Hedging of Options in Jump-Diffusion Markets with Correlated Assets

*Minoo Bakhshmohammadlou**

Iran University of Science and Technology, Tehran, Iran

ABSTRACT

We consider the hedging problem in a jump-diffusion market with correlated assets. For this purpose, we employ the locally risk minimizing approach and obtain the hedging portfolio as a solution of a multidimensional system of linear equations. This system shows that in a continuous market, independence and correlation assumptions of assets lead to the same locally risk minimizing portfolio. In addition, we investigate the sensitivity of the risk with respect to the variation of correlation parameters, this enables us to select the more profitable portfolio. The results show that the risk increases, with increasing the correlation parameters. This means that to reduce risk it is necessary to invest in low correlated assets.

1 Introduction

The stochastic processes with jump are very popular models for description of market fluctuations. The hedging problem in such environment is a very challenging issue in financial mathematics, because the market driven by these processes is incomplete; this means that associated with any hedging portfolio there is an unchangeable (residual) risk. Therefore, it is necessary to evaluate this risk and then try to reduce it. For this purpose, two properties of portfolio are very important: self-financing and admissibility. The local risk minimization (LRM) and the mean-variance (MV) are two major quadratic methods for hedging, which the former one focuses on the admissibility and the later one emphasizes on the self-financing property, respectively, for more approaches refer to chapter 10 of [1]. In both hedging strategies assets are assumed to be independent, while in real-word economy we deal with correlated assets that move up and down together. Therefor we can say that this assumption is very restrictive and does not correspond to what happens in real markets.

The correlation between assets and its impacts on investment and hedging are extremely important in financial mathematics, see [2-8]. Events such as wars, natural disasters and market crashes, more than ever reveal the significance of this matter. In a real-word economy, the optimal investment in an asset is dependent on not only its behavior but also the behavior of other assets that correlated with it. In this text we consider the hedging problem in a jump-diffusion market with m correlated assets. Since

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* Corresponding author. Tel.: +989102050565
E-mail address: *m.bakhshmohamadlou@gmail.com*

the LRM portfolio has an explicit form, we apply this approach and obtain the LRM portfolio as solution of a m-dimensional system of linear equations. We conclude that in a continuous word when the jumps are limited to 0, both assumptions of independence and correlation of assets lead to a same hedging portfolio. In addition, we study the behavior of the residual risk with respect to correlation parameter. This work helps us to choose the more profitable portfolio. The results show that the risk increases strictly, with increasing the correlation parameter. Consequently, in order to reduce risk, we must avoid investment in high correlated assets. This confirms the results of Markowitz [2] for portfolio selection on actual data.

The structure of this paper is as follows: Section 2 introduces our setup market. Section 3 obtains the LRM portfolio. Finally section 4 studies the sensitivity of variance of the residual risk with respect to perturbations of the correlation parameter for risk management.

2 Set Up

Our market is equipped with a complete filtered probability space $(\Omega, \mathcal{F}, P)$ where P is a probability measure and $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ is a flow of information produced by the price processes at time t. In this world, the bond and stock price processes are described as follows:

$$dB(t) = B(t)r(t)dt$$
$$dS_i(t) = S_i(t^-)\left(r(t)dt + \sigma_i(t)dW_i(t) + \int_{\square_0} x_i \tilde{N}_i(dt,dx)\right), \tag{1}$$

for $i = 1, \cdots, m$, respectively, where $W(t)$ is the P-standard Brownian motion, $\tilde{N}(dt,dx)$ is the P-compensated Poisson random measure and the jump intensity is $\zeta(dx)dt$ on $\square_0 := R \setminus \{0\}$. Also, we suppose that the stock price processes are correlated, i.e., $d<W_i, W_j>_t = \rho_{ij}dt$ for $i, j = 1, ..., m$, $i \neq j$ and $0 \leq \rho_{ij} \leq 1$.

Definition 1. *Portfolio is a* $\mathcal{F}_t$-*predictable process* $\psi(t) = \left(\kappa(t), \theta_1(t), \cdots, \theta_m(t)\right)$, *where* $\kappa(t)$ *and* $\theta_i(t)$, $i = 1, \cdots, m$, *notate the number of shares of the bond and i th stock, respectively, and* $\theta_i \in \Phi = \left\{\phi; E\left(\int \phi_i^2 d\langle S_i \rangle\right) < \infty, \ i = 1, \cdots, m\right\}$.

The value of the portfolio $\psi(t)$ is defined by $V_\psi(t) = \kappa(t)B(t) + \sum_{i=1}^{m} \theta_i(t)S_i(t)$ and the associated cost process is $C_\psi(t) = V_\psi(t) - \sum_{i=1}^{m} \int_0^t \theta_i(u)dS_i(u)$.

As well as by imposing a the Markov structure on the price process, we have

$$V_t = E\left[H\left(S_1(T), \cdots, S_m(T)\right) \middle| \mathcal{F}_t\right]$$
$$= E\left[H\left(S_1(T), \cdots, S_m(T)\right) \middle| \mathbf{s} = \left(S_1(t) = s_1, \cdots, S_m(t) = s_m\right)\right]$$
$$= E^s\left[H\left(S_1(T-t), \cdots, S_m(T-t)\right)\right] = \mathcal{V}(t, s_1, \cdots, s_m) = \mathcal{V}(t, \mathbf{s}),$$

for a contingent claim $H \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$. Then by Ito's lemma we get

$$
\begin{aligned}
d\mathcal{V} &= \frac{\partial \mathcal{V}}{\partial t} dt + \sum_{i=1}^{m} \frac{\partial \mathcal{V}}{\partial s_i} \Big(S_i(t) r(t) dt + S_i(t) \sigma_i(t) dW_i(t) \Big) \\
&+ \frac{1}{2} \sum_{i=1}^{m} \frac{\partial^2 \mathcal{V}}{\partial s_i^2} S_i^2(t) \sigma_i^2(t) dt + \frac{1}{2} \sum_{i,j=1, i \neq j}^{m} \frac{\partial^2 \mathcal{V}}{\partial s_i \partial s_j} S_i(t) S_j(t) \sigma_i(t) \sigma_j(t) \rho_{ij} dt \\
&+ \sum_{i=1}^{m} \int_{\square_0} \Big(\mathcal{V}(t, s_i(1+x_i)) - \mathcal{V}(t, s_i) \Big) \tilde{N}_i(dt, dx) \\
&+ \sum_{i=1}^{m} \int_{\square_0} \Big(\mathcal{V}(t, s_i(1+x_i)) - \mathcal{V}(t, s_i) - \frac{\partial \mathcal{V}}{\partial s_i} S_i(t) x_i \Big) \zeta_i(dx) dt.
\end{aligned} \tag{2}
$$

3 The LRM Portfolio

Definition 2. ([9,10]) *An admissible strategy is "Locally Risk Minimizing" if the associated cost process is a square-integrable P-martingale and orthogonal to the P-martingale part of the stock price processes S_i, $i = 1, \cdots, m$.*

Proposition 1. Suppose that $\{S_t\}_{t \in [0,T]}$ satisfies the stochastic differential equation 1 Then the LRM portfolio is the solution of a m-dimensional system of linear equations $M\Theta = L$, where

$$
M = \begin{pmatrix}
s_1 \sigma_1^2 + \int_{\square_0} s_1 x_1^2 \zeta_1(dx) & \rho_{12} s_2 \sigma_2 \sigma_1 & \cdots & \rho_{1m} s_m \sigma_m \sigma_1 \\
\rho_{12} s_1 \sigma_1 \sigma_2 & s_2 \sigma_2^2 + \int_{\square_0} s_2 x_2^2 \zeta_2(dx) & \cdots & \rho_{2m} s_m \sigma_m \sigma_2 \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{1m} s_1 \sigma_1 \sigma_m & \rho_{2m} s_2 \sigma_2 \sigma_m & \cdots & s_m \sigma_m^2 + \int_{\square_0} s_m x_m^2 \zeta_m(dx)
\end{pmatrix}_{m \times m},
$$

$$
\Theta = (\theta_1, \theta_2, \cdots, \theta_m)^T,
$$

and

$$
L = \begin{pmatrix}
\frac{\partial \mathcal{V}}{\partial s_1} s_1 \sigma_1^2 + \int_{\square_0} x_1 [\mathcal{V}(t, s_1(1+x_1)) - \mathcal{V}(t, s_1)] \zeta_1(dx) + \sum_{j \neq 1} \frac{\partial \mathcal{V}}{\partial s_j} s_j \sigma_j \sigma_1 \rho_{1j} \\
\frac{\partial \mathcal{V}}{\partial s_2} s_2 \sigma_2^2 + \int_{\square_0} x_2 [\mathcal{V}(t, s_2(1+x_2)) - \mathcal{V}(t, s_2)] \zeta_2(dx) + \sum_{j \neq 2} \frac{\partial \mathcal{V}}{\partial s_j} s_j \sigma_j \sigma_2 \rho_{2j} \\
\vdots \\
\frac{\partial \mathcal{V}}{\partial s_m} s_m \sigma_m^2 + \int_{\square_0} x_m [\mathcal{V}(t, s_m(1+x_m)) - \mathcal{V}(t, s_m)] \zeta_m(dx) + \sum_{j \neq m} \frac{\partial \mathcal{V}}{\partial s_j} s_j \sigma_j \sigma_m \rho_{mj}
\end{pmatrix}_{m \times 1}.
$$

Proof.

From definition 2, the cost process of a LRM portfolio Θ is orthogonal to the P-martingale part of the stock price process, i.e., $MP_i = S_i(t^-)\left(\sigma_i(t)dW_i(t) + \int_{\square_0} x_i \tilde{N}_i(dt,dx)\right)$. Then we have

$$0 = d\langle C, MP_i\rangle_t = d\langle \mathcal{V}, MP_i\rangle_t - d\left\langle \sum_{i=1}^m \theta_i dS_i, MP_i\right\rangle_t,$$

for $i = 1,2,...,m$. Substituting the price process and equation 2 into above equation, we get

$$d\langle C, M_i\rangle_t = \sum_{i=1}^m \int_0^t \left(S_i^2(u)\sigma_i^2(u)\left(\frac{\partial \mathcal{V}}{\partial s_i} - \theta_i(u)\right) + \sum_{j=1,j\neq i}^m S_i(u)S_j(u)\sigma_j(u)\sigma_i(u)\left(\frac{\partial \mathcal{V}}{\partial s_j} - \theta_j(u)\right)\rho_{ij}\right.$$
$$\left. + \int_{\square_0} S_i(u)x_i\left[\mathcal{V}(u,s_i(1+x_i)) - \mathcal{V}(u,s_i) - \theta_i(u)S_i(u)x_i\right]\zeta_i(dx)\right)dt = 0.$$

Therefore

$$s_i\sigma_i^2\left(\frac{\partial \mathcal{V}}{\partial s_i} - \theta_i\right) + \int_{\square_0} x_i\left[\mathcal{V}(t,s_i(1+x_i)) - \mathcal{V}(t,s_i) - \theta_i s_i x_i\right]\zeta_i(dx)$$

$$+ \sum_{j=1,j\neq i}^m s_j\sigma_j\sigma_i\left(\frac{\partial \mathcal{V}}{\partial s_j} - \theta_j\right)\rho_{ij} = 0, \qquad i = 1,\cdots,m$$

and

$$\theta_i\left(s_i\sigma_i^2 + \int_{\square_0} s_i x_i^2\zeta_i(dx)\right) + \sum_{j=1,j\neq i}^m s_j\sigma_j\sigma_i\theta_j\rho_{ij} =$$

$$s_i\sigma_i^2\frac{\partial \mathcal{V}}{\partial s_i} + \int_{\square_0} x_i\left[\mathcal{V}(t,s_i(1+x_i)) - \mathcal{V}(t,s_i)\right]\zeta_i(dx) + \sum_{j=1,j\neq i}^m s_j\sigma_j\sigma_i\frac{\partial \mathcal{V}}{\partial s_j}\rho_{ij},$$

for $i = 1,2,...,m$.

In matrix form, this yields the desired m-dimensional system $M\Theta = L$.

$\square$

Proposition 2. In a continuous world which jumps are limited to zero, the hedging portfolio has the following representation:

$$\Theta = (\theta_1,...,\theta_m)^T = \left(\frac{\partial \mathcal{V}}{\partial s_1},...,\frac{\partial \mathcal{V}}{\partial s_m}\right)^T.$$

Proof.

Setting jumps equal to 0 in the m-dimensional system $M\Theta = L$, we can derive

$$\begin{pmatrix} s_1\sigma_1^2 & s_2\sigma_2\sigma_1\rho_{12} & \cdots & s_m\sigma_m\sigma_1\rho_{1m} \\ s_1\sigma_1\sigma_2\rho_{12} & s_2\sigma_2^2 & \cdots & s_m\sigma_m\sigma_2\rho_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ s_1\sigma_1\sigma_m\rho_{1m} & s_2\sigma_2\sigma_m\rho_{2m} & \cdots & s_m\sigma_m^2 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{V}}{\partial s_1}s_1\sigma_1^2 + \sum_{j\neq 1}\frac{\partial \mathcal{V}}{\partial s_j}s_j\sigma_j\sigma_1\rho_{1j} \\ \frac{\partial \mathcal{V}}{\partial s_2}s_2\sigma_2^2 + \sum_{j\neq 2}\frac{\partial \mathcal{V}}{\partial s_j}s_j\sigma_j\sigma_2\rho_{2j} \\ \vdots \\ \frac{\partial \mathcal{V}}{\partial s_m}s_m\sigma_m^2 + \sum_{j\neq m}\frac{\partial \mathcal{V}}{\partial s_j}s_j\sigma_j\sigma_m\rho_{mj} \end{pmatrix},$$

and then

$$s_i \sigma_i^2 \theta_i + \sum_{j=1, j \neq i}^{m} s_j \sigma_j \sigma_i \theta_j \rho_{ij} = s_i \sigma_i^2 \frac{\partial \mathcal{V}}{\partial s_i} + \sum_{j=1, j \neq i}^{m} s_j \sigma_j \sigma_i \frac{\partial \mathcal{V}}{\partial s_j} \rho_{ij},$$

for $i = 1, \ldots, m$.

Hence

$$\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_m \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \mathcal{V}}{\partial s_1} \\ \vdots \\ \dfrac{\partial \mathcal{V}}{\partial s_m} \end{pmatrix}.$$

□

Remark 1.

- The current proposition presents an interesting result. It shows that in a continuous market, the hedging portfolio is independent of correlation parameters and is equal to $\partial \mathcal{V} / \partial s_i$, for any θ_i $(i = 1, \ldots, m)$, whether the assets are independent or correlated.

- If S_i s are independent, i.e., $\rho_{ij} = 0$ for $i, j = 1, \ldots, m$, from proposition 1 we get:

$$\theta_i = \frac{\dfrac{\partial \mathcal{V}}{\partial s_i} \sigma_i^2 s_i + \int_{\square_0} x_i \left[\mathcal{V}(t, s_i(1 + x_i)) - \mathcal{V}(t, s_i) \right] \zeta_i(dx)}{s_i \sigma_i^2 + \int_{\square_0} s_i x_i^2 \zeta_i(dx)}.$$

This is the LRM portfolio in independent assets case.

4 Sensitivity

In this section we investigate the behavior of the risk with respect to perturbation of the correlation parameter. In LRM approach, Follmer and Schweitzer [9,10] proposed the risk process of portfolio φ as follows:

$$R_t(\varphi) = E\left[\left(C_T(\varphi) - C_t(\varphi) \right)^2 \mid \mathcal{F}_t \right], 0 \leq t \leq T.$$

Then for the LRM portfolio θ

$$R_0(\theta) = E\left[\left(C_T(\theta) - C_0(\theta) \right)^2 \right] = Var\left[C_T(\theta) \right],$$

using that the cost process associate to θ is a martingale. Therefore, in aforesaid market we have

$$R_0(\theta) = Var[C_T(\theta)] = Var[\mathcal{V}(T) - \sum_{i=1}^{m} \int_0^T \theta_i(t) dS_i(t)]$$

$$= Var[\int_0^T (\frac{\partial \mathcal{V}}{\partial t} + \sum_{i=1}^{m} [\frac{\partial \mathcal{V}}{\partial s_i} S_i(t) r(t) + \frac{1}{2} \frac{\partial^2 \mathcal{V}}{\partial s_i^2} S_i^2(t) \sigma_i^2(t)$$

$$+ \frac{1}{2} \sum_{j=1, j \neq i}^{m} \frac{\partial^2 \mathcal{V}}{\partial s_i \partial s_j} S_i(t) S_j(t) \sigma_i(t) \sigma_j(t) \rho_{ij}$$

$$+ \int_{\mathbb{R}_0} (\mathcal{V}(t, s_i(1 + x_i)) - \mathcal{V}(t, s_i) - \frac{\partial \mathcal{V}}{\partial s_i} x_i) \zeta_i(dx)]) dt$$

$$+ \sum_{i=1}^{m} \int_0^T S_i(t) \sigma_i(t) (\frac{\partial \mathcal{V}}{\partial s_i} - \theta_i(t)) dW_i(t)$$

$$+ \sum_{i=1}^{m} \int_0^T \int_{\mathbb{R}_0} (\mathcal{V}(t, s_i(1 + x_i)) - \mathcal{V}(t, s_i) - \theta_i(t) S_i(t) x_i) \tilde{N}_i(dt, dx)]$$

$$= E[\sum_{i=1}^{m} (\int_0^T S_i(t) \sigma_i(t) (\frac{\partial \mathcal{V}}{\partial s_i} - \theta_i(t)) dW_i(t)$$

$$+ \int_0^T \int_{\mathbb{R}_0} (\mathcal{V}(t, s_i(1 + x_i)) - \mathcal{V}(t, s_i) - \theta_i(t) S_i(t) x_i) \tilde{N}_i(dt, dx))]^2$$

$$= \sum_{i=1}^{m} \int_0^T [S_i^2(t) \sigma_i^2(t) (\frac{\partial \mathcal{V}}{\partial s_i} - \theta_i(t))^2$$

$$+ 2 \sum_{j=1, j \neq i}^{m} S_i(t) S_j(t) \sigma_j(t) \sigma_i(t) (\frac{\partial \mathcal{V}}{\partial s_i} - \theta_i(t)) (\frac{\partial \mathcal{V}}{\partial s_j} - \theta_j(t)) \rho_{ij}$$

$$+ \int_{\mathbb{R}_0} (\mathcal{V}(t, s_i(1 + x_i)) - \mathcal{V}(t, s_i) - \theta_i(t) S_i(t) x_i)^2 \zeta_i(dx)] dt.$$

As you see, the risk process ρ_{ij} varies linearly in R $(j, i = 1, \ldots, m)$ and

$$\frac{\partial R}{\partial \rho_{ij}} = 4 s_i s_j \sigma_i \sigma_j (\frac{\partial \mathcal{V}}{\partial s_i} - \theta_i)(\frac{\partial \mathcal{V}}{\partial s_j} - \theta_j).$$

Behavior of R depends on the sign of two sentences $(\frac{\partial \mathcal{V}}{\partial s_i} - \theta_i)$ and $(\frac{\partial \mathcal{V}}{\partial s_j} - \theta_j)$, for

$j, i = 1, \ldots, m$. Since these sentences have the same sign, we conclude that for reducing the risk, it is necessary to avoid investment in high correlated assets. This is former confirmed by Markowitz's economic research [2] for analysis of portfolios of securities.

5 Conclusion

The hedging problem is a challenging subject in financial mathematics. In most of existing approaches assets are assumed to be independent, while in the real- world economy assets are influenced by each other. Therefore, we can say that this restriction is very strong. This defect motivates us to consider the hedging problem in a jump-diffusion market with m correlated assets. We use the locally risk minimizing approach and derive the hedging portfolio as solution of a m-dimensional system of linear equations $M \Theta = L$. Results show that in a continuous market the optimal portfolio is independent of correlation parameters as well as two assumptions of independence and correlation of assets have the same LRM portfolio. Furthermore, for risk management we investigate the sensitivity of

the risk with respect to correlation parameter. We conclude that investor should invest in low correlated assets to manage risk.

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