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Alternating Direction Explicit Method for a Nonlinear Model in Finance

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ABSTRACT

In this article, at first standard linear Black-Scholes model and then some nonlinear Black-Scholes models will be considered and thereupon alternating direction explicit (ADE) method is applied firstly for solving the standard Black-Scholes model and then for Barles and Soner model which is one of the most complete and comprehensive nonlinear Black-Scholes models. Furthermore, the stability of this method has been considered and its accuracy will be compared with other numerical methods such as finite difference methods. Since in solving nonlinear Black-Scholes models by the ADE methods, we need to solve only some scalar nonlinear equations instead of a full nonlinear system of equations that we should solve in implicit methods, so this method can be a suitable choice for solving such models.

1 Introduction

Standard Black-Scholes-Merton model is a linear partial differential equation which is introduced in 1973 by Fischer Black and Myron Scholes [3] and earlier by Robert Merton [16] for financial derivatives pricing such as options. This model is also called the Black-Scholes model. An option is a contract that gives the holder of the option a right (not obligation) to buy (call option) or sell (put option) a stock at a fixed price (called an exercise price or a strike price), at a fixed date (called the expiry date or maturity date). Myron Scholes and Robert Merton were awarded the Nobel Prize in 1973 for their remarkable work when unfortunately, Fisher Black passed away two years earlier. The standard Black-Scholes equation is assumed in a complete market where some parameters such as illiquid market, transaction cost and large investor performance were not taken into account.
In recent years several nonlinear Black-Scholes models have been introduced that considered one or some of these parameters to give a more accurate model for derivative security pricing. For more details refer to [1, 2, 8, 10, 11, 14]. One of the most accurate nonlinear Black-Scholes models called Barles and Soner model [2] will be considered in this work. Nonlinearity in the nonlinear Black-Scholes models arises from a nonlinear volatility function which not only depends on time t and underlying asset price S but also on the Greek Gamma that is the second derivative of the option price $V(S,t)$ with respect to S while in the classic Black-Scholes model the volatility of the underlying asset was assumed constant. However, the linear Black-Scholes equation has an analytical solution but the nonlinear Black-

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Scholes equations do not have. Therefore, we should look for numerical methods for solving such non-linear equations to find an approximation for their solutions. Several numerical methods have been applied for dealing with different nonlinear models such as the upwind finite difference method [12], a positivity-preserving scheme [9], fourth-order semi-discretization [5] and [7], standard and nonstandard finite difference methods [15], alternative direction implicit (ADI) scheme [6] and alternative direction explicit (ADE) scheme [4]. In the next section, the Barles and Soner model will be considered then, in Section 3, the ADE method will be applied for dealing with the Barles and Soner model. This scheme has been applied before for solving the linear Black-Scholes equation and Frey and Patie nonlinear model in [4] but to our knowledge, the efficiency of this method for the Barles and Soner model has not been considered yet. Therefore, in Section 4, the Barles and Soner model will be solved by the ADE method and compared with some other numerical methods. Finally, in the last section, some conclusions have been demonstrated.

2 Linear and Nonlinear Black-Scholes Models

A nonlinear Black-Scholes model has the following form:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\left(S,t,\frac{\partial^2 V}{\partial S^2}\right)S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,\ (S,t)\in(0,S_{max})\times[0,T] \quad (1)$$

$$V(S,T) = \begin{cases} \max\{S-K,0\} & ,call\ option \\ \max\{K-S,0\} & ,put\ option \end{cases} \quad (2)$$

$$V(0,t) = \begin{cases} 0 & ,call\ option \\ Ke^{-r(T-t)} & ,put\ option \end{cases} \quad (3)$$

$$V(S_{\max},t) = \begin{cases} S_{\max} - Ke^{-r(T-t)} & ,call\ option \\ 0 & ,put\ option \end{cases} \quad (4)$$

where two-variable function $V(S,t)$ is the value or price of the option for a value S of the underlying risky asset at time t. r,K and S_{max} are respectively the interest rate, strike price and the upper bound of S domain and σ is the volatility on an underlying risky asset which is assumed constant in the classic linear Black-Scholes model while in a nonlinear Black-Scholes model is a function of S,t and $\vartheta = \partial^2 V/\partial S^2$. Several nonlinear Black-Scholes models have been introduced in recent years such as the Leland model [11], Risk-Adjusted Pricing Methodology(RAPM) [10], Barles and Soner [2], Feedback and illiquid market [8], Parameterized Illiquid Model [1] where took into accounts one or some parameters such as illiquid market, transaction cost and large investor performance. Since the Barles and Soner model is one the most comprehensive nonlinear models for option pricing, this model will be considered in the rest of this work. By the way as most of the nonlinear models do not have analytical solutions therefore, we must solve them by numerical methods to achieve approximations for their solutions. While the linear Black-Scholes equation has the following analytical solution:

$$V(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \quad (5)$$

Where

$$d_{1,2} = \frac{\log\frac{S}{K} + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad (6)$$

and $N(.)$ indicates the standard joint normal distribution. In fact this $V(S,t)$ is the price of a European call option (sometimes is shown by $C+S,t,$) which is the solution of the linear form of (1) with constant volatility (ϖ). By put-call parity the European put option price will be computed as follow:

$$S\;.\;\;P(S,t)\,0\,C(S,t)\cong Ke^{0r(T0t)} \tag{7}$$

where $P(S,t)$ is the put option price.

2.1 The Barles and Soner Nonlinear Volatility Model

Barles and Soner [2] considered both the transaction cost and the risk from volatile portfolios. They took an approach based on utility maximization which results in the following adjustment of the volatility:

$$\varpi^2(t,S,V_{SS})\cong\varpi_0^2\Big[1.\;\;\therefore\Big(e^{r(T0t)}\pi^2RS^2\frac{\div^2V}{\div S^2}\Big)\Big] \tag{8}$$

where ϖ_0 is the original volatility constant, π is the Leland transaction cost [11] and R is a risk aversion factor. Finally, $\therefore(x)$ is the solution of the following nonlinear ODE:

$$\therefore\!/(x)\cong\frac{\therefore(x)\;.\;1}{2\sqrt{x\therefore(x)}\;0\,x},\qquad x\approx 0 \tag{9}$$

with the initial condition $\therefore(0)\cong 0$. We have used Maple's "fsolve" command to find specific values of $\therefore$. The implicit exact solution takes the form

$$\sqrt{|x|}\cong\begin{cases}\dfrac{0\sinh^{01}(\sqrt{\therefore(x)})}{\sqrt{\therefore(x)\;.\;1}}\;.\;\sqrt{\therefore(x)} & \text{for } x\text{ A}0\\[2ex] \dfrac{0\sinh^{01}(\sqrt{0\therefore(x)})}{\sqrt{\therefore(x)\;.\;1}}\;0\sqrt{0\therefore(x)} & \text{for } x\;?\;0\end{cases} \tag{10}$$

and it is shown in [5] that

$$\therefore(x)\angle\begin{cases}(0,\clubsuit) & \text{for } x\text{ A}0\\ (01,0) & \text{for } x\;?\;0\end{cases}\quad\text{and}\quad \therefore\!/(x)\text{ A}0 \text{ for } x\approx 0. \tag{11}$$

In appendix A of [2] the existence of a unique continuous viscosity solution to this problem has been shown.

3 Alternating Direction Explicit Schemes

Here different kinds of ADE schemes will be considered. These methods consists of two explicit substeps. One step is constructed from lower boundary to upper one which is called upward and another step vice versa which is called downward. Suppose we discretize the interval $[0,S_{max}]$ to J equidistant subinterval, hence the step size of this discretization is $h\cong S_{max}/J$ and the nodal points are $S_j\cong j\partial h$

for $j \cong 0,1,...,J$. Similarly the time interval $[0,T]$ will be discretized to N equidistant subinterval with step size $k \cong T / N$ with the nodal points $t_n \cong n \partial k$ for $n \cong 0,1,...,N$. Suppose u^n and d^n are the numerical solutions of the Black-Scholes equation where is solved upward and downward respectively. Therefore, the value of the ADE method is the average of these two values i.e. $v^n \cong (u^n \,.\, d^n)/2$, where v^n is the numerical approximation of the option value $V(S,t_n)$ at time t_n .Applying different approximations for the convection term in a partial differential equation causes different ADE schemes. For instance, Towler and Yang [19] used central differences for an approximation of the convection term as follows:

$$\frac{\div v(S_j,t_{n.\frac{1}{2}})}{\div S} \,|\, \frac{u^n_{j.1} \,0\, u^{n.1}_{\text{þ}1}}{2h}, \qquad j \cong 1,...,J\,0\,1,$$
$$\frac{\div v(S_j,t_{n.\frac{1}{2}})}{\div S} \,|\, \frac{d^{n.1}_{j.1} \,0\, d^{n}_{\text{þ}1}}{2h}, \qquad j \cong J\,0\,1,...,1. \qquad (12)$$

More accurate approximation is introduced by [17] and [18] as follows:

$$\frac{\div v(S_j,t_{n.\frac{1}{2}})}{\div S} \,|\, \frac{u^n_{j.1} \,0\, u^n_j \,.\, u^{n.1}_j \,0\, u^{n\ 1}_{\text{þ}1}}{2h}, \qquad j \cong 1,\dots,J\,0\,1,$$
$$\frac{\div v(S_j,t_{n.\frac{1}{2}})}{\div S} \,|\, \frac{d^{n.1}_{j.1} \,0\, d^{n\ 1}_j \,.\, d^n_j \,0\, d^n_{\text{þ}1}}{2h}, \qquad j \cong J\,0\,1,\dots,1. \qquad (13)$$

Where, the last discretization for the convection term have been implemented in the next section. For the partial derivative concerning to time in both upward and downward steps following approximation will be used:

$$\frac{\div v(S_j,t_{n.\frac{1}{2}})}{\div t} \,|\, \frac{u^{n.1}_j \,0\, u^n_j}{k}, \qquad n \cong 1,\dots,N\,0\,1,$$
$$\frac{\div v(S_j,t_{n.\frac{1}{2}})}{\div t} \,|\, \frac{d^{n.1}_j \,0\, d^n_j}{k}, \qquad n \cong 1,\dots,N\,0\,1 \qquad (14)$$

and for discretization of the diffusion term:

$$\frac{\div^2 v(S_j,t_{n.\frac{1}{2}})}{\div^2 S} \,|\, \frac{u^n_{j.1} \,0\, u^n_j \,0\, u^{n.1}_j \,.\, u^{n\ 1}_{\text{þ}1}}{h^2}, \qquad j \cong 1,\dots,J\,0\,1\,,$$
$$\frac{\div^2 v(S_j,t_{n.\frac{1}{2}})}{\div^2 S} \,|\, \frac{d^{n.1}_{j.1} \,0\, d^{n\ 1}_j \,0\, d^n_j \,.\, d^n_{\text{þ}1}}{h^2}, \qquad j \cong J\,0\,1,\dots,1. \qquad (15)$$

To obtain a symmetric scheme, following approximations for reaction term have been used:

$$v(S_j,t_{n.\frac{1}{2}}) \,|\, \frac{u^{n.1}_j \,.\, u^n_j}{2}, \qquad j \cong 1,\dots,J\,0\,1, \qquad (16)$$

$$v(S_j, t_{n\cdot\frac{1}{2}}) \mid \frac{d_j^{n\cdot 1} \cdot d_j^{n}}{2}, \qquad j \cong J\,0\,1,\ldots,1.$$

By substituting the discretizations (13)-(16) in (1) and the central approximation $V_{SS}(S_j, t_n) \mid \frac{v_{j\cdot 1}^n \,0\, 2v_j^n \cdot v_{j0\,1}^n}{h^2}$ in the Barles and Soner volatility function (8) for both the upward and the downward steps, for every $n \cong 0,\ldots, N\,0\,1$, two systems of equations will be obtained as follows:

$$\begin{aligned} &\text{Upward}: && A_u u^{n+1} = B_u v^n, \\ &\text{Downward}: && A_d d^{n\cdot 1} \cong B_d v^n, \\ &\text{ADE}: && v^{n\cdot 1} \cong \frac{u^{n\cdot 1} \cdot d^{n\ 1}}{2}. \end{aligned} \tag{17}$$

Where A_u and B_d are the lower and B_u and A_d are the upper triangular matrices of size $J\,0\,1$ with the following diagonal elements for $j \cong 1,\ldots, J\,0\,1$:

$$\begin{aligned} A_u[j,j] &\cong 1\cdot \frac{1}{2}\varpi_0^2 kj^2(1\cdot \therefore(j,n))\,0\,\frac{rjk}{2}\cdot\frac{rk}{2}, \\ B_u[j,j] &\cong 1\,0\,\frac{1}{2}\varpi_0^2 kj^2(1\cdot \therefore(j,n))\,0\,\frac{rjkj}{2}\,0\,\frac{rk}{2}, \end{aligned} \tag{18}$$

$$\begin{aligned} A_d[j,j] &\cong 1\cdot \frac{1}{2}\varpi_0^2 kj^2(1\cdot \therefore(j,n))\cdot\frac{rkj}{2}\cdot\frac{rk}{2}, \\ B_d[j,j] &\cong 1\,0\,\frac{1}{2}\varpi_0^2 kj^2(1\cdot \therefore(j,n))\cdot\frac{rkj}{2}\,0\,\frac{rk}{2}, \end{aligned}$$

where $\therefore(j,n) \cong \therefore + e^{r(T\,0\,t_n)}\pi^2 Rj^2(v_{j\cdot 1}^n\,0\,2v_j^n \cdot v_{j0\,1}),$. Since in this ADE scheme, the symmetric discretizations of convection, diffusion and reaction terms have been implemented, the scheme is unconditionally stable for the linear Black-Scholes equation, for more details refer to [4] and [13]. Here we will show that this scheme is conditionally stable for the nonlinear Barles and Soner Black-Scholes equation. It is sufficient to show that under some condition, the spectral radius of the matrix of the system of equations is less than one. i.e.

$$\upsilon \cong max\{| o_j |, j \cong 1,\cdots, J\,0\,1\}\ ?\ 1. \tag{19}$$

The system of equations in the upward and downward steps are $u^{n\cdot 1} \cong A_u^{0\,1}B_u v^n$ and $d^{n\cdot 1} \cong A_d^{0\,1}B_d v^n$ respectively, where both $A_u^{01}B_u$ and $A_d^{01}B_d$ are the tridiagonal matrices. Also, the eigenvalues of the lower diagonal matrices A_u^{01} and B_d, and the upper diagonal matrices B_u and A_d^{01} are their diagonal elements. Furthermore, the spectral radius of a product of matrices is less than or equal to the product of their spectral radiuses, then we have:

$$\begin{aligned} \upsilon_u &\propto \upsilon_{A_u^{01}} \wedge \upsilon_{B_u} \propto \max\left\{\left|\frac{B_u[j,j]}{A_u[j,j]}\right|, j \cong 1,\cdots, J\,0\,1\right\}, \\ \upsilon_d &\propto \upsilon_{A_d^{01}} \wedge \upsilon_{B_d} \propto \max\left\{\left|\frac{B_d[j,j]}{A_d[j,j]}\right|, j \cong 1,\cdots, J\,0\,1\right\}. \end{aligned} \tag{20}$$

From (11) we have $1.\ \therefore(x)\ \text{A}0$ for every x, therefore (18) implies that

$$\frac{B_u[j,j]}{A_u[j,j]}\ ?\ 1, \qquad j \cong 1,\cdots,J\ 0\,1.$$
$$\frac{B_d[j,j]}{A_d[j,j]}\ ?\ 1, \qquad j \cong 1,\cdots,J\ 0\,1. \tag{21}$$

Now we check under which condition the above fractions is greater than one for every $j \cong 1,...,J\ 0\,1.$

$$0\,1\ ?\ 0\,1.\ \frac{2\ 0\ rjk}{1.\ \frac{1}{2}\varpi_0^2 kj^2(1.\ \ \therefore(j,n))\ 0\ \frac{rjk}{2}.\ \frac{rk}{2}}.$$

$$0\,1\ ?\ \frac{B_u[j,j]}{A_u[j,j]} \Uparrow\ \ 0\,1\ ?\ \frac{1\,0\ \frac{1}{2}\varpi_0^2 kj^2(1.\ \ \therefore(j,n))\ 0\ \frac{rjkj}{2}\ 0\ \frac{rk}{2}}{1.\ \frac{1}{2}\varpi_0^2 kj^2(1.\ \ \therefore(j,n))\ 0\ \frac{rjk}{2}.\ \frac{rk}{2}} \Uparrow \tag{22}$$

Therefore, the upward step is conditionally stable if

$$0\ ?\ \frac{2\ 0\ rjk}{1.\ \frac{1}{2}\varpi_0^2 kj^2(1.\ \ \therefore(j,n))\ 0\ \frac{rjk}{2}.\ \frac{rk}{2}}, \qquad j \cong 1,...,J\ 0\,1. \tag{23}$$

Similarly, the downward step is stable if

$$0\,1\ ?\ 0\,1.\ \frac{2.\ rjk}{1.\ \frac{1}{2}\varpi_0^2 kj^2(1.\ \ \therefore(j,n)).\ \frac{rjk}{2}.\ \frac{rk}{2}}\ ?\ 1, \qquad j \cong 1,...,J\ 0\,1. \tag{24}$$

Since the above inequalities satisfy for every $j \cong 1,\ldots,J\ 0\,1$ and the time step k, hence the downward step is unconditionally stable and finally, the combination of the upward and the downward steps i.e. the ADE scheme is conditionally stable.

4 Numerical Solution for the Linear and Nonlinear Black-Scholes Equation

Here at first the standard linear Black-Scholes equation has been solved by the ADE scheme to show its efficiency for solving linear partial differential equations. Consider an European put option with parameters $\varpi \cong 0.2$, $r \cong 0.03$, $T \cong 1$, $K \cong 30$ and $S_{\max} \cong 90$. Table 1 shows the maximum error of the ADE scheme for finding the European put option price which is the solution of the linear Black-Scholes equation with different step sizes but with the same mesh ratio ($\frac{k}{h^2} \cong 0.0025$). This example as we expected shows the smaller step sizes in discretization cause more accurate solutions and then less numerical errors. Furthermore Fig. 1 and Fig. 2 indicate that maximum error occur around the strike price, which is assumed $K \cong\ 30$ here and its reason is the non-smooth final condition in this point. Now the Barles and Soner nonlinear Black-Scholes model will be solved by the ADE scheme and the accuracy of this method has been compared with two other finite difference methods.

Since the Barles and Soner model does not have any analytical solution, to evaluate the accuracy of the numerical methods and calculate the numerical errors, a solution with tiny step size will be considered as a reference solution and the difference absolute value of other solutions with the reference indicate approximation errors.

Table 1: Maximum Error of the ADE in the Last Iteration ($t \cong 0$) for linear B-S

asset step size (h)	time step size (k)	max error of ADE
4	0.04	0.0507025
2	0.01	0.022887
1	0.0025	0.00600658
0.5	0.000625	0.00245572
0.25	0.00015625	0.00119212

In the following, the price of a European put option in the Barles and Soner model has been computed by the ADE scheme with these parameter $\varpi_0 \cong 0.2$, $r \cong 0.1$, $T \cong 1$, $K \cong 40$, $S_{\max} \cong 80$ and the transaction cost $a \cong \pi^2 R \cong 0.02$. The reference solution is achieved with step sizes $h \cong 0.375$ for discretization in S and $k \cong 0.00009765625$ in time t i.e.. $J \cong 215$ and $N \cong 10240$ which is computed in 33 hours by a computer with 2.9 GHz Intel Core i5 and memory 8 GB.
S and t have been halved in every iteration so, the convergence order in the method will be achieved by $2^q \cong \frac{|e_h \, 0 \, e_{h/2}|}{|e_{h/2} \, 0 \, e_{h/4}|}$. The last column of Table 2 indicates by reducing the step sizes the order of convergency is approximately 2.

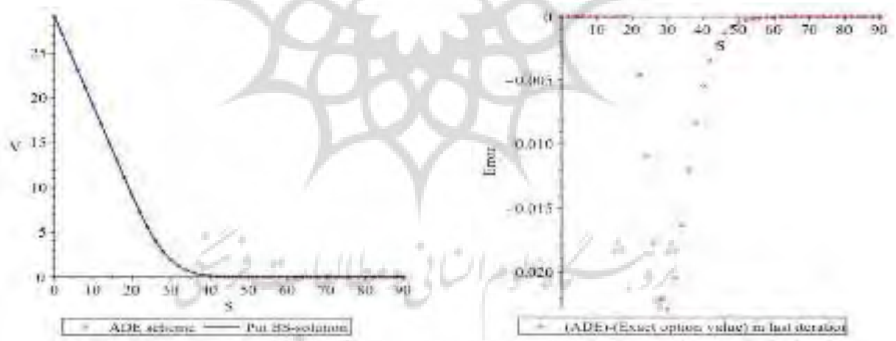

Fig. 1: Put Option Price (Left Figure) and its Numerical Errors (Right Figure) at $t \cong 0$ with $h \cong 2$ and $k \cong 0.01$.

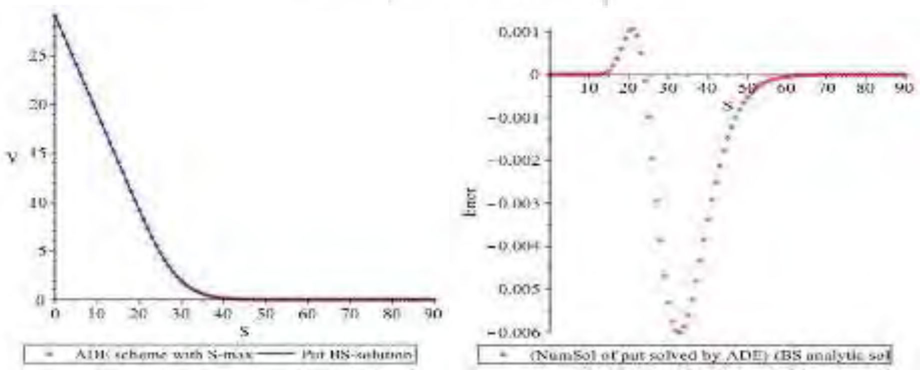

Fig. 2: Put Option Price (Left Figure) and its Numerical Errors (Right Figure) at $t \cong 0$ with $h \cong 1$ and $k \cong 0.0025$

Table 2: Maximum Error in the Last Iteration ($t \cong 0$) of the ADE for the Barles and Soner Model

asset step size (h)	time step size (k)	max error of ADE	error difference	error ratio
8	0.003125	0.15197770		
4	0.0015625	0.04869464	0.10328306	
2	0.00078125	0.01359658	0.04791339	2.94
1	0.000390625	0.00327254	0.01032404	3.40
0.5	0.0001953125	0.00037647	0.00289607	3.56

Figures 3, 4 and 5 show the approximation errors of the European put option in the nonlinear Barles and Soner model which is solved by the ADE scheme with different step sizes in comparison with the reference solution with $h \cong 0.375$ and $k \cong 0.00009765625$. Similar to the linear Black-Scholes model, the figures of the approximations errors indicate the maximum error in the Barles and Soner model and generally in the nonlinear Black-Scholes models takes place close to the strike price K.

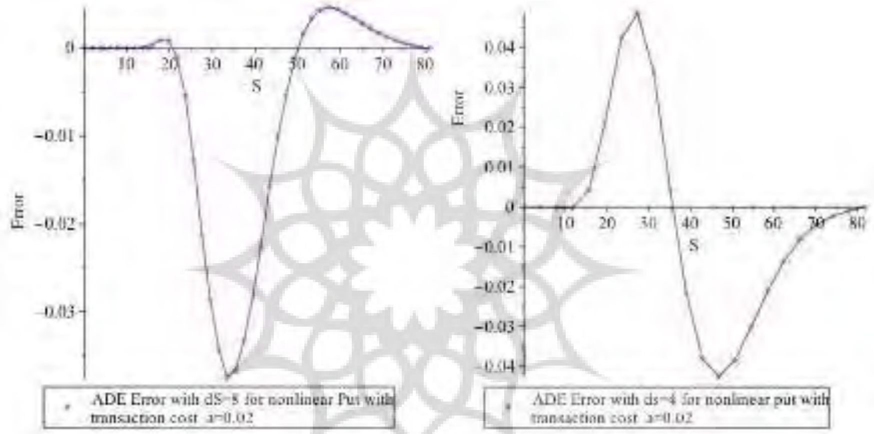

Fig. 3: Approximation Error of the Barles and Soner Put Option at $t \cong 0$ with $h \cong 8$ and $k \cong 0.003125$ (Left Figure) and $h \cong 4$ and $k \cong 0.0015625$ (right figure).

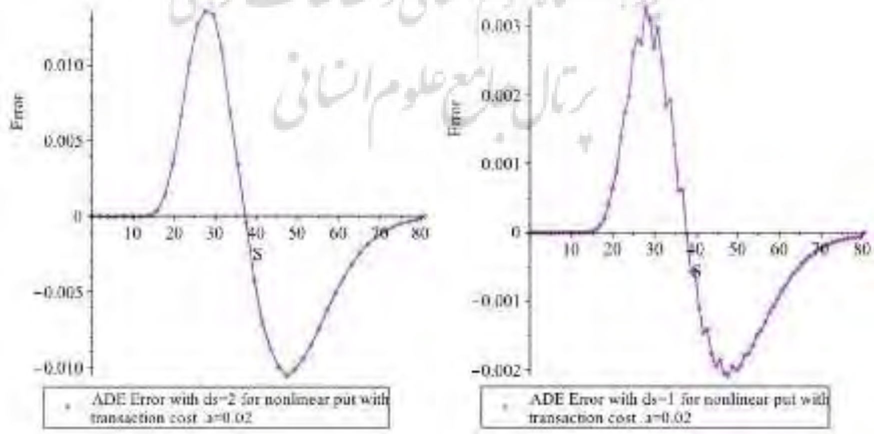

Fig. 4: Approximation Error of the Barles and Soner Put Option at $t \cong 0$ with $h \cong 2$ and $k \cong 0.00078125$ (left figure) and $h \cong 1$ and $k \cong 0.000390625$ (right figure).

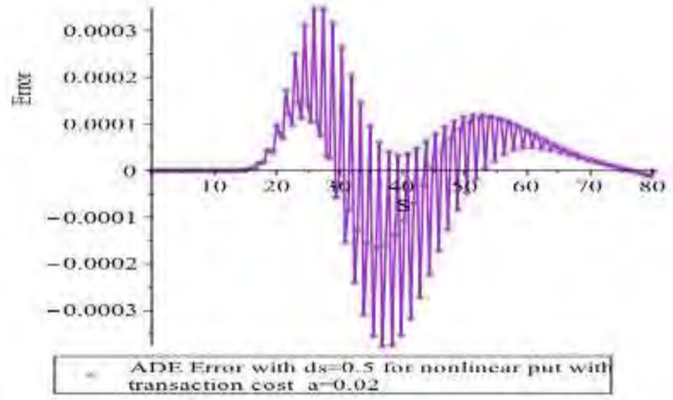

Fig. 5: Approximation Error of the Barles and Soner Put Option at $t \cong 0$ with $h \cong 0.5$ and $k \cong 0.0001953125$

Now the accuracy of the ADE scheme for solving the Barles and Soner model has been compared with two standard finite difference methods in Table 3 such as Crank-Nicolson with Rannacher time stepping (CNR) and forward Euler (explicit scheme with the first order difference in time and central second order in S (FtCS)) which indicates that however for the coarse meshes the ADE method has bigger error than other two methods but by halving step sizes the maximum error of the ADE method decreases more rapidly than CNR and FtCS methods and finally we can see for the finest mesh in this table, the ADE method has smaller maximum error than two others.

Table 3: Maximum Error in the last Iteration ($t \cong 0$) of the Barles and Soner Model

J	N	ADE error	CNR error	FtCS error
10	320	0.15197770	0.127837	0.126505
20	640	0.04869464	0.032952	0.032240
40	1280	0.01359658	0.009476	0.009176
80	2560	0.00327254	0.002836	0.002706
160	5120	0.00037647	00.001026	0.000986

5 Conclusions

In this work, the Barles and Soner nonlinear Black-Scholes model has been solved with the ADE method and compared with two standard finite difference schemes CNR and FtCs. We demonstrated the ADE scheme for the considered nonlinear model has the second order of convergency and conditionally stable and also is more accurate than CNR and FtCS methods for the finer meshes. Therefore the ADE method would be a suitable choice for solving such nonlinear equations to avoid suffering from computational times of implicit schemes.

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