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Optimal Portfolio Selection for Tehran Stock Exchange Using Conditional, Partitioned and Worst-case Value at Risk Measures

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Abstract

This paper presents an optimal portfolio selection approach based on value at risk (VaR), conditional value at risk (CVaR), worst-case value at risk (WVaR) and partitioned value at risk (PVaR) measures as well as calculating these risk measures. Mathematical solution methods for solving these optimization problems are inadequate and very complex for a portfolio with high number of assets. For these reasons, a combination of particle swarm optimization (PSO) and genetic algorithm (GA) is used to determine optimized weights of assets. Stocks' Optimized weight results show that

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proposed algorithm gives more accurate outcomes in comparison with GA algorithm. According to back-testing analysis, PVaR and WVaR overestimate risk value while VaR and CVaR give a rather accurate estimation. A set of companies in Tehran Stock Exchange are considered as a case study for empirical analysis.

Key words: *portfolio optimization, value at risk, CVaR, WVaR, PVaR, HGAPSO*

***JEL Classification**: G10, G11, G19*

1. Introduction

Optimal portfolio allocation is an important issue for risky asset holders who are exposed to financial risks. Markowitz (1952) proposed optimal portfolio selection based on mean-variance optimization in which the concept of portfolio diversification was formally introduced for the first time. In his model, portfolio's mean and variance are considered as return and risk respectively in which the optimal portfolio is the one with maximum return at a given level of risk or with minimum risk at determined return. A set of optimum portfolios constitute an efficient frontier that an investor chooses an appropriate portfolio regarding his/her position.

Many researchers found that two main assumptions of Markowitz approach are violated in empirical studies with real data. The first assumption (which is related to a tradeoff between risk and return) is criticized by Tobin (1958) and Chamberlin (1983) who believed that this assumption is valid if the returns distribution is symmetric. Other empirical evidence by Mandelbrot (1963), Fama (1965), Simkowitz and Beedles (1983) and Alles and Kling (1994) indicated that returns have asymmetric distribution and a high kurtosis. Second assumption of Markowitz approach is that investors are indifferent between positive and negative mean deviations. But, many empirical observations demonstrate that investors reflect asymmetric behavior facing upside and downside risks [Mittone and Vorkink (2007), Barbies and Huang (2008), Gollier and Parker (2007) and etc.]. Therefore, using portfolio variance as risk measure faces some limitations. In this method, positive returns (with far distance from mean) are desirable for an investor who considers them as risks because variance is a symmetrical criterion and does not consider the change of directions. On the other hand, considering variance as risk criteria is not tangible for investors and requires statistical information (Giorgi, 2002). Some studies (Quirk 1966 and Mao 1970) show that investors focus more on undesirable risks in comparison with risks with negative and positive volatilities. Regarding problems of variance, Markowitz (1959) suggested semi-variance as risk measure which only considered negative volatilities but it needs too much information and is uncertain for explaining returns probability distribution. Konno & Ymazaki (1991) proposed mean absolute deviation (MAD) as risk measure by expanding Markowitz model. This criterion was also criticized by many. For example Simaan (1997) shows that information about variance-covariance matrix is disregarded in MAD calculation.

The correct definition of risk and presenting an accurate indicator to measure its value are two fundamental issues in financial economy which have been consistently taken into account from Markowitz's work until now. In general, the modern theory of risk is highly expanded through introduction of value at risk (VAR) by J.P. Morgan in 1994. VaR measures maximum expected loss which is related to market risk. Since this measure present risk value in a single number, it is accepted by many financial institutions and insurance companies as an international measure. However, VaR is not a coherent risk measure, some of risk criteria including conditional value at risk (CVaR), worst value at risk (WVaR) and partitioned value at risk (PVaR) were proposed with coherent risk measure properties. Therefore, these four risk measures are considered instead of variance to improve portfolio optimization. The purpose of this paper is to use these alternative approaches for portfolio optimization of Tehran stock exchange. In fact, share of each stock is determined by minimization of these risk measures subject to constraints. Optimization problems are solved by a hybrid genetic algorithm and particle swarm optimization (HGAPSO) method for different companies of Tehran stock exchange.

The structure of the paper is as follows. Main principles of four mentioned approaches are described in section 2. Particle swarm optimization (PSO) and genetic algorithm methods are explained in section 3. Empirical results are analyzed in section 4 and finally conclusion of paper is presented in section 5.

2. Portfolio Optimization Approaches Based on VaR Measures

In this section, four new approaches of portfolio optimization based on VaR measures are presented.

2.1. Value at risk (VaR)

VaR is considered as the most common criterion by banks and financial institutions under Basel Accord. VaR can be defined as maximum expected loss of financial position during a given period in future (a day, a week or a month) at a specified probability level. By definition, VaR has two main parameters: time horizon which is shown as the number of days and confidence level. In general, by assuming N days as time horizon and α percent as probability level, we are (100-α) percent sure that loss will not be more than VaR during next N days. Mathematically, VaR can be demonstrated as follows:

$$P_r\{(V_1 - V_0) \leq -VaR\} \leq \alpha \quad (1)$$

Where V_0 and V_1 are portfolio value at t=0 and t=1, respectively. There are three approaches of VaR calculation and estimation including parametric, non-parametric and semi-parametric methods. In non-parametric approach, there is no constraint on portfolio return distribution and VaR is calculated based on historical return quintile or predicted return quintile. Historical simulation and Monte Carlo simulation are in this method. In fully parametric approach, VaR is calculated by imposing pre-assumptions on return distribution and dynamics of volatility model. GARCH type models and Riskmetrics model are examples of this approach. Semi-parametric approach combines two previous methods in order to impose assumptions only on dynamics of volatility model. Volatility-weighted historical simulation, filtered historical simulation and Conditional autoregressive VaR models are in this category.

As this study focuses on portfolio optimization problem, parametric approach or variance-covariance method are applied to form it based on VaR. Variance-covariance technique which was described in J.P Morgan Riskmetrics programming assumes that returns follow normal distribution and correlations between risky assets are constant. Under this method, VaR is estimated simply through calculation of returns variance-covariance. In this case, VaR is measured mathematically as follows:

$$VaR_{1-\alpha} = \mu_p - \phi^{-1}(1-\alpha).\sigma_p \tag{2}$$

Where μ_p and σ_p are conditional mean and variance of portfolio respectively and ϕ^{-1} is an inverse cumulative density function at α probability level. For portfolio optimization problem, VaR is defined as a minimum real number (γ) that does not exceed $-w'r$ with α probability. This definition is expressed as:

$$VaR_{\alpha}(w) = \min\{\gamma: P(\gamma \leq -w'r) \leq \alpha\} \tag{3}$$

Where r and $w$are returns and weights vectors of n risky assets and $\bar{R}_p = w'r$ is portfolio mean. Also, P indicates probability distribution of asset returns. Thus the portfolio optimization problem based on VaR definition can be written as:

$$\min_w \quad \gamma \tag{4}$$

$$\text{s.t.} \quad w'e = 1$$

Where e is a vector of ones and budget constraint indicates that sum of assets weights equals 1. Note that this optimization problem is a chance-constrained stochastic program and requires accurate and complete returns probability distribution which is complex. Due to VaR non-convexity, its solution is difficult and therefore, objective function of equation (4) is approximated. Usually in finance framework, it is assumed that assets returns follow normal distribution with mean vector of μ_r and variance-covariance matrix of Ω_r. Considering this assumption and using parametric approach (equation 2), approximation of portfolio optimization problem is specified as:

$$\min_w \quad VaR_\alpha(w) = -w'\mu_r + \phi^{-1}(\alpha).\sqrt{w'\Omega_r w} \tag{5}$$

$$\text{s.t.} \quad 1)\ w'e = 1$$

$$2)\ \mu_p = w'\mu_r = \tau$$

Where τ is target gain and second constraint shows that expected mean of portfolio should be equal to τ. μ_r and Ω_r are achieved simply regarding equations 6 and 7 respectively.

$$\mu_r = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}, \quad \Omega_r = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix} \tag{6}$$

Where

$$\mu_i = E(r_i), \quad \sigma_{ij} = E[(r_i - \mu_i)(r_j - \mu_j)] \tag{7}$$

2.2. Conditional value at risk (CVaR)

One of the important limitations of VaR is that it does not satisfy coherent risk characteristics proposed by Artzner et al. (1999). With respect to VaR disadvantages such as non-Subadditivity and also due to the fact that VaR does not give any information about losses more than itself, another risk measure was proposed by Uryasev and Rockafellar (1999) which is called conditional value at risk (CVaR). This risk indicator has all properties of coherent risk and can measure expected losses in case of an unfavorable economic situation. In other words, VaR only measures maximum loss at specified confidence level in normal market state and gives no information about the exact amount of loss in critical conditions while CVaR calculates expected loss in an unanticipated situation. CVaR provides additional information about loss in the left side of its distribution curve when expected loss exceeds VaR. Mathematical representation of CVaR is shown as:

$$CVaR_{1-\alpha} = E(x \mid x > VaR_{1-\alpha}) \tag{8}$$

According to the VaR definition and assuming f(x) as density function of x, CVaR can be specified as:

$$CVaR_{1-\alpha} = \frac{1}{\alpha}\int_{-\infty}^{VaR_{1-\alpha}} xf(x)dx \tag{9}$$

In special case, considering f(x) as a normal density function, CVaR is obtained as:

$$CVaR_{1-\alpha} = \mu_p - \alpha^{-1}.\varphi[\phi^{-1}(1-\alpha)].\sigma_p \tag{10}$$

Where φ is normal standard density and ϕ is its cumulative distribution function. It is obvious that CVaR is larger than VaR.

Optimization problem using CVaR measure can be approximated as:

$$\begin{aligned} \min_w \quad & CVaR_{1-\alpha} = -w'\mu_r + \alpha^{-1}\varphi(\phi^{-1}(1-\alpha)).\sqrt{w'\Omega_r w} \\ \text{s.t.} \quad & 1)\ \mu_p = w'\mu_r = \tau \\ & 2)\ w'e = 1 \end{aligned} \tag{11}$$

In this model, mean vector and variance-covariance matrix is calculated as VaR model.

3.2. Worst value at risk (WVaR)

Precise Calculation of parametric VaR and CVaR necessitates complete information about returns distribution due to lack of information. On the other hand, return distribution in these optimization models are usually pre-determined while in reality it is not. In this situation, it may lead to underestimation of true VaR and CVaR values with optimization approximations in equations (5) and (11). Ghaoui et al. (2003) argued that financial data face error and it is possible that calculated VaR is lower than true one. Due to this reason, they [same as Bertsimas and Popescu (2000)] believed that standard deviation coefficient in VaR model is not valid and an upper bound is taken into account for this coefficient. In fact, considering a given mean and variance-covariance of risky assets returns, they offered a new conservative risk measure with a pessimistic approximation of VaR named as worst-case value at risk (WVaR). Assuming that P_{r} is set of all probability distributions in P^m space with mean of μ_r and covariance of Ω_{r}, mathematical definition of WVaR can be represented as:

$$WVaR_{\alpha} = \min\{\gamma:\ \sup P(\gamma \leq -w'r) \leq \alpha, P \epsilon P_{\mathrm{r}}\}$$
$$\mathrm{P}_r(\mu_r, \Omega_r) = \{\mathrm{P}: \mu_r = E(r), E(r-\mu_r)(r-\mu_r)' = \Omega_r\} \tag{12}$$

Ghaoui et al., specified WVaR model as:

$$WVaR_{\alpha}(w) = w'\mu_r - k(\alpha).\ \sqrt{w'\Omega_r \mathrm{w}} \tag{13}$$

Where $k(\alpha) = \sqrt{((1-\alpha)/\alpha)}$ is the upper bound which is replaced instead of $\phi^{-1}(\alpha)$ in VaR model. They suggested although returns distribution is unknown, WVaR can be specified using mean and variance-covariance of assets returns in a probability distributions set. Indeed, it is better to take one class of probability distributions into account instead of considering joint distribution of assets returns and parameter estimations using empirical data. So, the portfolio optimization problem based on WVaR is approximately formed as:

$$\min_w \; WVaR_\alpha(w) = w'\mu_r - k(\alpha). \|\Omega_r{}^{0.5} x\|_2 \tag{14}$$

$$\text{s.t.} \quad 1)\ \mu_p = w'\mu_r = \tau$$

$$2)\ w'e = 1$$

Where $\|.\|_2$ indicates 2-norm and the descriptive statistics are calculated the same as VaR and CVaR approaches. Note that WVaR is a symmetric risk measure while VaR and CVaR are downside ones.

4.2. Partitioned value at risk (PVaR)

Goh et al (2012) introduced a new coherent risk measure via expanding WVaR which is called partitioned value at risk (PVaR). Unlike VaR and CVaR measures that define on single probability distribution, PVaR defines over one class of probability distributions. This new risk measure can be identified through additional statistical information so that random returns are divided to two half spaces of negative and positive. Thus, statistical information such as mean and variance-covariance are calculated for gain and loss half spaces of assets returns separately. Like WVaR, This measure satisfies all property of coherent risk. It can be shown that for returns with asymmetric distribution, PVaR evaluates risk value lower than WVaR.

Assuming r_i is time series returns of i^{th} asset with dimension of t×1, due to dimension adaption of two half spaces, it is divided into two non-negative $r_i^1 = \max\{r_i, 0\}$ and non-positive $r_i^2 = \max\{r_i, 0\}$ partitions in such a way that $r_i = r_i^1 + r_i^2$. Generally, portfolio returns vector *r* is divided into two gain and loss vectors as$= (\mathrm{r}^1, \mathrm{r}^2)$. Where r^1 involves positive and zero returns while r^2 includes negative and zero ones. It is obvious that $r = r^1 + r^2$. Moreover, mean of r^1 and r^2 are μ^1 and μ^2 respectively and therefore, the mean portfolio is $\mu_r = \mu^1 + \mu^2$. In this situation, statistical information is located in R^{2n}. Finally, variance-covariance matrix $\widehat{\Omega}_r$ dimension is 2n×2n which is represented as follows:

$$\widehat{\Omega}_{r_{2n.2n}} = \begin{bmatrix} var(r_1^1) & \dots & \text{cov}(r_1^1, r_n^1) & \text{cov}(r_1^1, r_1^2) & \dots & \text{cov}(r_1^1, r_n^2) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \text{cov}(r_n^1, r_1^1) & \dots & var(r_n^1) & \text{cov}(r_n^1, r_1^2) & \dots & \text{cov}(r_n^1, r_n^2) \\ \text{cov}(r_1^2, r_1^1) & \dots & \text{cov}(r_1^2, r_n^1) & var(r_1^2) & \dots & \text{cov}(r_1^2, r_n^2) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \text{cov}(r_n^2, r_1^1) & \dots & \text{cov}(r_n^2, r_n^1) & \text{cov}(r_n^2, r_1^2) & \dots & var(r_n^2) \end{bmatrix} \quad (15)$$

Where,

$$cov(r_i^k, r_j^k) = E\{(r_i^k - \mu_i^k)(r_j^k - \mu_j^k)\} \quad (16)$$

In this equation, r_i^k indicates the i^{th} asset return in k^{th} half space (k=1, 2 and i, j=1, 2,…n). In short, based on these partition statistics, $P_r(\mu^1, \mu^2, \widehat{\Omega})$ can be defined as one set of probability distributions.

$$P_r(\mu^1, \mu^2, \widehat{\Omega}_r) = \{P: P \epsilon P_r(\mu^1, \mu^2, \widehat{\Omega}_r), \mu_r = E\begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix}, \\ E\begin{pmatrix} r^1 - \mu^1 \\ r^2 - \mu^2 \end{pmatrix}\begin{pmatrix} r^1 - \mu^1 \\ r^2 - \mu^2 \end{pmatrix}' = \widehat{\Omega}_r\} \quad (17)$$

With respect to above clarification, PVaR definition over probability distribution P_r is presented as bellows:

$$PVaR_\alpha(w) = -w'\mu_r + \min_{t,s \geq 0}\{k(\alpha) \| \widehat{\Omega}_r^{\,0.5}\begin{pmatrix} w-s \\ w+t \end{pmatrix} \|_2 + \mu^{1\prime} s - \mu^{2\prime} t\} \quad (18)$$

Accordingly, portfolio optimization problem based on PVaR measure is written as:

$$\min_{w} \; PVaR_{\alpha}(w) = w'\mu_r - k(\alpha).\left\| \Omega_r{}^{0.5} \begin{pmatrix} w-s \\ w+t \end{pmatrix} \right\|_2 + \mu^{1'}s - \mu^{2'}t$$

$$\text{s.t.} \quad 1)\ \mu_p = w'\mu_r = \tau \quad (19)$$

$$2)\ w'e = 1$$

$$3)\ t, s \geq 0$$

In this model, s and t are auxiliary variables which are considered only for forming of optimization problem and do not contain any information.

3. Heuristic Algorithms

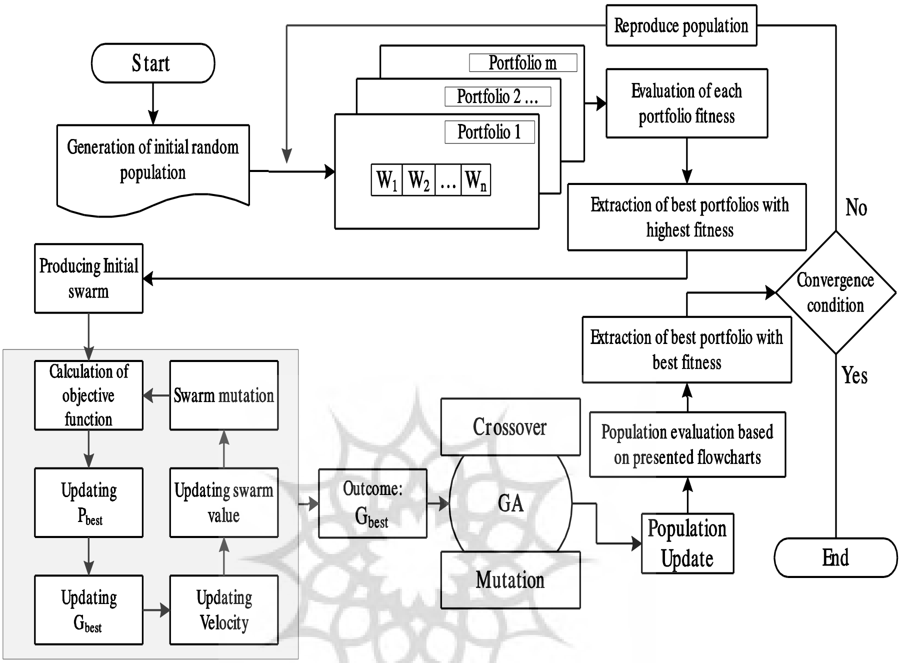
Solution of an optimization problem via mathematical methods is very complex when number of variables increase. Therefore, the computational methods are used to obtain the optimal solution. In this study, genetic algorithm (GA) and hybrid genetic algorithm with particle swarm optimization (HGAPSO) is applied to find large number of assets weights.

GA which was proposed by Holland (1975) is a computational search method based on the structure of genes and chromosomes. This algorithm starts with initial population of random individuals that each of them is considered as optimum solution and is driven over serial generations using three operator including selection, crossover and mutation. In four above mentioned optimization problems based on VaR measures, portfolio is considered as individuals or chromosomes. Risky assets weights are considered as genes which their encoding is real number because objective functions and constraints are continuous.

Particle swarm optimization (PSO) is one of population based algorithms that were suggested by Kennedy and Eberhart (1995). This method is originated from birds' flight to find foods and nests. It involves population with different swarms so that each swarm is a solution. For portfolio optimization problem, set of assets weights is considered as population and each asset is assumed as swarm. Two main characteristics of each swarm are position and velocity. At first swarms are positioned in search space randomly and weighted based on a random number. The velocity indicates distance and direction of swarm. In fact, the main foundation for this approach is that each swarm remembers its and others' previous best position and moves to the direction of the best swarm.

Although GA and PSO methods act parallel in exploration space and track different locations of solution space, they have some limitations. GA advantages are its robustness and adaptability while its disadvantages are long run time and lowering convergence speed. On the other hand, PSO has an advantage of high convergence to achieve optimum solution but its disadvantage is its dependency to initial conditions. In order to utilize the benefits of both methods, a combination of these two algorithms is presented in this study. The hybrid algorithm enjoys both GA accuracy and PSO convergence speed. The general framework of HGAPSO is shown in Figure 1 where hybrid algorithm begins with initial population of m portfolios that each of them involves n assets shares. Next, each random portfolio fitness is evaluated through objective functions values and then portfolios with highest fitness are selected. These selected portfolios are considered as initial swarms and positioned in a six steps loop. This loop consists of fitness function estimation, updating of p-best (local optimum), g-best (global optimum), swarm velocity, swarm value and swarm mutation. As a result, global optimum weights are obtained. Afterward, new population is reproduced by crossover and mutation operators of GA algorithm. This new population is fitted and individuals with highest fitness are extracted. If these solutions are converged to g-best results, the optimum weights are found. Otherwise, all the previous steps are repeated until the convergence condition is eventually established to achieve a global optimum.

Figure 1: Hybrid Algorithm Flowchart



4. Empirical Results

This study concentrates on finding optimum portfolio based on VaR, CVaR, WVaR and PVaR optimization approaches using GA and HGAPSO algorithms and calculation of these mentioned risk measures. In this section, statistical population and its descriptive data are presented at first and then, empirical results of assets optimum weights are analyzed. Finally, back test statistics in order to evaluate calculated risk measures is discussed.

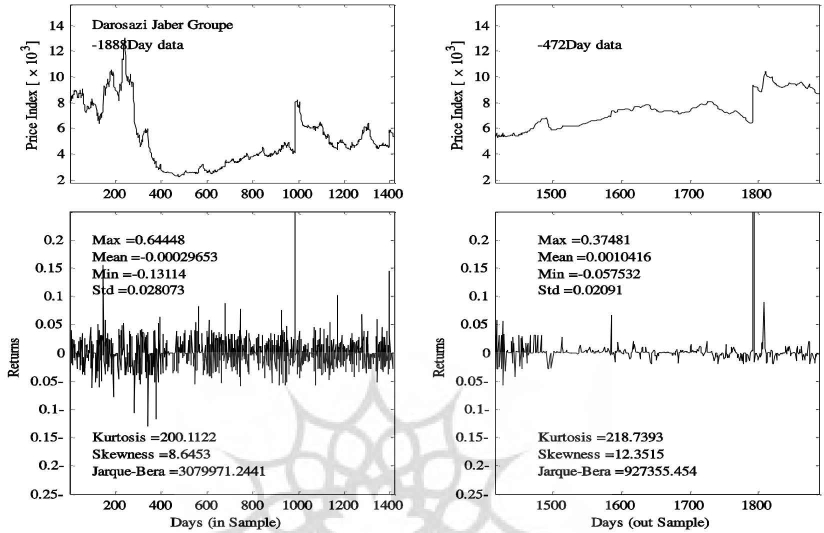
4.1. Statistical description

In this study, stocks of 17 companies of Tehran stock exchange are selected as portfolio. In order to reduce the correlations between stocks, these companies are chosen from among various industries. Time series closed price of this portfolio is collected from archive of Finance information processing of Iran (FIPIRAN) between 21/12/2006 to 12/12/2014. Daily returns of this data are obtained via: $r_t = Ln(\frac{P_t}{P_{t-1}})$, where P_t and P_{t-1} are closed price at time t and t-1 respectively. Returns time series data consists of 2360 observations which are divided to 1880 in sample observations for optimization and 472 out sample observations in order to evaluate estimated risk measures. Table 1 demonstrates names of these companies and some descriptive statistics of them. Also, figures of daily closed price and returns for 4 companies as examples are shown in Figure 2.

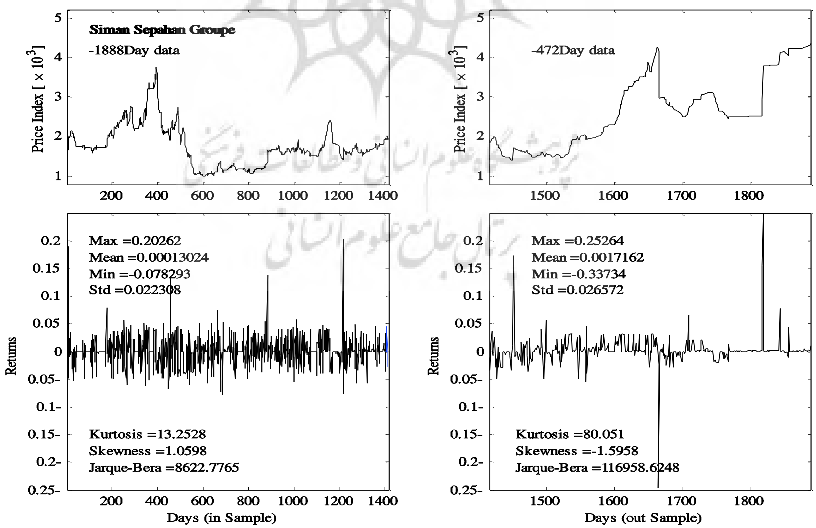
Table1: Company Names and Their Descriptive Statistics

i	Company name	Mean	Std. dev.	Skewness	Kurtosis	Jarque–Bera
1	Iran Transfo (IT)	0.00030	0.05000	25.91	859.05	57880990.24
2	Traktorsazi Iran (TI)	-0.00080	0.02230	-1.270	13.940	9912.42
3	Nosazi & Sakhteman (NS)	-0.00130	0.02818	0.736	9.6900	3687.15
4	Darosazi Jaber (DJ)	-0.00029	0.02807	8.640	200.11	3079971.24
5	Siman Sepahan (SS)	0.00013	0.02220	1.060	13.260	8622.78
6	Pertol Abadan (PD)	-0.00010	0.03990	18.090	544.03	23129982.54
7	Mes Shahid Bahonar (MSB)	-0.00039	0.02588	-0.990	13.870	9619.58
8	Tooka Fulad (TF)	-0.00110	0.02816	2.290	31.100	63785.63
9	Sarmayegozari Alborz (SM)	-0.00047	0.02403	2.520	28.980	55128.94
10	Pars Khodro (PK)	0.00035	0.04033	13.34	374.36	10905084.69
11	Saipa (S)	0.00012	0.03159	5.980	93.580	656819.85
12	Iran Khodro (IK)	-0.00052	0.02661	0.092	39.260	103467.09
13	Hamlonaghle Tooka (HK)	-0.00030	0.03315	7.430	164.63	2072546.64
14	Dadepardazi Iran (DI)	-0.00069	0.04208	3.810	230.42	4073381.19
15	Toseye Sanaye Behshahr (TSB)	-0.00070	0.02469	1.020	17.480	16833.19
16	Sanaye Lastik Sahand (SLS)	-0.00140	0.02998	0.170	46.830	151199.08
17	Nafte Behran (NB)	-0.00080	0.03785	10.73	186.71	2691348.22

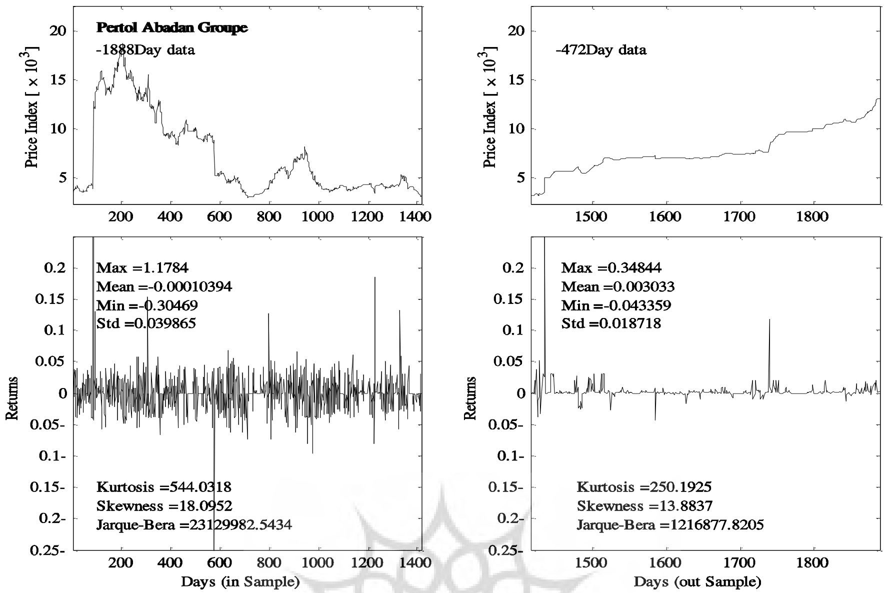
Figure 2: Closing Price and Returns Time Series Data for a)DJ, b)SS, c)PA and d)IK Stocks



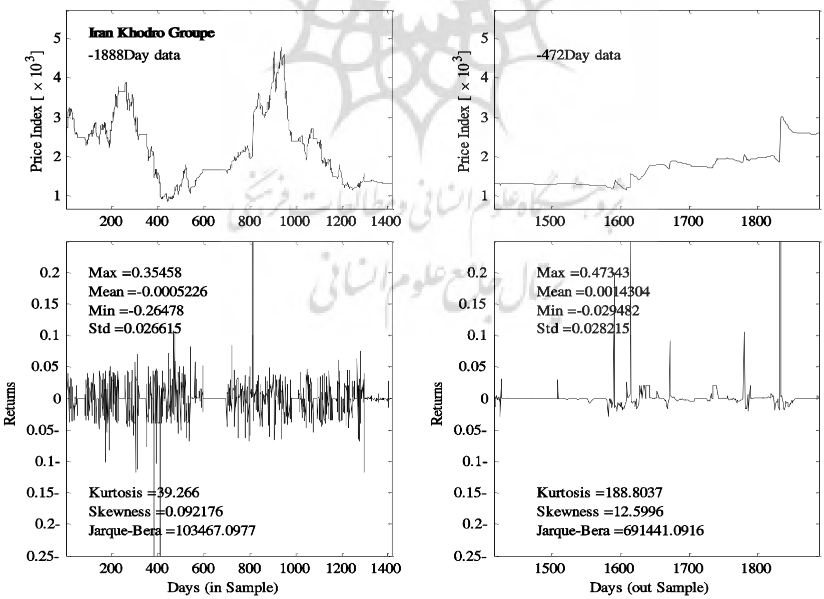
(a)



(b)


Pertol Abadan Groupe
-1888Day data
Price Index [× 10^3]
-472Day data
Max =1.1784
Mean =-0.00010394
Min =-0.30469
Std =0.039865
Kurtosis =544.0318
Skewness =18.0952
Jarque-Bera =23129982.5434
Max =0.34844
Mean =0.003033
Min =-0.043359
Std =0.018718
Kurtosis =250.1925
Skewness =13.8837
Jarque-Bera =1216877.8205
Returns
Days (in Sample)
Days (out Sample)

(c)


Iran Khodro Groupe
-1888Day data
Price Index [× 10^3]
-472Day data
Max =0.35458
Mean =-0.0005226
Min =-0.26478
Std =0.026615
Kurtosis =39.266
Skewness =0.092176
Jarque-Bera =103467.0977
Max =0.47343
Mean =0.0014304
Min =-0.029482
Std =0.028215
Kurtosis =188.8037
Skewness =12.5996
Jarque-Bera =691441.0916
Returns
Days (in Sample)
Days (out Sample)

(d)

According to Table 1, mean returns of all stocks are negative except IT, SS, S and PK companies. Return deviations of all companies are nearly close to each other. Based on skewness statistics, return distribution of SLS and IK companies is nearly symmetric. MSB and TI have negative skewness. Returns frequency curves of other companies are positively skewed. High values of kurtosis measures of all companies indicate that their returns distributions are fat tailed. Also, high values of Jarque–Bera statistics shows that null hypothesis of normality is rejected for return distribution of everyone.

4.2. Portfolio optimization results

In this section, mentioned portfolio optimization problems are computed for above case study with GA and HGAPSO. Solving optimization problems via these two heuristic methods begins with 100 random portfolios as initial population and optimizing process will be ended after 100 iterations. MATLAB Simulation of these algorithms is done in 100 runs. Optimized weights of mentioned companies and four calculated risk criteria through GA and HGAPSO are shown in table 2 and table 3 respectively at confidence levels of 0.95 and 0.99. Note that stocks weights are as percentage.

Table 2: Optimized Results (GA)

	C=0.95				**C=0.99**			
	VaR	**CVaR**	**WVaR**	**PVaR**	**VaR**	**CVaR**	**WVaR**	**PVaR**
W_1	2.319	1.830	2.356	4.382	3.301	2.453	2.625	10.760
W_2	11.334	10.770	9.572	11.765	11.785	7.416	9.454	4.836
W_3	3.350	4.206	6.480	3.040	4.762	6.127	4.316	5.033
W_4	4.765	7.517	7.093	5.569	4.582	8.900	8.086	7.749
W_5	12.411	11.248	9.793	7.531	8.064	12.876	9.213	2.499

	C=0.95				C=0.99			
	VaR	**CVaR**	**WVaR**	**PVaR**	**VaR**	**CVaR**	**WVaR**	**PVaR**
W_6	3.487	2.400	4.174	2.644	3.219	5.575	3.850	6.416
W_7	6.583	3.103	6.893	3.866	6.875	7.162	6.868	9.505
W_8	5.449	5.925	1.525	5.811	7.405	6.027	2.857	2.974
W_9	11.104	11.121	11.581	8.562	9.493	8.846	10.045	10.510
W_{10}	2.440	4.674	4.339	3.035	6.447	2.998	4.502	1.502
W_{11}	5.463	4.983	3.530	10.225	6.175	4.110	5.288	0.884
W_{12}	7.478	6.519	7.308	5.477	3.948	6.470	6.376	3.959
W_{13}	5.287	7.504	6.400	5.216	5.630	3.186	7.412	2.589
W_{14}	2.779	3.685	3.895	2.189	2.820	1.293	3.508	6.852
W_{15}	7.903	5.537	7.245	7.940	9.735	9.006	6.800	9.952
W_{16}	4.555	4.493	4.608	6.311	3.129	3.866	6.722	5.117
W_{17}	3.294	4.485	3.208	6.436	2.629	3.689	2.078	8.862
Risk Value	0.01246	0.01559	0.03250	0.02971	0.01780	0.01993	0.07408	0.05558

Table 3: Optimized Results (HGAPSO)

	C=0.95				C=0.99			
	VaR	**CVaR**	**WVaR**	**PVaR**	**VaR**	**CVaR**	**WVaR**	**PVaR**
W_1	2.830	2.855	2.832	14.493	2.840	2.889	2.721	10.372
W_2	9.778	9.894	9.953	8.261	9.987	9.942	9.776	5.814
W_3	5.430	5.396	5.054	11.322	5.343	5.448	5.128	12.190
W_4	6.351	6.439	6.174	5.331	6.435	6.385	6.169	3.467
W_5	11.803	11.736	11.283	7.971	11.656	11.640	11.316	4.625
W_6	3.553	3.522	3.747	2.852	3.586	3.585	3.699	8.635

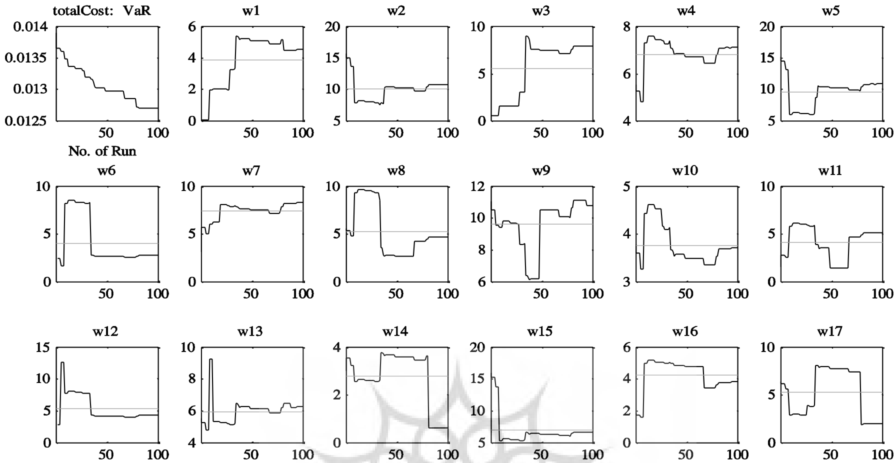
	C=0.95				C=0.99			
	VaR	**CVaR**	**WVaR**	**PVaR**	**VaR**	**CVaR**	**WVaR**	**PVaR**
W_7	6.739	6.683	6.816	5.028	6.666	6.707	6.664	3.556
W_8	4.845	4.855	4.685	4.227	4.841	4.830	4.629	3.700
W_9	9.602	9.547	9.742	7.609	9.590	9.538	9.849	5.242
W_{10}	3.732	3.699	3.571	2.786	3.690	3.773	3.767	5.717
W_{11}	4.239	4.338	4.234	3.193	4.285	4.177	4.163	9.305
W_{12}	6.511	6.519	6.856	5.530	6.516	6.543	7.109	3.747
W_{13}	5.129	5.206	5.693	4.046	5.256	5.199	5.629	2.742
W_{14}	2.686	2.705	2.584	2.309	2.702	2.734	2.633	1.719
W_{15}	8.241	8.082	8.107	6.986	8.132	8.265	8.221	12.062
W_{16}	4.652	4.683	4.768	4.804	4.626	4.517	4.720	4.529
W_{17}	3.878	3.841	3.901	3.254	3.849	3.827	3.805	2.578
Risk Value	0.01229	0.01513	0.03193	0.02926	0.01717	0.01941	0.07231	0.05477

Figure 3 indicates obtained solutions trend using GA and HGAPSO over 100 runs for VAR and PVAR approaches.

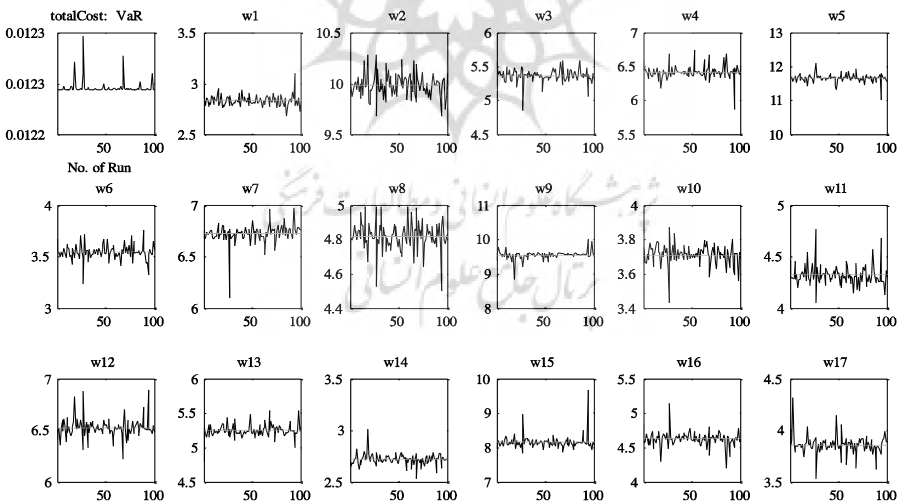
According to results of Tables 2 and 3 and Figure 3, some points can be expressed:

First, according to figure 3 (a and b), convergence process to optimum solutions with GA algorithm is associated with more fluctuation than hybrid algorithm. For example, variation of 1st asset weights over 100 runs via GA is between 0 and 5 but these results through HGAPSO are fluctuated between 2.5 to 3.5. Moreover, the amount of obtained VaR by GA varies between 0.0125 to 0.0135 while this variation is between 0.0122 and 0.0123. These results indicate that, although GA is robust, its convergence speed is low. In fact combining GA with PSO, new algorithm (HGAPSO) considers the GA accuracy as well as PSO convergence speed.

Figure 3: Optimized Weight and c Trends for (a) GA, (b) GA-PSO



(a)



(b)

Second, As it can be observed, optimized weights of 1st asset with GA(or HGAPSO) based on VaR, CVaR ,WVaR and PVaR approaches are 2.31(3.2), 1.83(2.45), 2.35(2.65) and 4.38 (14.49) respectively at 95 percent confidence level. These results are 2.83(2.84), 2.85(2.88), 2.83(2.72) and 14.49(10.37) at 99 percent confidence level. It shows that results are almost the same for VaR, CVaR and WVaR measures but variation of these results are high according to PVaR method. Thus, it can be understood that VaR, CVaR and WVaR approaches are almost close together at both confidence levels using mentioned algorithms. But, related results are different due to auxiliary variable of s and t in PVaR approaches.

Third, as it is clear, by substituting obtained solutions on each objective function, optimized values of four mentioned risk measures are achieved. These computed risk measures are shown in the last row of tables 2 and 3 at two confidence levels of 95 and 99 percent. It is obvious that absolute values of these calculated risk criteria at 99 percent confidence level are higher than ones at 95 percent confidence level. Also, results of risk values ensure $\text{VaR} \leq \text{CVaR} \leq \text{PVaR} \leq \text{WVaR}$ which confirms mathematical and theoretical expressions.

Fourth, absolute values of these risk measures through HGAPSO method are slightly smaller than GA algorithm. Thus, it can be found that hybrid algorithm is more accurate than GA that confirms explanation of previous sections. As mentioned above, although GA has a benefit of robustness and adaptability, its convergence speed is low. So, in addition to considering GA advantages, HGAPSO also enjoys an advantage of high convergence to achieve optimum solution from PSO. In other words, the hybrid algorithm enjoys from accuracy as well as PSO convergence speed.

Finally, in most cases optimized results of the above two tables show that stocks of SS and DI have the highest and lowest shares based on three approaches of VaR, CVaR and WVaR. But because of variability of PVaR approach, these results are different depending on application methods and confidence levels.

4.3. Back-testing

As it is shown in pervious section, in addition to extraction of companies' optimized weights, four risk measures including VaR, CVaR, WVaR and PVaR are calculated for mentioned portfolio at 0.95 and 0.99 confidence levels. For evaluation and comparison of these measures, four following back-testing statistics are applied.

Kupiec's proportion of failure (POF): This test examines equality hypothesis of expected failure rate and actual failure rate. Assuming that each failure occurrence probability is constant, then total exemptions (X) is followed binomial distribution as B (T,). Therefore likelihood ratio statistics of this test which has $\chi^2(1)$ distribution is as follows:

$$LR_{POF} = 2Ln\left[\frac{\widehat{\alpha}^{x}(1-\widehat{\alpha})^{T-x}}{\alpha^{x}(1-\alpha)^{T-x}}\right] \tag{20}$$

Where α is probability level or expected failure rate ($\widehat{\alpha} = \frac{X}{T}$) is actual failure rate and X is number of failures. Indeed, X is number of days when occurred loss is more than estimated loss (risk).

Kupiec's time until first failure (TUFF): it is considered that the number of exemptions has a binomial distribution same as Kupiec's POF test. Null hypothesis of this test is specified as $H_0: \alpha = \widehat{\alpha} = \frac{1}{v}$, where v is the first day of failure occurrence. Likelihood ratio statistics of Kupiec's (TUFF) test also has $\chi^2(1)$ distribution which is presented as:

$$LR_{TUFF} = 2Ln\left[\frac{\frac{1}{v}\left(1-\frac{1}{v}\right)^{v-1}}{\alpha(1-\alpha)^{v-1}}\right] \tag{21}$$

Christoffersen's independence test: This test does not take into account equality of expected and observed failure rates and investigates serial independence of failures. In fact, if risk measure is computed correctly, there is not a correlation between failures. Likelihood ratio statistics of this test is shown as:

$$LR_{IND} = -2Ln\left(\frac{(1-\pi)^{n_{00}+n_{10}}\pi^{n_{01}+n_{11}}}{(1-\pi_0)^{n_{00}}\pi^{n_{01}}(1-\pi)^{n_{10}}\pi^{n_{11}}}\right) \tag{22}$$

$$\pi_0 = \frac{n_{01}}{n_{00}+n_{01}}, \pi_1 = \frac{n_{11}}{n_{10}+n_{11}}, \pi = \frac{n_{01}+n_{11}}{n_{00}+n_{01}+n_{10}+n_{11}}$$

Where $n_{i,j}$ is the number of observations that situation j occurs after situation i. for example, n_{01} is the number of observations that previous day success accompanies with today failure. π_i indicates the probability of occurring conditional exemption on state *i* for the previous day. Note that LR_{IND} is asymptotically χ^2 distributed with one degree of freedom.

Joint test: This test is combination of Kupiec's POF and Christoffersen's independence tests. It evaluates equality of expected and observed failures simultaneous with failures serial independence. Hence, likelihood ratio statistics of this test is obtained by sum of two mentioned statistics as

$$LR_{mix} = LR_{\text{POF}} + LR_{ind} \tag{23}$$

LR_{mix} has χ^2 distribution with two degrees of freedom.

Evaluation of computed risk measures of VaR family is based on the fact that the value of calculated back-testing statistics should be lower than standard distribution ones. Critical values of χ^2 distribution are shown at the end of table 4. For tests of Kupiec's POF, Kupiec's TUFF and Christoffersen's independence, if calculated statistics are lower than 3.84(6.63) at 95 (99) percent confidence levels, four mentioned risk measures are estimated correctly. For joint test, these statistics should be lower than 5.99(9.2).

Table 4: Results of Back-testing Statistics

	C=0.95				C=0.99			
	VaR	**CVaR**	**WVaR**	**PVaR**	**VaR**	**CVaR**	**WVaR**	**PVaR**
LR_{POF}	25.76	29.66	39.89	39.89	0.71981	0.71981	Na	Na
H_0	reject	reject	reject	Reject	accept	accept	Reject	reject
LR_{TUFF}	2.66	8.73	18.40	8.73	0.15207	0.15207	Na	Na
H_0	accept	reject	reject	Reject	accept	accept	Reject	reject
LR_{IND}	0.069	0.038	0.0043	0.0043	0.038544	0.038544	Na	Na
H_0	accept	accept	accept	accept	accept	accept	Reject	reject
LR_{CC}	25.83	29.70	39.90	39.90	0.75835	0.75835	Na	Na
H_0	reject	reject	reject	Reject	accept	accept	Reject	reject

Note that table's critical values of χ^2 distribution is as:

$$\chi^2(0.05,1) = 3.84,\ \chi^2(0.05,2) = 5.99,\ \chi^2(0.01,1) = 6.63,\ \chi^2(0.01,2) = 9.2$$

These back-testing statistics results for mentioned calculated risk measures via optimization are shown in Table 4 at 0.95 and 0.99 confidence levels. Note that although estimated risk measures with GA algorithm are slightly bigger than hybrid method, it does not affect likelihood statistics values. So, these results are related to both algorithms.

Referring to above table and comparison of obtained statistics with standard ones, it can be observed that accuracy of all calculated risk measures is approved at 0.95 confidence level based on Christoffersen's independence test. On the other hand, only VaR is computed correctly subject to Kupiec's TUFF statistic at 0.95 confidence level. Kupiec's POF and Joint tests do not confirm the accuracy of all measures at mentioned confidence level. Increasing confidence level to 0.99, it can be found that based on all tests, VaR and CVaR are measured correctly but WVaR and PVaR values are not valid.

In fact, considering the results of back-testing, it is observed that in most cases, values of likelihood ratio statistics for VaR and CVaR are smaller than

3.84(6.63) at 0.95(0.99) confidence level which imply the accuracy of these two computed risk measures. But, these results for PVaR and WVaR are almost high which do not suggest the validity of these two estimated risk measures. For back-testing analysis, when occurred failures are significantly lower or higher than expected failures, the back-testing statistics are high. According to this empirical result, occurred failures based on PVaR and WVaR are very low at 0.95 confidence level and even zero at 0.99 confidence level. Note that, when no failure occurs, back-testing statistics is not available (Na). So, high values of back-testing statistics for PVaR and WVaR indicates that these two measures compute risk value pessimistically. In short, it is found that PVaR and CVaR overestimate risk value while VaR and CVaR estimate it relatively correct.

5. Conclusion

Unlike the classic portfolio optimization problem in which variance is considered as a risk measure, this study employed optimization approaches based on VaR, CVaR, WVaR and PVaR to find optimized portfolio and calculate four mentioned risk measures. These optimization problems are formed for sample portfolio which consists of stocks of 17 companies of Tehran stock exchange that are solved by GA and HGAPSO heuristic algorithms. Empirical results indicate that obtained solutions via HGAPSO method are more accurate than GA algorithm. For evaluation of computed risk measures, four types of back-testing including Kupiec's POF, Kupiec's TUFF, Christoffersen's independence and joint tests are applied. According to back-testing statistics results, the accuracy of VaR and CVaR estimation via optimization are approved, but it is not confirmed for PVaR and WVaR.

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