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Identifiability of Dynamic Stochastic General Equilibrium Models with Covariance Restrictions

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In this paper we study the identification problem of parameters of Dynamic Stochastic General Equilibrium Models with emphasis on structural constraints, in order to make the number of observable variables is equal to the number of exogenous variables. We derive a set of identifiability conditions and suggest a procedure for a thorough analysis of identification at each point in the parameters space. The procedure can be applied, before DSGE models are estimated, to determine where identification fails. We also use a Monte Carlo simulation and study the effect of restrictions on the estimate. The results show that the use of restrictions for estimation, when identification is reduced, leads us to inaccurate estimates and unreliable inference even when the number of observations is large.

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JEL Classification: B41, C13, C15

1 Introduction

Dynamic Stochastic General Equilibrium (DSGE) models use modern macroeconomic theory to explain and predict co-movements of aggregate time series over the business cycle. The term DSGE model encloses a broad class of macroeconomic models that spans the standard neoclassical growth model discussed in King et al, (1988) as well as New Keynesian monetary models with numerous real and nominal frictions argued in Christiano et al, (2005) and Smets and Wouters (2003). A common feature of these models is that decision rules of economic agents are derived from assumptions about preferences, technologies, information, and the prevailing fiscal and monetary policy regime by solving inter-temporal optimization problems. In a nutshell, the DSGE model paradigm delivers empirical models with a strong degree of

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theoretical coherence that is attractive for policy experiments. Modern DSGE models are flexible enough to accurately track and forecast macroeconomic time series. They have become one of the most widely used tools for monetary policy analysis in central banks.

Despite the development of DSGE models, a problem that is rarely addressed in the studies is parameter identifiability. Identification is a prerequisite for estimation of the parameters, and the ability to do it for full-edged structural models is believed to be one of the main achievement of this line of research. Parameter identification is a potentially serious issue for DSGE models and is not a new concern. Among the authors who bring up this point are Sargent (1976) and Pesaran (1989). More recently Beyer and Farmer (2004) provide several examples of commonly used models that are unidentifiable. They argue that the problem is likely to be common in DSGE models.

In most empirical DSGE papers the question of parameter identification is not confronted directly. Usually, if some of the parameters are considered to be of lesser interest, and/or with potentially problematic identifiability, their values are calibrated and assumed known, instead of being estimated. Furthermore, since DSGE models are frequently estimated using Bayesian methods, potential identification problems remain hidden due to the use of priors. As a result, it is often unclear to what extent the reported estimates reflect information in the data instead of subjective beliefs or other considerations reflected in the choice of prior distribution of the parameters. One reason why this is an important issue is that DSGE models are increasingly being used for analyzing policy-relevant questions, for instance, the design of optimal monetary policy. Such analysis often hinges crucially on the values assigned to the parameters of the model. It is, therefore, important to know how informative the data is for the parameters of interest, and whether there are any benefits from estimating instead of calibrating the models.

In this paper, we present an efficient approach for determining identifiability of the parameters of linearized DSGE models, by using restrictions on the reduced form. In particular, we present necessary and sufficient conditions for local identification of deep parameters and discuss when its global identifiability can be ascertained. The conditions for partial identification are also examined. That is, when only a subset of all parameters is of interest, or some of the parameters are unidentifiable. Second, we utilize the maximum likelihood estimator of the parameters in DSGE models and

compare its performance to different degrees of identifiability by using Monte Carlo simulations.

The early literature on identification of rational expectations (RE) models started with the works by Sargent (1976) and McCallum (1979) on observational equivalence and was extended to more general setups by Wallis (1980), Pesaran (1981), and Pudney (1982). Interest in the identification of RE models then waned as models tended to be calibrated rather than estimated. But recent interest in estimation of DSGE models has prompted a return to the problem of identification of RE models in general and that of DSGE models in particular. The issue on the identification of the new Keynesian Phillips curve is discussed by Mavroeidis (2005), Nason and Smith (2008), Kleibergen and Mavroeidis (2009), and Dees et al, (2009), and of the Taylor rule by Cochrane (2011). More generally, Canova and Sala (2009) conclud "it appears that large class of popular DSGE structures are only very weakly identified;" and Iskrev (2010) conclude, "The results indicate that the parameters of the Smets and Wouters (2007) model is quite poorly identified in most of the parameter space." Other recent papers that consider determining the identification of DSGE models are those by Andrle (2010), Iskrev and Ratto (2010), Komunjer and Ng (2011).

Unlike all of the above papers, where only identification of the structural parameters is considered, we present conditions for identifiability for both the structural coefficients and the underlying deep parameters, which is robust from observations and is typically what DSGE modelers are interested in estimating. Moreover, we present identification conditions for models with future expectations and covariance restrictions, which is also in line with the models in the current DSGE literature.

This article is organized as follows. In section 2 we introduce the general model and notation. We also discuss parameter identification in general terms and introduce a result from Rothenberg (1971) which will serve as a basis for our approach for identification of DSGE models. In Section 3 we present conditions for identifiability of the parameters in DSGE models. We follow a two-step approach in which the identification of the deep parameters is conditioned first on the identifiability of the structural parameters, and then on the uniqueness of the mapping from structural parameters to it. We also make a distinction between identification only based on the cross-equation restrictions and identification when covariance restrictions are used too. Conditions for identification are given for the complete system of equations. Finally, we discuss how identification failures can be detected in practice, and describe an algorithm for a thorough identification analysis of DSGE models.

In section 4 we investigate the effect of different degree of identification on the maximum likelihood estimator (ML) of the parameters in DSGE models through the Monte Carlo simulation study. Section 5 concludes.

2 Linearized Models of DSGE

A conventional DSGE model is summarized by a system of non-linear equations. However, most analyses including either simulation or estimation of DSGE models use linear approximations of the original models. Especially, the model is first expressed in terms of stationary variables, and then linearized or log-linearized around the steady-state values of these variables. In general, a linearized model of DSGE can be expressed in the following form:

$$\Gamma_0(\alpha)X_t = \Gamma_1(\alpha)E_tX_{t+1} + \Gamma_2(\alpha)X_{t-1} + \Gamma_3(\alpha)U_t \quad (1)$$

where X_t is the m −dimensional vector of endogenous variables, and the structural errors, U_t, are independent and identically distributed n-dimensional random vectors with $E[U_t] = 0, E[U_t\acute{U}_t] = I$. The elements of the matrices $\Gamma_0, \Gamma_1, \Gamma_2$ and Γ_3 are functions of the $k \times 1$ vector of deep parameters α.

Here, we define the structural parameters as unknown components of the coefficient matrices $\Gamma_0, \Gamma_1, \Gamma_2$ and Γ_3. More precisely, we show the unknown elements of $\Gamma_0, \Gamma_1, \Gamma_2$ with γ_1 and the unknown components of Γ_3 with γ_2. Also, let $\gamma = [\acute{\gamma_1}, \acute{\gamma_2}]$ be a vector of all structural parameters, so that the mapping from α to γ is given by the function $\tilde{g}$, i.e.

$$\gamma = \tilde{g}(\alpha) \quad (2)$$

To generalize the results, as with most models, the structural shocks are considered independent of each other. Therefore, Γ_3 is a diagonal matrix with the standard deviations of the shocks in the diagonal. Therefor there is no distinction between the deep parameters that enter in Γ_3 and the structural parameters in γ_2. Hence we can write $\alpha = [\acute{\alpha_1}, \acute{\alpha_2}]$, where $\alpha_2 = \gamma_2$. This implies that the function $\tilde{g}$ can be written as $\tilde{g} = [g, I_n]'$, where g maps α_1 into γ_1 and I_n is the n-dimensional identity matrix.

In the literature, various approaches are proposed for solving linear rational expectations models like (1) (see for instance Blanchard and Kahn (1980), Anderson and Moore (1985), Klein (2000) and Sims (2002)). Solving the equation (1) for α there may exist zero, one, or many stable solutions.

Assuming that a unique solution exists, it can be cast in the following form

$$X_t = AX_{t-1} + BU_t \quad (3)$$

Where the $m \times m$ matrix A and the $m \times n$ matrix B are functions of α, and are unique for each value of α. I let $\Lambda = BB'$ and define $\tau_1 = vec(A)$ and $\tau_2 = vech(\Lambda)$ Lastly, let $\tau = [\tau_1', \tau_2']'$ be the vector of all reduced-form parameters. Throughout the paper we assume that the vector of reduced-form parameters τ is unique for each admissible value of α and the structural shocks $U_t \sim N(0, \Sigma)$ where $\Sigma = \Gamma_3\Gamma_3'$, for $t = 1,2, \dots, T$.

2.1 Definition of Parametric Identification

Let Z be a vector-valued random variable in R^m representing the outcome of some random experiments. A structure S is a complete specification of the probability distribution function of Z. The set of all a priori possible structures, M, is called a model. In most applications, Z is assumed to be generated by a parametric probability distribution function $F(z, \alpha)$, where parametric probability distribution function F is assumed known, but the $k \times 1$ parameter α is unknown. Hence, a structure is described by a parametric point α and a model is a set of points $A \subseteq R^k$. In this framework we have the following definitions.

Definition 1. Two parameter points (structures) $\alpha^1$and α^2 are said to be observationally equivalent if $F(z, \alpha^1) = F(z, \alpha^2)$, for (almost) all possible Z.

Definition 2. A parameter point α^0 in A is said to be globally identifiable if there is no other α in A which is observationally equivalent.

Sometimes a weaker concept of identifiability is useful.

Definition 3. A parameter point α^0 is said to be locally identifiable if there exists an open neighborhood of α^0 containing no other α in A which is observationally equivalent.

The above definitions imply that in principle a structure must include all parameters that completely characterize the distribution of the data. Unless we do so, we can never conclude that a model is not identified. In what follows we are going to use only parameters that specify the first two moments of the data. Thus the identification conditions that we find are in general only sufficient, and not necessary. However, if it is assumed that the data is generated from a normal distribution, then the conditions for identifiability become necessary and sufficient. The reason is that a normal distribution is completely described by its first two moments and it is very common in the literature on estimation of DSGE models, most of which follows the likelihood approach.

2.2 Criteria for Identification

Rothenberg (1971) advances a general condition for identification of parametric models. The condition is that the information matrix must be non-singular at the true value of the parameters. Such achievements can be applied for detecting identification problems for other extremum estimators as well and not only likelihood-based methods. The general criterion for identifiability is that the Hessian of the objective function is of full rank. Due to its generality, however, the information matrix approach has some important restitution. First, in many situations, and in particular in the case of DSGE models, the Hessian is very difficult, and often impossible to obtain in analytical form. As a result, researchers have to use numerical methods to compute second derivatives, which inevitably lead to inaccuracy in the results. Moreover, this approach is in most cases limited to local identification only and therefore does not allow researchers to determine whether their model is globally or only locally identified. And finally, due to the intricacies involved in obtaining the Hessian, it is usually hard to pinpoint the source of identification problems in the underlying model.

Another, less general result in Rothenberg provides the basis for an alternative method for determining identification in parametric models, without involving the information matrix. The approach is based on the relationship between the parameters of interest, and characteristics of the probability distribution of the data. It boils down to a question of uniqueness of a solution to a system of equations. The well-known rank and order conditions for identification of systems of linear simultaneous equations are an example of this approach. A very useful feature of this approach, when applied to DSGE models, is that it allows for a more transparent and intuitive interpretation of the necessary identification condition, and makes it easier for researchers to locate sources of potential identification problems. Moreover, this identification framework extends, in a natural way, into an estimation procedure that could potentially be useful for empirical validation of DSGE models. We generalize this result to account for situations where the mapping between the parameters in α and those in τ is given by an implicit function, such as

$$f(\alpha, \tau) = 0 \tag{4}$$

Now, if we show the Jacobian of f respect to α with f_α, the above discussion can be generalized as follows:

Proposition. If the data density depends on the parameter vector α through the reduced-form parameter τ while τ is globally identifiable. Assume also that the mapping from α to τ be defined implicitly as in (4). Then,

(P1) If the Jacobian f_α, appraised at α^0 has a full column rank, then α^0 is locally identifiable.

(P2) If the density of Z depends on α only through τ, and α^0 is a regular point of f_α, then full rank matrix f_α is a necessary condition for identification of α^0

(P3) If f_α has a full column rank for all values of α then α^0 is globally identifiable.

The first proposition (P1) tells us that the existence of a locally unique mapping from α to τ is a sufficient condition for the identification of α When the data density depends on α only through τ the latter is also a necessary condition, as implied by the second result (P2). Otherwise it is not. We will make use of this distinction in the following way. A density function is completely determined by its moments. If τ contains the parameters necessary to characterize all moments of the data, then we are in a situation when (P2) is applicable, and the rank condition is both necessary and sufficient. However, we often cannot use all moments, either because some of the higher moments are difficult to estimate accurately, or because we do not want to assume a particular density function, and, therefore, do not know what those moments are. If that is the case, we can still use the fact that, if some of the moments we do use imply a unique value of α then this is sufficient for identification.

The proposition (P3) is useful when the mapping from the reduced form parameters to the parameters of interest α is linear. So, if the rank condition holds, the solution for α will be globally unique. This result is analogous to Theorem 5 in Rothenberg (1971).

Presenting these results for an implicit instead of an explicit function makes them convenient for studying the identification of the parameters in DSGE models.

3 Identification in DSGE Models

Examining articles related to DSGE models, especially internal articles, clearly shows that the problem of parameter identification has been largely neglected in the current empirical DSGE literature. There are two main reasons for that. First, because the mapping from structural to reduced form parameters is extremely complicated, and, except for very special cases, not

available in an analytical form, using the information matrix criterion for identification is very difficult. Lubik and Schorfheide (2004) argue that "It is difficult to directly detect identification problems in large DSGE models, since the mapping from the vector of structural parameters - into the state-space representation that determines the joint probability distribution of Z is highly nonlinear and typically can only be evaluated numerically". Computing the Information matrix requires the second derivative of the log-likelihood function which cannot be obtained directly if the structural function mapping into reduced-form parameters is not available. The second reason is that the large majority of papers in this literature apply the Bayesian approach to the estimation of DSGE models. Identification problems can be alleviated with the use of priors for the structural parameters. Intuitively, a skillfully selected prior can make a parameter identifiable by placing a low probability on values that would render it unidentifiable.

In this section, we show that identification of DSGE models can be studied directly, as a problem of uniqueness of a solution to a system of equations, instead of trying to do that through the Information matrix. We start by deriving the cross-equation and covariance restrictions that relate the parameters of interest to those of the reduced form model.

3.1 Conditions for Identification of γ

Initially, we provide requirements for identification of the structural parameters. It is important for two reasons: First, the identifiability conditions for γ we conclude here will be useful in the next section, where the identification of the deep parameters is studied. And second, the identification and estimation of the structural parameters may be special interest for researchers. If for instance, the goal is to study the economy's response to different shocks, the structural model is all we need. Furthermore, note that with $\Gamma_1 = 0$ the model in (1) reduces to the canonical structural vector autoregressive model (SVAR) (Revenna, 2007), which is widely used in the empirical macroeconomic literature. Thus, the results regarding the identification of γ are directly applicable to identification in SVAR models.

To find conditions for identification, we use Equation (3). It implies that the expectation of X_{t+1} is$E_t X_{t+1} = AX_t$. Using this in Equation (1) yields

$$(\Gamma_0 - \Gamma_1 A)X_t = \Gamma_2 X_{t-1} + \Gamma_3 U_t \quad (5)$$

Comparing it with the one in Equation (3), we find the following two sets of restrictions

$$(\Gamma_0 - \Gamma_1 A)A - \Gamma_2 = 0, \quad (6)$$
$$(\Gamma_0 - \Gamma_1 A)\Lambda(\Gamma_0 - \Gamma_1 A)' - \Sigma = 0. \quad (7)$$

Vectorising both sides of the Equation (6), we have

$$(A' \otimes I_m)vec(\Gamma_0) - (A'^2 \otimes I_m)vec(\Gamma_1) - vec(\Gamma_2) = 0_m, \quad (8)$$

Where I_m and 0_m are $m \times m$ identity matrix and a zero matrix, respectively. Each of the three Γ matrices contains zeros, other known constants, and elements of γ. Let $\tilde{\gamma} = [1, \gamma_1']'$. Then, for $i = 0,1,2$ we can write $vec(\Gamma_i) = G_i\tilde{\gamma}$, where G_i is a $m^2 \times (l+1)$ matrix containing only zeros and ones.

Thus Equation (8) becomes

$$((A' \otimes I_m)G_0 - (A'^2 \otimes I_m)G_1 - G_2)\tilde{\gamma} = 0_m, \quad (9)$$

Where $\tilde{\gamma} = \tilde{\gamma}(\alpha)$. Denoting the first column of $((A' \otimes I_m)G_0 - (A'^2 \otimes I_m)G_1 - G_2)$ with $-\Psi_1$ and the remaining l columns with Ψ_2. Equation (9) becomes:

$$\Psi_2\gamma_1 = \Psi_1, \quad (10)$$

Where the matrices Ψ_1 and Ψ_2 contain known constants and reduced-form parameters τ, which are identified by assumption.

Hence, it is straightforward to see from restriction (7) that, if γ_1. is identified, γ_2 will be as well. The following theorem follows immediately the identifiability of γ_1 from utilization P1- P3 in cross-equation restrictions (6).

Theorem 1. The cross-equation restrictions concludes that the vector of structural parameters γ is globally identifiable if the rank of Ψ_2 is equal l, the dimension of γ_1.

According to the above Theorem 1, if the rank of Ψ_2 is not equal to l, then it is not possible to decide on the identification of structural parameters γ. Also, we can see this result following directly from the well-known rank condition for uniqueness of a solution of a system of linear equations, which, in our notation, is Ψ_2 that has a full column rank.

We can also use the covariance constraint as another implicit function mapping γ to τ. To apply this function, we need to have some a priori restrictions on the structural covariance matrix Σ. For this purpose, we assume Σ is known to be diagonal, in this case restriction (7) could be useful for identifying γ_1, and, in fact, could help identify structural parameters which are otherwise unidentifiable. The difficulty with using the covariance restrictions

is that in general, the system of equations (7) is nonlinear in γ_1, and therefore there are no general conditions for global uniqueness of its solution. however, it is possible to find necessary and sufficient conditions for local uniqueness of the solution. To state a more general condition for identifiability of γ_1, we need some additional notation. Let f_1. and f_2 be the vectorized versions of (6) and (7) respectively.

By rewriting the elements of f_2 so that the ones corresponding to a priori constrained components of Σ, denoted with f_{2c}, come before those corresponding to the unconstrained components, denoted with f_{2c}. The identification of γ_1, when all available restrictions are used, depends on the rank of the Jacobian of the following matrix:

$$\begin{bmatrix} f_1 \\ f_{2c} \end{bmatrix} \tag{11}$$

We state this point in following result.

Result 1. The cross-equation restrictions and the covariance restrictions conclude that the vector of structural parameters γ is locally identifiable, when both the cross-equation and the covariance restrictions are used, on condition that the Jacobian with respect to γ_1 of the matrix in (11) has full column rank. This condition is also necessary for local identification, if the density of the structural shocks is assumed to be Gaussian.

Another expression to understand this result is the implicit function theorem. We can write the mean and covariance restrictions in a form as

$$h(\gamma, \tau) = 0 \tag{12}$$

In other words, the implicit function theorem expresses that a solution (γ_0, τ_0) of (12) is locally unique if the Jacobian $h_\gamma(\gamma, \tau) = 0$ has full column rank when evaluated at (γ_0, τ_0).

It should be noted that Theorem 1 can be concluded of Result 1. In this case, note that from the definition of $f_1$and equation (10), we have $f_1 = \Psi_2\gamma_1 - \Psi_1$. Hence the Jacobian of f_1, with respect to γ_1 is equal to Ψ_2. If Ψ_2 has full column rank, the matrix in (11) will also have full column rank. Also, since Ψ_2 does not depend on the structural parameters, identification in this case is global.

Example 1. Consider the following simple rational expectations model.

$$y_t = \theta E_t y_{t+1} + (1-\theta)x_t + u_t$$
$$z_t = \beta z_{t-1} + \varepsilon_t$$

where the endogenous variable y is determined by its expected value in the next period, and by the current values of the observed exogenous variable z, and by an i.i.d shock u. The exogenous variable z is governed by a stationary $AR(1)$ process. Both θ and β are assumed to be positive and smaller than 1.

Initially, by rewriting the above model to the standard form of (1), we have $X = [y, z]'$ and Γ_0, Γ_1 and Γ_2 as follows:

$$\Gamma_0 = \begin{pmatrix} 1 & -(1-\theta) \\ 0 & 1 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}$$

Solving the model, we find that the reduced-form matrix A is given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{(1-\theta)\beta}{1-\theta\beta} \\ 0 & \beta \end{pmatrix}$$

It is straightforward to express the cross-equation restrictions (6) in the form of (10)

$$\underbrace{\begin{pmatrix} a_{22}(1-a_{12}) & 0 \\ 0 & -1 \end{pmatrix}}_{\Psi_2} \underbrace{\begin{pmatrix} \theta \\ \beta \end{pmatrix}}_{\gamma_1} = -\underbrace{\begin{pmatrix} a_{12} - a_{22} \\ a_{22} \end{pmatrix}}_{\Psi_1}$$

For this example there is no difference between structural and reduced form parameters, that is, $\gamma = \alpha$ The rank condition for identification requires that the rank Ψ_2. is equal to 2 at the true value of A. Therefore the mean restrictions will be sufficient for identification of θ and β only if $a_{22}(1 - a_{12}) \neq 0$. Using $a_{12} = \frac{(1-\theta)\beta}{1-\theta\beta}$ and $a_{22} = \beta$, we find that the last condition is equivalent to $\beta(1-\beta) \neq 0$ or $\beta \neq 0$ and $\beta \neq 1$ These two conditions are also necessary if there are no a priori restrictions on Σ.

To review the effect of having restrictions on Σ, suppose that ε and u are uncorrelated, which implies that Σ is diagonal. Then, from the covariance restrictions (7) we obtain an additional equation that could be useful for the identification of θ and β. The equation is:

$$\Lambda_{12}(1 - \theta a_{11}) - \Lambda_{22}(1 - \theta(1 - a_{12})) = 0$$

Thus, the Jacobian of matrix in (11) is:

$$\begin{pmatrix} \Psi_2 \\ \dfrac{\partial f_{2c}}{\partial \gamma_1} \end{pmatrix} = \begin{pmatrix} a_{22}(1-a_{12}) & 0 \\ 0 & -1 \\ \Lambda_{22}(1-a_{12}) & 0 \end{pmatrix}$$

As a result, the model is identified if and only if the rank of this matrix is equal to 2. We have the following three cases:

1) $\beta \neq 0$ and $\beta \neq 1$, then θ is identified by both the cross-equation and the covariance restrictions;
2) $\beta = 0$, then $a_{12} = a_{22} = 0$and since $\Lambda_{22} = \sigma_\varepsilon^2 \neq 0$, θ is identified by the covariance restrictions only;
3) $\beta = 1$, then $a_{12} = a_{22} = 1$ therefore θ is not identified

An interesting feature of this example is that identification, even if it depends on the covariance restrictions, is always global. This is not generally true as we noted above. It is true here because of the recursive structure of the model, which makes $\Gamma_0 - \Gamma_1 A$ a triangular matrix, and, therefore the equation inferred by the restriction on Σ is linear in the structural parameters.

3.2 Conditions for Identification of α

As we discussed in the previous section, the coefficient matrices Γ_0, Γ_1 and Γ_2 of the model in (1) are typically functions of behavioral or deep parameters. In the current empirical DSGE literature, researchers are usually interested in estimating deep parameters, and not the structural parameters, which their identification was studied in the previous section. In this section, we study the identification of the deep parameters $\alpha = [\alpha_1', \alpha_2']'$. As in the previous section, we primarily concerned with the identification of α_1 given that, if it is identified, α_2 will be identified as well.

We know from section 2 that the identifiability of α depends on whether the equations in (6) and (7) have unique solutions for α. A sufficient condition for that is given in the following Theorem.

Theorem 2. The cross-equation restrictions concludes that the vector of deep parameters α is locally identifiable if the rank of the matrix

$$\Psi_2 \frac{\partial g(\alpha_1)}{\partial \alpha_1}, \tag{13}$$

equals to k, the dimension of α_1. This condition is sufficient for global identifiability, only if the rank is equal to k for all admissible values of α_1.

As defined in section 2, g is the function mapping α_1 into γ_1; To understand the Theorem, suppose first that γ_1 is identifiable by the mean restrictions only,

which, by virtue of Theorem 1 implies that Ψ_2 is of full rank. Then α_1 will be identified if the equation $\gamma_1 = g(\alpha_1)$ has a unique solution. For a general function g we can only determine whether a solution α_1^* is locally unique, that is, for any other α_1^{**} in some neighborhood of α_1^*, $g(\alpha_1^{**}) \neq \gamma_1$The condition is that the Jacobian$\partial g(\alpha_1)/\partial \alpha_1$, is of full column rank when evaluated at α_1^*.

It is important to note, however, that, unless γ_1 and α_1 are of the same dimension, we do not need to know the whole vector γ_1 to identify α_1 In other words, Ψ_2 does not have to be of full column rank, as long as the matrix in (13) is. Using the fact that the rank of a product of two matrices is no greater than the smaller of the two ranks, we obtain the following result:

Result 2. A necessary condition for α_1 to be locally identifiable by the cross-equation restrictions only, is that both Ψ_2. and $\partial g(\alpha_1)/\partial \alpha_1$have ranks greater or equal to *k*, the dimension of α_1.

Theorem 2 tells us when the mean restrictions (6) are sufficient for the identification of α. As before, the covariance restrictions (7) are also potentially useful for identifying α_1, if there are a priori restrictions on the structural covariance Σ. As in the previous section, the identifiability of α, when both the mean and the covariance restrictions are used, depends on the rank of the Jacobian of the matrix in (11).

Result 3. The vector of deep parameters α is locally identifiable when both the cross-equation and the covariance restrictions are used, on condition that the Jacobian with respect to α_1 of the matrix in (11) is equal to k, the dimension of α_1 This condition is also necessary for local identifiability of α_1 if the density of the structural shocks is assumed to be Gaussian.

The difference between Result 3 and Result 1 is that here the derivatives are taken with respect to the deep parameters α_1, instead of γ_1. As with Theorem 2 the following corollary follows immediately.

Result 4. A necessary condition for α to be locally identifiable by the cross-equation and covariance restrictions is that both the Jacobian with respect to γ_1 of the matrix in (11), and $\partial g(\alpha_1)/\partial \alpha_1$ have ranks greater or equal to *k*, the dimension of α_1.

Clearly, there is order conditions associated with the rank conditions given above and in Result 2. For a matrix to have a rank equal to *k*, its row and column dimensions must be greater or equal to *k*. This implies that the row dimension of the matrix in (11) must not be less than *k*, and also, that there must be at least *k* structural parameters, i.e. we must have $l \geq k$.

Since γ depends on α_1 only through γ_1, it should be clear that the covariance restrictions are useful for the identification of α only as far as they

help identify γ_1. If the cross-equation restrictions are sufficient for the identification of γ_1, that is, if Ψ_2 is of full column rank, all one needs to determine is whether there is a unique solution of the equation$\gamma_1 = g(\alpha_1)$. In this case, the rank condition on the Jacobian of g is both necessary and sufficient for identification of α_1.

Example 2. Consider the following simple rational expectations model.

$$y_t = \gamma_{11}E_t y_{t+1} + \gamma_{12}y_{t-1} - \gamma_{13}(r_t - E_t\pi_{t+1}) + \upsilon_t$$
$$\pi_t = \gamma_{21}E_t\pi_{t+1} + \gamma_{22}y_t + \gamma_{23}\pi_{t-1} + e_t$$
$$r_t = \gamma_{31}\pi_{t-1} + \gamma_{32}y_{t-1} + \gamma_{33}r_{t-1} + \xi_t$$
$$\upsilon_t = \gamma_{41}\upsilon_{t-1} + \nu_t$$
$$e_t = \gamma_{51}e_{t-1} + \varepsilon_t$$

where the vector of structural parameters γ_1 is related to the deep parameters α_1 as follows:

$$\gamma_{11} = \frac{\chi}{1+\chi}, \quad \gamma_{12} = \frac{1}{1+\chi}, \quad \gamma_{13} = -\frac{1}{\psi};$$

$$\gamma_{21} = \frac{\beta}{1-\varpi\beta}, \quad \gamma_{22} = \frac{(\psi+\nu)(1+\zeta\beta)(1-\zeta)}{(1+\varpi\beta)\zeta}, \quad \gamma_{23} = \frac{\varpi}{1+\varpi\beta};$$

$$\gamma_{31} = (1-\lambda_r)\lambda_\pi, \quad \gamma_{32} = (1-\lambda_r)\lambda_y, \quad \gamma_{33} = \lambda_r;$$
$$\gamma_{41} = \rho_1, \quad \gamma_{51} = \rho_2.$$

There are 14 deep parameters in this model, 11 of them are $\alpha_1 = [\beta, \chi, \varpi, \psi, \nu, \zeta, \lambda_r, \lambda_\pi, \lambda_y, \rho_1, \rho_2]'$ and the other 3 are $\sigma = [\sigma_\nu, \sigma_\varepsilon, \sigma_\xi]'$.

First, we will review the necessary condition for the rank of $\partial g(\alpha_1)/\partial\alpha_1$. According to Result 4, if the rank is less than 11, then α is not identified. Because there are 11 structural parameters γ_1, the dimension of $\partial g(\alpha_1)/\partial\alpha_1$ is 11×11 so it passes the order condition. The necessary rank condition fails due to the fact that two of the structural parameters γ_{11} and γ_{12} depend on single deep parameter χ. Because of this, the first two rows of $\partial g(\alpha_1)/\partial\alpha_1$ are not linearly independent, and, therefore, the rank is less than 11. Another way to see this is by noting that two of the deep parameters ζ and ν appear in only one of the structural parameters γ_{22}. Therefore, unless additional information is available, ζ and ν are not identifiable. Proceeding in the same manner, we find that all other parameters can be solved for if γ_1 is known. All except β and ϖ are globally uniquely determined, and, β and ϖ are only locally uniquely determined.

This implies that $\chi, \psi, \lambda_r, \lambda_\pi, \lambda_y, \rho_1, \rho_2, \sigma_\nu, \sigma_\varepsilon, \sigma_\xi$ will be globally identifiable, β and ϖ locally identifiable, and, ζ and ν - unidentifiable, if γ_1 can be identified. From Theorem 2 a sufficient condition for identification of γ_1 is that matrix Ψ_2 from (10) has a full column rank. As we show in the Appendix, Ψ_2 is of full column rank for the parameter values used in Canova and Sala (2009). Therefore, for those parameter values, γ_1 is globally identified by the mean restrictions only.

4 Simulation

In this section, we report the results from several Monte Carlo experiments. The goal of the experiments is to obtain evidence on the relative performance of the ML estimator for models with the weakly identified structure to evaluate the practical importance of using covariance restrictions in estimation.

4.1 The Model

The simulated data for all experiments comes from the log-linearized version of a small monetary New Keynesian model for Iranian economy, as an Islamic country; it consists of the following three equations:

$$x_t = \theta_f E_t x_{t+1} - \theta_b (i_t - E_t x_{t+1}) + \varepsilon_t, \quad (14)$$

$$\pi_t = \psi E_t x_{t+1} + \beta_f x_t + \nu_t, \quad (15)$$

$$m_t = \frac{\beta_b}{\theta_b} c_t - \frac{\beta_b}{\lambda} i_t, \quad (16)$$

$$M_t = m_t - m_{t-1} + \pi_t, \quad (17)$$

$$M_t = \rho_y M_{t-1} + \rho_\pi \pi_t + \rho_r x_t + \zeta_t. \quad (18)$$

Equation (14) is a log-linearized Euler equation at time t, equation (15) is a Phillips curve and equation (16) is the monetary policy rule. All structural shocks - ε, ν and ζ are assumed to be mutually and serially uncorrelated white noise processes.

Table 1

Details on the Monte Carlo Design

Parm.	Cond	θ_f	θ_b	ψ	β_f	β_b	λ	ρ_π	ρ_y	ρ_r	σ_ε	σ_ν	σ_ζ
P1	8.8	.60	.40	.15	.50	.30	.50	.200	.010	.900	1	1	1
P2	220	.63	.00	.01	.48	.52	.01	.254	.095	.845	1	1	1
P3	552	.10	.00	.17	.50	.30	.50	.254	.095	.845	1	1	1

note. Cond refers to the condition number of matrix Ψ_2 and is a measure for parameter identifiability; high values indicate weak identification. *Source*: Authors' Findings.

The vector of structural parameters that will be estimated is $\gamma = [\theta_f, \theta_b, \psi, \beta_f, \beta_b, \lambda, \rho_\pi, \rho_y, \rho_r, \sigma_\varepsilon, \sigma_\nu, \sigma_\zeta]'$. The Monte Carlo experiments differ in the values of used to generate data for $X = [x, \pi, m]'$. We present results for the three different parameterizations P1, P2, and P3, shown in Table 1. They were chosen as examples of environments with different degrees of parameter identifiability. The second column in the table shows the condition number of matrix Ψ_2; We use this as an indicator of how well γ is identified. The condition number of Ψ_2 tells us how far will the estimate $\hat{\gamma}_1$ be when $\widehat{\Psi}_1$ and $\widehat{\Psi}_2$ are different from the true values Ψ_1 and Ψ_2. We use the @rcommand cond instruction in MATLAB software to calculate the condition number; larger values of condition number mean that the identification of the models is weaker. Each experiment consists of generating 1000 samples of size T=150. The samples are obtained by solving the model (14) - (18) and using the reduced-form matrices to generate 1200 observations of Z; only the last 200 observations are used in estimation to eliminate the influence of the initial conditions.

4.2 Estimation

We estimate the parameters of the model in (14) - (18) using maximum likelihood estimator (ML). Each estimator is applied with and without making use of the covariance restrictions, resulting in two different estimates for γ When the covariance restrictions are not used in estimation, we first estimate the parameters in γ without the three σ's, and then use the covariance restrictions to solve for the σ's. The ML estimator is obtained by directly maximizing the likelihood of reduced form VAR, subject to the restriction implied in the model. The likelihood is computed assuming that the structural shocks follow a multivariate Gaussian distribution.

4.3 Monte Carlo Results

The Monte Carlo results are presented in Table 2. For each of the three experiments, we report the Mean Bias and the Root Mean Squared Error (RMSE). In this Table, the first column depicts the results when the covariance restrictions are not used in the estimation, and the second column shows results when all restrictions are used.

Wherever the covariance restrictions are used in estimation, affects not only the ranking of the estimator but also improves their performance substantially in terms of Bias and RMSE. For most parameters, Bias and

RMSE decreases when the covariance restrictions are used. Notable exceptions are the parameters θ_f and ψ when data generated by P3; for this case, RMSE increases from 5.9 and .67 to 6.66 and .93 respectively. This observation is interesting because the weak identification in parameterizations P3 is mostly due to weak identifiability of the parameters in equation (14). This can be seen by comparing the RMSE's for the parameters in that equation for the three parameterizations. It suggests that using only the cross-equation restrictions may be better for the ML estimator when some of the parameters are weakly identifiable.

Table 2
Monte Carlo Results

Par.	P1		P2		P3	
	Bias					
θ_f	-0.116	-0.081	-0.831	-0.457	-0.269	-0.377
θ_b	0.124	0.107	0.024	0.018	0.122	0.106
ψ	-0.053	-0.048	-0.178	-0.199	-0.104	-0.200
β_f	0.027	0.092	0.023	0.023	0.103	0.022
β_b	0.012	-0.003	-0.045	-0.031	-0.022	-0.015
λ	0.023	-0.051	-0.016	-0.017	-0.077	-0.045
ρ_π	-0.022	-0.099	-1.581	-0.001	-1.335	0.113
ρ_y	0.064	0.106	0.048	0.003	0.247	-0.018
ρ_r	0.007	0.000	-0.012	0.004	0.035	0.050
σ_ε	0.691	0.020	0.805	0.008	0.625	0.036
σ_ν	0.683	0.089	0.149	-0.047	-0.084	-0.302
σ_ζ	0.537	-0.026	3.535	-1.170	4.413	-0.116
	RMSE					
θ_f	0.779	0.665	3.635	2.447	5.897	6.532
θ_b	0.534	0.445	0.094	0.070	0.776	0.641
ψ	0.493	0.465	0.877	0.859	0.675	0.932
β_f	0.985	0.677	0.303	0.296	1.084	0.415
β_b	0.275	0.255	0.417	0.411	0.422	0.037
λ	0.754	0.555	0.230	0.243	0.560	0.414
ρ_π	1.355	1.349	6.156	0.828	6.838	0.887
ρ_y	1.133	0.890	0.180	0.049	1.314	0.266
ρ_r	0.868	0.810	0.866	0.862	0.970	0.948
σ_ε	2.945	1.180	5.734	1.019	5.077	2.025
σ_ν	3.480	1.314	3.461	1.290	3.542	1.337
σ_ζ	4.850	1.235	5.778	0.967	6.655	2.153

Source: Authors' Findings.

5 Conclusion

In this paper, we take a starting point with the cross-equation and covariance restrictions that characterize linearized DSGE models and show how they can be used to study the identification of the parameters of such models. We derive a set of identifiability conditions and suggested a procedure for a thorough analysis of identification at each point in the parameters space. The procedure can be applied before DSGE models are estimated, to determine where identification fails, where it likely to be weak, and where the identification problems originate. We also use a Monte Carlo simulation and study the effect of restrictions on the estimate. The results show that the use of restrictions for estimation, when identification is reduced, leads us to inaccurate estimates and unreliable inference even when the number of observations is large.

A useful extension of the identification results presented here would be to find a systematic way of attributing detected identification issues to specific model parameters. We can generalize the results of this paper only to specific equations. For instance, if a rank deficiency, indicating identification failure, or near rank deficiency, suggesting a weak identification, is found, we are able to find the particular equations, but not the causative parameters in those equations.

References

Anderson, G., & Moore, G. (1985). A Linear Algebraic Procedure for Solving Linear Perfect Foresight Models. *Economics Letters*, 17(3), 247-252.

Andrle, M. (2010). A Note on Identification Patterns in DSGE Models. *ECB Working Paper* No. 1235, Frankfurt: European Central Bank.

Beyer, A., & Farmer, R. E. A. (2004). On The Indeterminacy of New-Keynesian Economics. *Working Paper Series* 323, European Central Bank.

Blanchard, O. J., & Kahn, C. M. (1980). The Solution of Linear Difference Models under Rational Expectations. *Econometrica*, 48(5), 1305-11.

Canova, F., & Sala, L. (2009). Back to Square One: Identification Issues in DSGE Models. *Journal of Monetary Economics*, 56, 431–449.

Christiano, L. J., Eichenbaum, M., & Evans, C. L. (2005). Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy. *Journal of Political Economy*, 113(1), 1-45.

Cochrane, J. (2011). Determinacy and Identification with Taylor Rules. *Journal of Political Economy*, 119, 565–615.

Dees, S., Pesaran, M. H., Smith, L. V., & Smith, R. P. (2009). Identification of New Keynesian Phillips Curves from a Global Perspective. *Journal of Money. Credit and Banking*, 41, 1481–1502.

Iskrev, N. (2010). Local Identification in DSGE Models. *Journal of Monetary Economics*, 57, 189–202.

Iskrev, N. and Ratto, M. (2010). Analysing Identification Issues in DSGE Models. *MONFISPOL papers*, Stressa, Italy.

King, R. G., Plosser, C. I., & Rebelo, S. (1988). Production, Growth, and Business Cycles: I the Basic Neoclassical Model. *Journal of Monetary Economics*, 21(2-3), 195-232.

Klein, P. (2000). Using the Generalized Schur Form to Solve a Multivariate Linear Rational Expectations Model. *Journal of Economic Dynamics and Control*, 24(10), 1405-1423.

Kleibergen, F., & Mavroeidis, S. (2009). Weak Instrument Robust Tests in GMM and the New Keynesian Phillips Curve. *Journal of Business and Economic Statistics*, 27, 293–311.

Komunjer, I., & Ng, S. (2011). Dynamic Identification of DSGE Models. *Econometrica*, 79, 1995–2032.

Lubik, T. A., & Schorfheide, F. (2004). Testing for Indeterminacy: An Application to U.S. Monetary Policy. *American Economic Review*, 94(1). 190-217.

Mavroeidis, S. (2005). Identification Issues in Forward-Looking Models Estimated by GMM, With an Application to the Phillips Curve. *Journal of Money. Credit and Banking*, 37, 421–448.

McCallum, B. (1979). On the Observational Inequivalence of Classical and Keynesian Models. *Journal of Political Economy*, 87, 395–402.

Nason, J., & Smith, G. W. (2008). Identifying the New Keynesian Phillips Curve. *Journal of Applied Econometrics*, 23, 525–551.

Pesaran, M. H. (1981). Identification of Rational Expectations Models. *Journal of Econometrics*, 16, 375–398.

Pesaran, M. H. (1989). *The Limits to Rational Expectations*. Oxford: Blackwell.

Pudney, S. (1982). The Identification of Rational Expectations Models under Structural Neutrality. *Journal of Economic Dynamics and Control*, 4, 117–121.

Revenna, F. (2007), Vector Autoregressions and Reduced Form Representations of Dynamic Stochastic General Equilibrium Models. *Journal of Monetary Economics*, 54, 2048-2064.

Rothenberg, T. J. (1971). Identification in Parametric Models. *Econometrica*, 39(3), 577-9.

Sargent, T. J. (1976). The Observational Equivalence of Natural and Unnatural Rate Theories of Macroeconomics. *Journal of Political Economy*, 84, 631–640.

Sims, C. A. (2002). Solving Linear Rational Expectations Models. *Computational Economics*, 20(1-2), 1-20.

Smets, F., & Wouters, R. (2003). Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach. *American Economic Review*, 97, 588–606.

Smets, F., & Wouters, R. (2007). Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach. *American Economic Review*, 97, 588–606.

Wallis, K. F. (1980). Econometric Implications of the Rational Expectations Hypothesis. *Econometrica*, 48. 49–73.