac{\partial^2 L}{\partial u^2(t)} = -\epsilon(\lambda + 1) \tag{24}$$

Letting B = $\frac{\sum_{i=1}^{M} \alpha_i(P_i - C_i + \mu_i(t))\left(p_i + q_i \frac{x_i(t)}{\bar{X}_i}\right)(\bar{X}_i - x_i(t))}{\epsilon(\lambda+1)}$, we can infer the following three cases:

Case a: If $\frac{\partial L}{\partial u(t)} > 0$ then the Lagrangian L is an increasing function of $u(t)$ and $u(t) = minimum\{B, u^u\}$

Case b: If $\frac{\partial L}{\partial u(t)} = 0$ then L is maximised for the value of $u(t)$ as $u^*(t) = B$ (for $\lambda \geq 0$)

Case c: If $\frac{\partial L}{\partial u(t)} < 0$ then L is a decreasing function of $u(t)$ and $u(t) = maximum\{B, u^l\}$

Numerical Illustration

A numerical example is presented here to demonstrate the usage of the above proposed model. The equivalent crisp control theory problem given by Equations (9) - (11) is transformed to a discrete form by using the technique proposed by Rosen (1968). The following discrete formulation of the problem is considered for its numerical application:

$$Maximize\ J = \sum_{k=1}^{T}\left(\sum_{i=1}^{M}\left[(P_i - C_i)(x_i(k+1) - x_i(k)) - \frac{\epsilon_i}{2}u_i^2(k)\right] - \frac{\epsilon}{2}u^2(k)\right) \tag{25}$$

subject to

$$x_i(k+1) = x_i(k) + \left(p_i + q_i \frac{x_i(k)}{\bar{X}_i}\right)\left(u_i(k) + \alpha_i u(k)\right)\left(\bar{X}_i - x_i(k)\right), \tag{26}$$

$$x_i(0) = x_{i0};\ u_i^l \leq u_i(k) \leq u_i^u\ ;\ u^l \leq u(t) \leq u^u;\ \forall\, i = 1 \dots M, \forall\, k = 1 \dots T \tag{27}$$

for all scenarios and

for Scenario 1:

$$\frac{\sum_{i=1}^{M}\frac{\epsilon_i}{2}u_i^2(k)+\frac{\epsilon}{2}u^2(k)-W_1'}{W_2'-W_1'}\leq 1-\eta_1\ \forall\ k=1\ldots T \tag{28}$$

for Scenario 2:

$$\frac{W_3'-\left(\sum_{i=1}^{M}\frac{\epsilon_i}{2}u_i^2(k)+\frac{\epsilon}{2}u^2(k)\right)}{W_3'-W_2'}\geq \eta_2\forall\ k=1\ldots T \tag{29}$$

The above model is solved by using Lingo11. The value of M which denotes the number of segments into which the population is divided is taken as 3. The entire time interval is assumed to be divide into 23 time periods of equal duration. The values taken for the different parameters used is tabulated below:

Table 1: Values of Parameters

Segments	$\overline{X_i}$	p_i	q_i	α_i	P_i	C_i	ϵ_i	Initial Sales
1	169107	0.00166	0.4176	0.32	4000	3400	227500	11760
2	152460	0.004161	0.480575	0.32	4400	3700	227500	8231
3	157581	0.00138	0.540395	0.32	4200	3400	227500	8260

The value of ϵ is taken as 500000 and T is taken as 23. The total available budget which is assumed to be fuzzy in nature, is represented by the triangular fuzzy number and the value of (W_1', W_2', W_3') is taken as (775000, 870000, 975000).

- ***Scenario 1***

Using the value of η_1 as 0.05, the values of optimal sales obtained in each market segment are given as follows:

Table 2: The Optimal Sales in segmented market

Time	T1	T2	T3	T4	T5	T6	T7	T8
$x_1(t)$	11760	14988.11	18985.32	23894.85	29863.88	37029.36	45496.45	55308.91
$x_2(t)$	8231	9958.304	11977.24	14328.98	17057.46	20208.21	23826.77	27956.29
$x_3(t)$	8260	10663.04	13692.26	17485.52	22195.44	27980.94	34991.66	43343.14
Time	T9	T10	T11	T12	T13	T14	T15	T16
$x_1(t)$	66413.62	78625.46	91603.74	104855.3	117777.6	129744.3	140217.1	148848.3
$x_2(t)$	32634.39	37889.13	43734.28	50164	57147.43	64623.86	72499.34	80646.03
$x_3(t)$	53082.06	64143.91	76311.05	89186.25	102201.7	114680.2	125948.8	135476.8
Time	T17	T18	T19	T20	T21	T22	T23	
$x_1(t)$	155535.6	160408.3	163756	165935	167288.7	168118.1	168577.1	
$x_2(t)$	88905.23	97095.36	105025.1	112511.1	119402.5	125699.1	131138.2	
$x_3(t)$	142987	148493	152248.3	154638.2	156072.6	156972.9	157338.4	

Table 3. Optimal promotional efforts in the segmented market

Time	T1	T2	T3	T4	T5	T6	T7	T8
$u_1(t)$	0.2053868	0.2059914	0.206755	0.2077188	0.2089337	0.2104622	0.2123804	0.2147789
$u_2(t)$	0.602532	0.604741	0.607542	0.61109	0.615573	0.62122	0.628306	0.637152
$u_3(t)$	0.332071	0.332603	0.33323	0.333968	0.334835	0.335852	0.337042	0.338431
$u(t)$	0.4760204	0.4774322	0.4792483	0.4815818	0.484576	0.4884107	0.4933093	0.4995469
Time	T9	T10	T11	T12	T13	T14	T15	T16
$u_1(t)$	0.2177633	0.2214527	0.2259741	0.2314525	0.2379929	0.2456567	0.2544322	0.2642119
$u_2(t)$	0.648125	0.661619	0.678031	0.697708	0.720872	0.747517	0.777299	0.80946
$u_3(t)$	0.340048	0.341924	0.344095	0.346596	0.349464	0.352734	0.356435	0.360585
$u(t)$	0.5074549	0.5174233	0.5298943	0.5453416	0.5642305	0.586952	0.6137395	0.6445974
Time	T17	T18	T19	T20	T21	T22	T23	
$u_1(t)$	0.2747952	0.2859491	0.2976174	0.3110294	0.350836	0.3553921	0.4066611	
$u_2(t)$	0.842832	0.875994	0.907651	0.937739	0.987423	1	0.0574	
$u_3(t)$	0.365181	0.370192	0.375559	0.381239	0.388278	0.393908	2.690314	
$u(t)$	0.6793172	0.71775	0.760866	0.8172122	1	1	0.0453	

The optimal value of total profit for Scenario 1 is 313094100 units.

Scenario 2

Taking the value of η_2 as 0.5, the values of optimal sales obtained in each market segment are given as follows:

Table 4: The Optimal Sales in segmented market

Time	T1	T2	T3	T4	T5	T6	T7	T8
$x_1(t)$	11760	14900.93	18770.19	23500.05	29226.58	36077.5	44153.6	53503.38
$x_2(t)$	8231	10746.23	13878.15	17751.07	22498.71	28255.32	35140.65	43237.72
$x_3(t)$	8260	10586.29	13499.6	17125.33	21602.14	27074.91	33681.96	41534.6
Time	T9	T10	T11	T12	T13	T14	T15	T16
$x_1(t)$	64092.37	75771.81	88255.71	101118.3	113823.6	125792.7	136497.1	145554.3
$x_2(t)$	52563.29	63034.17	74436.39	86409.23	98458.52	110009.8	120498.9	129477.8
$x_3(t)$	50688.45	61107.86	72629.07	84933.28	97545.12	109871.8	121287.8	131250.1
Time	T17	T18	T19	T20	T21	T22	T23	
$x_1(t)$	152790.6	158250	162147.3	164790.3	166503	167570.8	168246	
$x_2(t)$	136699.9	142151.1	146015.1	148595.2	150226.1	151208.3	151817	
$x_3(t)$	139406.5	145656.1	150136.1	153146.5	155050.7	156191	157035.7	

Table 5. The Optimal promotional efforts in the segmented market

Time	T1	T2	T3	T4	T5	T6	T7	T8
$u_1(t)$	0.5785115	0.5805325	0.58308	0.5862854	0.590309	0.5953445	0.6016221	0.6094091
$u_2(t)$	0.5075558	0.5091364	0.5111462	0.5136978	0.5169315	0.5210203	0.5261742	0.5326452
$u_3(t)$	0.452744	0.4540116	0.4556297	0.4576926	0.4603189	0.4636561	0.4678857	0.4732289
$u(t)$	0.2240509	0.2247599	0.225659	0.2267976	0.2282367	0.230051	0.2323313	0.2351852
Time	T9	T10	T11	T12	T13	T14	T15	T16
$u_1(t)$	0.619006	0.6307354	0.6449175	0.6618295	0.6816434	0.7043433	0.7296338	0.7568635
$u_2(t)$	0.5407277	0.5507549	0.5630852	0.5780737	0.5960236	0.6171125	0.6412976	0.6682171
$u_3(t)$	0.4799498	0.4883555	0.4987901	0.5116162	0.5271801	0.5457532	0.5674492	0.5921263
$u(t)$	0.2387379	0.2431295	0.248509	0.2550212	0.2627857	0.2718656	0.2822282	0.2937053
Time	T17	T18	T19	T20	T21	T22	T23	
$u_1(t)$	0.7850061	0.8127405	0.8386484	0.8614937	0.8804888	0.9255937	0.9477261	
$u_2(t)$	0.6971222	0.7268866	0.7561292	0.7834413	0.8076557	0.8796074	0.9552448	
$u_3(t)$	0.6193001	0.648115	0.6774158	0.7059356	0.7325522	1	1	
$u(t)$	0.305968	0.3185352	0.3308314	0.3422867	0.3524534	0.4112431	0.4278221	

The optimal value of total profit for Scenario 2 is 310339800 units.

Discussion

Optimal control theoretic models are of immense importance to the software engineers as well as marketing managers as they deal with many sophisticated decision-making activities. The goal of all optimization problems is to either maximize benefits and/or minimize efforts. The resource and other constraints used in optimal control problems are assumed to be deterministic or stochastic in nature, which is not true in general. In most real-life problems, it is observed that some of the model constants can only be computed roughly due to their dependence on various non-deterministic factors. Hence the optimal solution of the problem so obtained, is not actually representative of the complete system information and can provide us with the incorrect picture of the system. In such cases defining the problem under fuzzy environment offer the opportunity to model subjective imaginations of the decision maker as precisely as a decision maker will be able to describe it. Optimal control theory approach under fuzzy environment is gaining even more importance in the era of globalization due to some additional sources contributing to bringing uncertainties in the problem definitions.

To summarize the results here, a total expenditure on promotion in the first scenario is 3913472.174 units whereas in the second scenario it is 3821125.088 units. Thus, we observe that a higher level of expenditure on promotion results in increase in the profit for a fuzzy (variable) budget although the relationship of increase in expenditure on promotion and corresponding profit is not linear. t is also seen that the "necessity" constraint requires greater efforts for promotion than "possibility" constraints. urther, we note that increased promotional effort rates over a period of time results in an increase in sale of a product.

Figure 6 and Figure 7 depict the graphs of optimal sales $(x_i^*(t))$ from the potential market in each segment. The graphs of optimal values of promotional effort rate $((u^*(t), u_i^*(t))$ for both mass and differentiated promotions are shown in Figure 8 and Figure 9 in both scenarios.

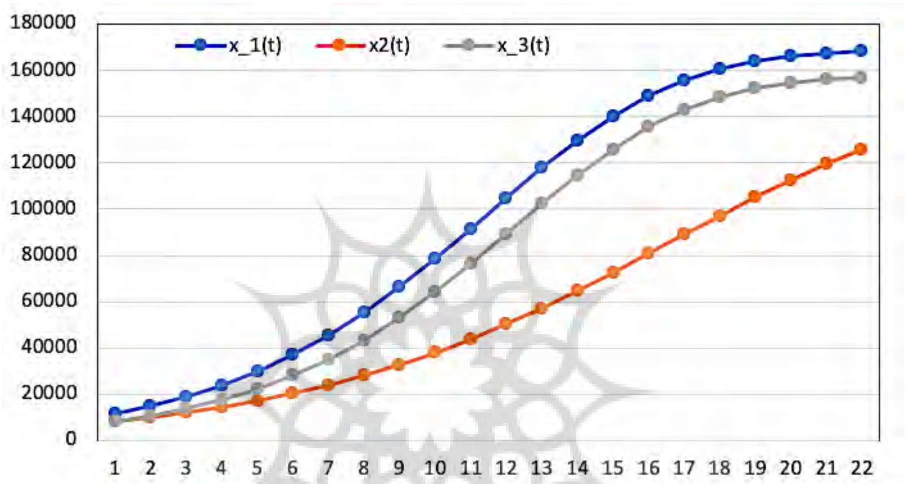

Figure 6. Optimal values of sales in first scenario

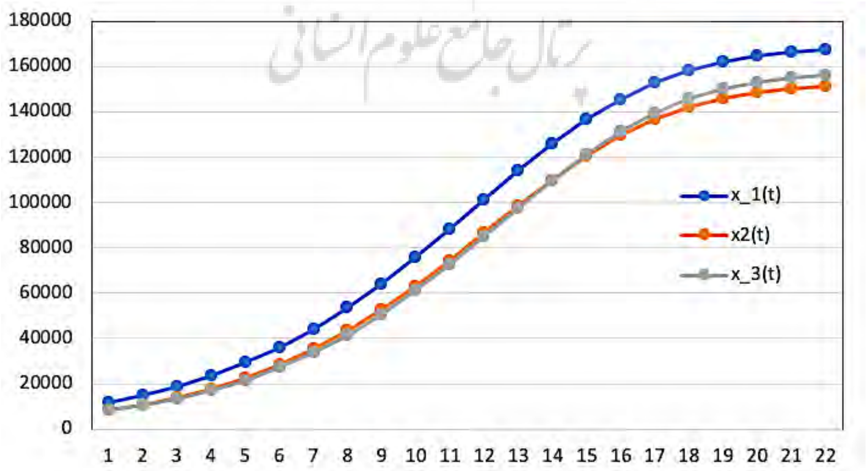

Figure 7. Optimal values of sales in second scenario

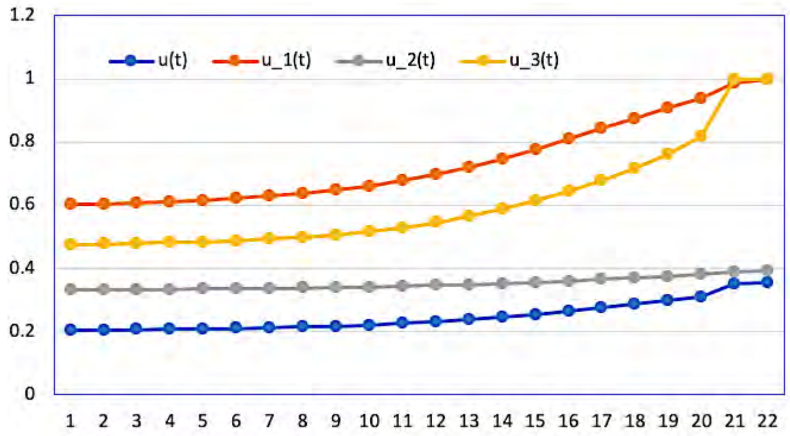

Figure 8. Optimal promotional effort rates in first scenario

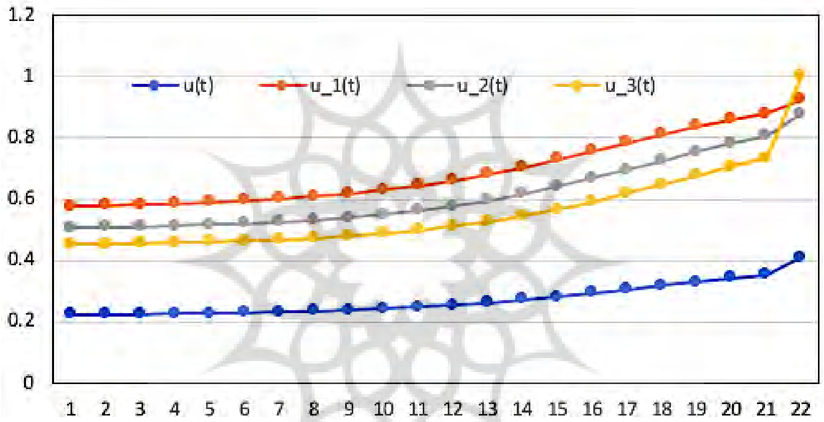

Figure 9. Optimal promotional effort rates in second scenario

From Figure 6 and Figure 7, it can be concluded that sales $(x_i^*(t))$ starts from initial value and then increase as time increases and follows a s-shaped curve which is a common sales pattern in innovation-diffusion models. Figure 8 and Figure 9 display that initially optimal promotional effort rates are increasing over time. This result shows that promotional efforts are used to generate awareness and sales of the product in planning period.

Conclusion and Scope for Future Research

With fast-paced technical developments, we witness more and more new products being produced or earlier products upgraded. However, with the same pace corresponding set of substitutes also catch-up with these changes. Under these circumstances, a firm has to put in more efforts to popularize its goods. Here we have selected constraints based on real-life situation and use optimal control theory to determine how a company can take decisions regarding allocation of funds for the purpose of promoting its product. The market has been divided into groups of customers having similar preferences, so that promotion can be done in an efficient manner. In addition, the company can also keep aside some funds for promotion of its product to the entire population as a whole. To handle the fluctuation in funds, the budget available is taken as a triangular fuzzy number. These constraints assimilate real-life situations faced by any industry. The fuzzy programming problem is first converted to a crisp one by the use of Necessity and Possibility constraints and the model is further discretized to analyze a given numerical example. The use of Lingo11 software has been made to achieve the results. The results show optimal use of resources and determine maximum profit that can be obtained in real-time for a variable budget to promote a product in both homogenous and segment-wise market. Similar models can be studied in future where the firm has more than one product to promote and the fuzzy parameter denoting the budget is a general fuzzy number.

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