Advances in Mathematical Finance
& Applications, 5(3), (2020), 331-345
DOI: 10.22034/amfa.2020.1885000.1340

Published by IA University of Arak, Iran
Homepage: www.amfa.iau-arak.ac.ir

On Integral Operator and Argument Estimation of a Novel Subclass of Harmonic Univalent Functions

Z. Dehdast*, Sh. Najafzadeh, M.R. Foroutan

Department of mathematics, Payame noor University, p.o.box 19395-3697, Tehran, Iran

ARTICLE INFO

Article history:
Received 09 December 2019
Accepted 05 January 2020

Keywords:
Harmonic function,
Integral operator,
Extreme point,
Distortion bounds,
Convolution

ABSTRACT

Financial Mathematics is the application of mathematical methods to financeial problems. It is shown that Harmonic Univalent Functions play important roles in Financial Mathematics. This paper introduced and established a subclass of harmonic univalent functions involving the argument of complex-value functions of the form $f = h + \overline{g}$. Additionally, this study investigates some properties of this subclass such as necessary and sufficient coefficient bounds, extreme points, distortion bounds and Hadamard product.

1 Introduction

Although, Louis Bachelier is considered the author of the first scholarly work on mathematical finance, published in 1900, mathematical finance emerged as a discipline in the 1970s, following the work of Fischer Black, Myron Scholes and Robert Merton [1,2] on option pricing theory [3]. Mathematical finance, also known as quantitative finance and financial mathematics, is a field of applied mathematics, concerned with mathematical modeling of financial markets. Generally, mathematical finance will derive and extend the mathematical or numerical models without necessarily establishing a link to financial theory, taking observed market prices as input [4]. Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics. Harmonic functions have been studied in many areas such as differential geometers [5-9]; mathematical finance [10-14]. The field of Mathematical Finance has undergone a remarkable development since the seminal papers by Black and Scholes [1] and Merton [2], in which the famous "Black-Scholes Option Pricing Formula" was derived. In 1997 the Nobel prize in Economics was awarded to Merton and Scholes for this achievement, thus also honoring the late Black [3]. Silverman [15] provided sufficient coefficient conditions for normalized harmonic functions to map onto either starlike or convex regions. These conditions were also shown to be necessary when the coefficients are negative. Ahuja [14] investigated harmonic analogs and formed certain harmonic functions which preserve close-to-convexity under convolution. Ahuja [16] determined representation theorems, distortion bounds, convolutions, convex combinations, and neighbourhoods for harmonic functions. Yalçin [17] defined and investigated a new class of Salagean-type harmonic univalent functions and obtained coefficient conditions, extreme points, distortion

* Corresponding author. Tel.: *+989028483422*
E-mail address: *z.dehdast59@gmail.com*

bounds, convex combination and radii of convex for the above class of harmonic univalent functions. Muir [18] constructed a weak subordination chain of convex univalent harmonic functions using a harmonic de la Vallée Poussin mean and a modified form of Pommerenke's criterion for a subordination chain of analytic functions. Ang et al. [19] derived several sufficient conditions of the linear combinations of harmonic univalent mappings to be univalent and convex in the direction of the real axis. Li and Ponnusamy [20] investigated the subject of disk of convexity of sections of univalent harmonic functions. Ho [21] established the mapping properties of integral operators on space of bounded mean oscillation and Campanato spaces. Berra et al. [22] provided the mapping properties of some integral operators on space of bounded mean oscillation BMO, Campanato spaces and Lipschitz spaces. They play several roles on the studies of harmonic analysis. Li et al. [23] provided approximation of functions by linear integral operators on variable exponent spaces associated with a general exponent function on a domain of a Euclidean space. Approximation of functions by positive linear operators is a classical topic in approximation theory starting with the Bernstein operators [24], for approximating functions in the space C[0, 1] of continuous functions on [0, 1]. Ruzhansky and Sugimoto [25] presented a criterion for the global boundedness of integral operators which are known to be locally bounded. Grinshpan [26] applied a result of Warschawski and improved the estimation of the argument in the considered class for the important case when the mapping is nearly circular. CHO and SRIVASTAVA [27] presented a method to derive some inclusion properties and argument estimates of certain normalized analytic functions in the open unit disk, which were defined by means of a class of multiplier transformations. Aouf [28] obtained some argument properties of meromorphically multivalent functions associated with generalized hypergeometric function and derived the integral preserving properties in a sector. Some other subclasses of harmonic univalent functions investigated by many authors, for example see [29,30]. According to the above mentioned concepts, the main objective of this paper is to provide a more precise definition of these concepts in finance applications and define and verify a subclass of harmonic univalent functions involving the argument of complex-value functions and investigate some properties of this subclass.

2 Preliminaries

Here, we tend to investigate some important concepts of Harmonic Functions that are useful in theory of mathematical finance [31] In order to put forward our methodology, we start with introducing the following important concepts that are used throughout the paper. Hence, let H denote the class of functions which are complex-valued, harmonic, univalent, sense-preserving in $\Delta = \{z \in C : |z| < 1\}$ normalized by $f(0) = h(0) = fz(0) - 1 = 0$.

Definition 1: Each $f \in H$ can be expressed as $f = h + g \in H$, where h and g are analytic in Δ. Therefore if $f \in H$, then

$$h(z) = z + \sum_{n=2}^{+\infty} a_n z^n \quad , \qquad g(z) = \sum_{n=1}^{+\infty} b_n z^n \quad , \quad |b_1| < 1 \tag{1}$$

are the analytic and co-analytic part of f respectively.

With respect to the definition 1, we assume that H be the subfam̄ily of H consisting harmonic functions $f = h + g$ where

$$f(z) = z - \sum_{n=2}^{+\infty} |a_n| z^n \quad and \quad g(z) = \sum_{n=1}^{+\infty} |b_n| z^n \quad , \quad |b_1| < 1 \tag{2}$$

In this case, if co-analytic part of $f = h + \bar{g}$ is identically zero, then H reduces to the class of S of normalized analytic univalent functions.

For $0 < \lambda \leq 1$, $0 \leq \beta, r < 1$, $k \in N0 = N \cup \{0\}$, $0 \leq t \leq 1, \alpha, \theta \in R$, we have the following useful definition.

Definition 2 ([32-34]): The class $H_{\lambda,k}(\alpha, \beta, t)$ is a set of all functions $f \in$ H satisfying the relation

$$Re\left\{\left(1 + te^{i\alpha}\right)\frac{\partial}{\theta\partial}\left[\arg\left(T_\lambda^k(re^{i\theta})\right)\right] - te^{i\alpha}\right\} \geq \beta \tag{3}$$

Where

$$T_\lambda^k f(z) = T_\lambda^k h(z) + \overline{T_\lambda^k g(z)} \tag{4}$$

$$= z + \sum_{n=2}^{+\infty} \frac{a_n}{\left(1 + \lambda(n-1)\right)^k} z^n + \sum_{n=1}^{+\infty} \overline{\left[\frac{b_n}{1 + \lambda(n-1)^k}\right]} z^n$$

With a simple calculation on (4) we get the relation (5), see [35-37].

$$\frac{\partial}{\theta\partial}\left[\arg\left(T_\lambda^k(re^{i\theta})\right)\right] = Im\frac{\partial}{\theta\partial}\left[\log\left(T_\lambda^k\left(f(re^{i\theta})\right)\right)\right] \tag{5}$$

$$= Re\left\{\frac{z\left(T_\lambda^k h(z)\right)' - \overline{z\left(T_\lambda^k g(z)\right)'}}{T_\lambda^k h(z) + T_\lambda^k g(z)}\right\}$$

Let the subclass $H_{\lambda,k}(\alpha,\beta,t)$ consisting of functions $f = h + \bar{g} \in$ H and (3) holds true. Another purpose of this paper is to show and explain some applications of harmonic functions in economics and mathematical finance [38].

3 Main Results

Mathematical modelling have gained popularity in financial modeling due to the dependence structure of their increments and the roughness of their results [4]. In the present section, we investigate to obtain coefficient bounds for functions in the subclasses Hλ,k(α,β,t) and $\overline{H}$λ,k(α,β,t) and their possible roles in mathematical finance. These properties consist of necessary and sufficient coefficient bounds, extreme points, distortion bounds and Hadamard product. The following theorem reveals an important property for a function to be harmonic univalent.

Theorem 1. *Let* $f = h + \bar{g} \in \overline{H}$ and also

$$\sum_{n=1}^{+\infty}\left[\frac{k(n-1) + n - \beta}{1 - \beta}|a_n| + \frac{k(n+1) + n + \beta}{1 - \beta}|b_n|\right]\frac{1}{\left(1 + \lambda(n-1)\right)^k} \leq 2. \tag{6}$$

where $a_1 = 1$ *and* $0 \leq \beta \leq 1$, then f is harmonic univalent in Δ and $f \in H_{\lambda,k}(\alpha,\beta,t)$.

Proof.

According to the fact that

$$Re\,w \geq \beta \Longleftrightarrow |1-\beta+w| \geq |1+\beta-w|, \qquad (w \in \mathrm{C}, \beta \in \mathrm{R}),$$

for proving $f \in \mathrm{H}_{\lambda,k}(\alpha,\beta,t)$, we must show that (3), or equivalently

$$Re\left\{\left(1+te^{i\alpha}\right)\left[\frac{z\left(T_\lambda^k h(z)\right)' - \overline{z\left(T_\lambda^k g(z)\right)'}}{T_\lambda^k h(z) + T_\lambda^k g(z)} - te^{i\alpha}\right]\right\} \geq \beta \tag{7}$$

holds true. For this purpose, we set

$$X = \left(1+te^{i\alpha}\right)\left[z\left(T_\lambda^k h(z)\right)' - \overline{z\left(T_\lambda^k g(z)\right)'}\right] - te^{i\alpha}\left[T_\lambda^k h(z) + T_\lambda^k g(z)\right],$$

$$Y = T_\lambda^k h(z) + \overline{T_\lambda^k g(z)} \quad and \quad w = \frac{X}{Y'}$$

Then, it is enough to show that

$$|X-(1-\beta)Y| - |X-(1+\beta)Y| \geq 0.$$

This issue can be proved according to the following relations:

$$|X+(1-\beta)Y| - |X-(1+\beta)Y|$$

$$= \left(1+te^{i\alpha}\right)\left[\sum_{n=2}^{+\infty}\frac{a_n}{\left(1+\lambda(n-1)\right)^k}a_n z^n - \sum_{n=1}^{+\infty}\frac{a_n}{\left(1+\lambda(n-1)\right)^k}\overline{b_n}\,(\overline{z})^n\right]$$

$$-te^{i\alpha}\left[z+\sum_{n=2}^{+\infty}\frac{a_n}{\left(1+\lambda(n-1)\right)^k}a_n z^n - \sum_{n=1}^{+\infty}\frac{a_n}{\left(1+\lambda(n-1)\right)^k}\overline{b_n}\,(\overline{z})^n\right]$$

$$+(1-\beta)\left[z+\sum_{n=2}^{+\infty}\frac{a_n}{\left(1+\lambda(n-1)\right)^k}a_n z^n - \sum_{n=1}^{+\infty}\frac{a_n}{\left(1+\lambda(n-1)\right)^k}\overline{b_n}\,(\overline{z})^n\right]$$

$$-|\left(1+te^{i\alpha}\right)\left[z+\sum_{n=2}^{+\infty}\frac{a_n}{\left(1+\lambda(n-1)\right)^k}a_n z^n - \sum_{n=1}^{+\infty}\frac{a_n}{\left(1+\lambda(n-1)\right)^k}\overline{b_n}\,(\overline{z})^n\right]$$

$$-te^{i\alpha}\left[z+\sum_{n=2}^{+\infty}\frac{1}{\left(1+\lambda(n-1)\right)^k}a_n z^n - \sum_{n=1}^{+\infty}\frac{1}{\left(1+\lambda(n-1)\right)^k}\overline{b_n}\,(\overline{z})^n\right]$$

$$-(1+\beta)\left[z+\sum_{n=2}^{+\infty}\frac{1}{\left(1+\lambda(n-1)\right)^k}a_n z^n - \sum_{n=1}^{+\infty}\frac{1}{\left(1+\lambda(n-1)\right)^k}\overline{b_n}\,(\overline{z})^n\right]$$

$$= \Bigg| \left(1 + te^{i\alpha} - te^{i\alpha} + 1 - \beta\right) z + \left\{ \sum_{n=2}^{+\infty} \left(n\left(1 + te^{i\alpha}\right) - te^{i\alpha} + 1 - \beta\right) a_n z^n - \sum_{n=2}^{+\infty} \left(n\left(1 + te^{i\alpha}\right) - te^{i\alpha} + 1 - \beta\right) \overline{b_n}\, (\overline{z})^n \right\} \frac{1}{\left(1 + \lambda(n-1)\right)^k} \Bigg|$$

$$-|\left(1 + te^{i\alpha} - te^{i\alpha} + 1 - \beta\right) z - \{\sum_{n=2}^{+\infty} \left(n\left(1 + te^{i\alpha}\right) - te^{i\alpha} + 1 - \beta\right) a_n z^n$$

$$- \Bigg| \left(1 + te^{i\alpha} - te^{i\alpha} + 1 - \beta\right) z - \left\{ \sum_{n=2}^{+\infty} \left(n\left(1 + te^{i\alpha}\right) - te^{i\alpha} + 1 - \beta\right) a_n z^n + \sum_{n=2}^{+\infty} \left(n\left(1 + te^{i\alpha}\right) - te^{i\alpha} + 1 - \beta\right) \overline{b_n}\, (\overline{z})^n \right\} \frac{1}{\left(1 + \lambda(n-1)\right)^k} \Bigg|$$

$$\geq (2 - \beta)|z| - \sum_{n=2}^{+\infty} (t(n+1) + n - 1 + \beta)|b_n|\, |z^n| \left(\frac{1}{\left(1 + \lambda(n-1)\right)^k} \right) - \beta z$$

$$- \sum_{n=2}^{+\infty} (t(n+1) + n - 1 + \beta)|a_n|\, |z^n| \left(\frac{1}{\left(1 + \lambda(n-1)\right)^k} \right)$$

$$- \sum_{n=2}^{+\infty} (t(n+1) + n - 1 + \beta)|b_n|\, |z^n| \left(\frac{1}{\left(1 + \lambda(n-1)\right)^k} \right)$$

$$= 2(1 - \beta)|z| \left\{ 1 - \sum_{n=2}^{+\infty} \frac{(t(n-1) + n - \beta)}{1 - \beta} |a_n| \left(\frac{1}{\left(1 + \lambda(n-1)\right)^k} \right) - \sum_{n=2}^{+\infty} \frac{(t(n+1) + n + \beta)}{1 - \beta} |a_n| \left(\frac{1}{\left(1 + \lambda(n-1)\right)^k} \right) \right\} \geq 0 \quad , \quad (by\ (6)).$$

The latter inequality indicates that $f \in H_{\lambda,k}(\alpha,\beta,t)$. On the other side, for every λ we have

$$\left| \left(T_\lambda^k h(z)\right)' \right| = \left| 1 + \sum_{n=2}^{+\infty} \frac{n a_n}{\left(1 + \lambda(n-1)\right)^k} z^{n-1} \right|$$

$$\geq 1-\sum_{n=2}^{+\infty}\frac{|na_n|}{\left(1+\lambda(n-1)\right)^k}r^{n-1}$$
$$> 1-\sum_{n=2}^{+\infty}n|a_n|\frac{1}{\left(1+\lambda(n-1)\right)^k}$$
$$\geq 1-\sum_{n=2}^{+\infty}\frac{k(n-1)+n+\beta}{1-\beta}|a_n|\ \frac{1}{\left(1+\lambda(n-1)\right)^k}$$
$$\geq \sum_{n=2}^{+\infty}\frac{k(n+1)+n+\beta}{1-\beta}|b_n|\ \frac{1}{\left(1+\lambda(n-1)\right)^k}$$
$$\geq \sum_{n=2}^{+\infty}n\ |b_n|\ \frac{1}{\left(1+\lambda(n-1)\right)^k}$$
$$\geq \sum_{n=2}^{+\infty}n\ |b_n|\ \frac{1}{\left(1+\lambda(n-1)\right)^k}r^{n-1}$$
$$\geq \left|\left(T_\lambda^k h(z)\right)'\right|,$$

Thus, if $\lambda = 0$ we conclude that $f(z)$ is sense-preserving in Δ. Now, for univalency of f we consider two cases:

(i) $g(z) = 0$,

In this case, $f(z) = g(z)$ is analytic and the univalency of f follows by a result of [39-41].

(ii) $g(z) \neq 0$.

In this case, we show that if $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$. By letting Δ be a simply connected and convex domain, for $0 \leq j \leq 1$, we have :

$$z(j) = (1-j)z_1 + jz_2 \in \Delta,$$

and also if $z_1, z_2 \in \Delta$, then $z_1 \neq z_2$. But, we know that

$$T_\lambda^k f(z_2) - T_\lambda^k f(z_1) = \int_0^1 \left[(z_2-z_1)\left(T_\lambda^k h(z(t))\right)' + \left((z_2-z_1)T_\lambda^k g(z(t))\right)'\right]dt.$$

That leads to the following formula

$$Re\left[\frac{T_\lambda^k f(z_2) - T_\lambda^k f(z_1)}{z_2-z_1}\right] = \int_0^1 Re\left[\left(T_\lambda^k h(z(t))\right)' + \frac{\overline{z_2-z_1}}{z_2-z_1}\left(T_\lambda^k g(z(t))\right)'\right]dt$$
$$> \int_0^1 Re\left[\left(T_\lambda^k h(z(t))\right)' - \left|\left(T_\lambda^k g(z(t))\right)'\right|\right]dt.$$

From the above relation we can obtain the following relation

$$Re\left\{\left(T_\lambda^k h(z(t))\right)' - \left|\left(T_\lambda^k g(z(t))\right)'\right|\right\}$$
$$\geq Re\left(T_\lambda^k h(z(t))\right)' - \sum_{n=2}^{\infty}n\ |b_n|\ \frac{1}{\left(1+\lambda(n-1)\right)^k}$$
$$= 1-\sum_{n=2}^{\infty}n\ |b_n|\ \frac{1}{\left(1+\lambda(n-1)\right)^k} - \sum_{n=1}^{\infty}n\ |b_n|\ \frac{1}{\left(1+\lambda(n-1)\right)^k}$$

$$\geq 1-\sum_{n=2}^{+\infty}\frac{k(n-1)+n-\beta}{1-\beta}|a_n|\,\frac{1}{\left(1-\lambda(n-1)\right)^k}$$
$$-\sum_{n=1}^{+\infty}\frac{k(n+1)+n+\beta}{1-\beta}|b_n|\,\frac{1}{\left(1-\lambda(n-1)\right)^k}\geq 0\,,\quad \big(by\quad(6)\big).$$

In this step and simply by putting $\lambda = 0$, we obtain univalency of f. □

Remark: The function

$$F(z)=z+\sum_{n=2}^{+\infty}\frac{(1-\beta)\left(1+\lambda(n-1)\right)^k}{k(n+1)+n-\beta}x_n z^n + \sum_{n=1}^{+\infty}\frac{(1-\beta)\left(1-\lambda(n-1)\right)^k}{k(n+1)+n+\beta}\overline{y}_n(\overline{z})^n \qquad (8)$$

Show that the coefficient bound given by (6) is sharp where

$$\frac{1}{2}\left(\sum_{n=2}^{+\infty}|x_n|+\sum_{n=2}^{+\infty}|y_n|\right)=1$$

Based on the above mentioned definitions and theorem, we provide the following important theorem that presents the necessary and sufficient conditions.

Theorem 2. *Let* $f=g+\overline{h}=z-\sum_{n=2}^{\infty}|a_n|\,z^n+\sum_{n=1}^{\infty}|b_n|\,z^n$, Then $f(z)\in H_{\lambda,k}(\alpha,\beta,t)$, if and only if

$$\sum_{1}^{\infty}\left[\frac{k(n-1)+n-\beta}{1-\beta}|a_n|+\frac{k(n+1)+n+\beta}{1-\beta}|b_n|\right]\frac{1}{\left(1-\lambda(n-1)\right)^k}\leq 2 \qquad (9)$$

where $a_1 = 1$ *and* $0\leq\beta\leq 1$.

Proof.

Since $H_{\lambda,k}(\alpha,\beta,t)\subset H_{\lambda,k}(\alpha,\beta,t)$, then the "if" part can be directly concluded from theorem 1.

In order to prove the "only if" part, we show that

$$f\in H_{\lambda,k}(\alpha,\beta,t)\Rightarrow (6) \text{ holds true},$$

or equivalently

$$(6) \text{ isn't hold} \Rightarrow f\notin H_{\lambda,k}(\alpha,\beta,t).$$

Suppose that $f\in H_{\lambda,k}(\alpha,\beta,t)$, therefore we must have

$$0\leq Re\left\{\frac{\left(1+te^{i\alpha}\right)\left(T_\lambda^k h(z)\right)'-\overline{\left(T_\lambda^k g(z)\right)'}}{T_\lambda^k h(z)+\overline{T_\lambda^k g(z)}}-te^{i\alpha}-\beta\right\}$$

$$= Re\left\{\frac{\left(1+te^{i\alpha}\right)\left[z-\sum_{n=2}^{+\infty}\frac{na_n}{\left(1+\lambda(n-1)\right)^k}z^n-\sum_{n=1}^{+\infty}\frac{nb_n}{\left(1+\lambda(n-1)\right)^k}(\overline{z})^n\right]}{-\sum_{n=2}^{+\infty}\frac{a_n}{\left(1+\lambda(n-1)\right)^k}z^n-\sum_{n=1}^{+\infty}\frac{b_n}{\left(1+\lambda(n-1)\right)^k}(\overline{z})^n}\right.$$

$$\left.-\frac{\left(te^{i\alpha}-\beta\right)\left[z-\sum_{n=2}^{+\infty}\frac{a_nz^n}{\left(1+\lambda(n-1)\right)^k}+\sum_{n=1}^{+\infty}\frac{b_n}{\left(1+\lambda(n-1)\right)^k}\right]}{z-\sum_{n=2}^{+\infty}\frac{a_n}{\left(1+\lambda(n-1)\right)^k}z^n+\sum_{n=1}^{+\infty}\frac{b_n}{\left(1+\lambda(n-1)\right)^k}(\overline{z})^n}\right\}$$

$$= Re\left\{\frac{(1-\beta)-\sum_{n=2}^{\infty}\left[n+\beta+e^{i\alpha}(nk-k)\right]\frac{|a_n|}{\left(1+\lambda(n-1)\right)^k}z^{n-1}}{1-\sum_{n=2}^{\infty}\frac{|a_n|}{\left(1+\lambda(n-1)\right)^k}z^{n-1}+\frac{\overline{z}}{z}\sum_{n=2}^{\infty}\frac{|b_n|}{\left(1+\lambda(n-1)\right)^k}(\overline{z})^{n-1}}\right.$$

$$\left.-\frac{\frac{\overline{z}}{z}\sum_{n=2}^{\infty}\left[n+\beta+e^{i\alpha}(nk-k)\right]\frac{|b_n|}{\left(1+\lambda(n-1)\right)^k}(\overline{z})^{n-1}}{1-\sum_{n=2}^{\infty}\frac{|a_n|}{\left(1+\lambda(n-1)\right)^k}z^{n-1}+\frac{\overline{z}}{z}\sum_{n=2}^{\infty}\frac{|b_n|}{\left(1+\lambda(n-1)\right)^k}(\overline{z})^{n-1}}\right\}\geq 0.$$

The last inequality must hold for all z, $|z| = r < 1$. By choosing the values of z on the real axis such that $0 \leq |z| = r < 1$, the following relation should be held

$$Re\left\{\frac{1-\beta-\left[\sum_{n=2}^{\infty}(n-\beta)\frac{|a_n|}{\left(1+\lambda(n-1)\right)^k}r^{n-1}+\sum_{n=2}^{\infty}(n+\beta)\frac{|b_n|}{\left(1+\lambda(n-1)\right)^k}r^{n-1}\right]}{1-\sum_{n=2}^{\infty}\frac{|a_n|}{\left(1+\lambda(n-1)\right)^k}r^{n-1}+\sum_{n=1}^{\infty}\frac{|b_n|}{\left(1+\lambda(n-1)\right)^k}r^{n-1}}\right.$$

$$\left.-\frac{e^{i\alpha}\left[(nk-k)\frac{|a_n|}{\left(1+\lambda(n-1)\right)^k}r^{n-1}+\sum_{n=1}^{\infty}(nk-k)\frac{|b_n|}{\left(1+\lambda(n-1)\right)^k}r^{n-1}\right]}{1-\sum_{n=2}^{\infty}\frac{|a_n|}{\left(1+\lambda(n-1)\right)^k}r^{n-1}+\sum_{n=1}^{\infty}\frac{|b_n|}{\left(1+\lambda(n-1)\right)^k}r^{n-1}}\right\}$$

$$\geq 0.$$

Or

$$\frac{1-\beta-\sum_{n=1}^{\infty}(k(n-1)+n-\beta)\,|a_n|\,\frac{1}{\left(1+\lambda(n-1)\right)^k}r^{n-1}}{1-\sum_{n=2}^{\infty}\frac{|a_n|}{\left(1+\lambda(n-1)\right)^k}r^{n-1}+\sum_{n=1}^{\infty}\frac{|b_n|}{\left(1+\lambda(n-1)\right)^k}r^{n-1}}$$

$$-\frac{\sum_{n=1}^{\infty}(k(n+1)+n+\beta)\,|b_n|\,\frac{1}{\left(1+\lambda(n-1)\right)^k}r^{n-1}}{1-\sum_{n=2}^{\infty}\frac{|a_n|}{\left(1+\lambda(n-1)\right)^k}r^{n-1}+\sum_{n=1}^{\infty}\frac{|b_n|}{\left(1+\lambda(n-1)\right)^k}r^{n-1}}\geq 0$$

If the inequality (9) isn't hold then when $r \to 1$ the numerator is negative and this is a contradiction for $f(z) \in H_{\lambda,k}(\alpha,\beta,t)$ and so the proof is complete. □

4 Main Properties

In this sense, this section is devoted to show some properties of the developed subclass that are useful in the mathematical theory of finance and economics. Mainly, in this section, some concepts such as extreme points and distortion bounds for functions in the new developed subclass $H_{\lambda,k}(\alpha,\beta,t)$ are going to be introduced. Despite of these concepts, the convolution preserving property is also investigated in this section. First of all we need to prove the following important theorem that determines the extreme points.

Theorem 3. *$f = h + g \in H_{\lambda,k}(\bar{\alpha},\bar{\beta},t)$ if and only if*

$$f(z)=\sum_{n=1}^{\infty}(v_n h_n(z)+w_n g_n(z)) \tag{10}$$

Where

$$h_1(z)=z\,, \tag{11}$$

$$h_n(z)=z-\frac{(1-\beta)\big(1+\lambda(n-1)\big)^k}{(k(n-1)+n-\beta)}(z)^n\,, n=\ 1,2,3,\ldots, \tag{12}$$

$$g_n(z)=z-\frac{(1-\beta)\big(1+\lambda(n-1)\big)^k}{(k(n+1)+n+\beta)}(\overline{z})^n\,, n=\ 1,2,3,\ldots, \tag{13}$$

$$v_n \geq 0,\quad w_n \geq 0, \textstyle\sum_{n=1}^{\infty}(v_n+w_n)=1 \tag{14}$$

Proof.

Assume that f is defined according to (10), then we have

$$f(z)=u_1 h_1(z)+\sum_{n=2}^{\infty}u_n h_n(z)+\sum_{n=2}^{\infty}w_n g_n(z)$$

$$=u_1 z+\sum_{n=1}^{\infty}\left[\frac{(1-\beta)\big(1+\lambda(n-1)\big)^k}{(k(n-1)+n-\beta)}z^n\right]v^n$$

$$+\sum_{n=1}^{\infty}\left[z+\frac{(1-\beta)\big(1+\lambda(n-1)\big)^k}{(k(n+1)+n+\beta)}(\overline{z})^n\right]w^n$$

$$=z-\sum_{n=1}^{\infty}\left[\frac{(1-\beta)\big(1+\lambda(n-1)\big)^k}{(k(n-1)+n-\beta)}z^n\right]v^n\ +\sum_{n=1}^{\infty}\left[z+\frac{(1-\beta)\big(1+\lambda(n-1)\big)^k}{(k(n+1)+n+\beta)}(\overline{z})^n\right]w_n.$$

Then, the following relation is obtained

$$\sum_{n=2}^{+\infty}\frac{k(n-1)+n-\beta}{1-\beta}\times\frac{1}{\big(1-\lambda(n-1)\big)^k}\left(\frac{(1-\beta)\big(1+\lambda(n-1)\big)^k}{(k(n-1)+n-\beta)}U_n\right)$$
$$+\sum_{n=1}^{+\infty}\frac{k(n+1)+n+\beta}{1-\beta}\times\frac{1}{\big(1-\lambda(n-1)\big)^k}\left(\frac{(1-\beta)\big(1+\lambda(n-1)\big)^k}{(k(n+1)+n+\beta)}w_n\right)$$
$$=\sum_{n=2}^{+\infty}u_n+\sum_{n=1}^{+\infty}w_n\;=1-u_1\le 1,$$

This relation indicates $f(z)\in H_{\lambda,k}(\alpha,\beta,t)$.

Conversely, let's assume that $f(z)\in \overline{H}_{\lambda,k}(\alpha,\beta,t)$. In this case and according to (9) we set

$$u_1=1-\sum_{n=2}^{+\infty}u_n+\sum_{n=1}^{+\infty}w_n,$$
$$u_n=\frac{k(n-1)+n-\beta}{1-\beta}\left(\frac{1}{\big(1-\lambda(n-1)\big)}\right)^k|a_n|,\quad n=\;2,3,\dots,$$

and

$$w_n=\frac{k(n+1)+n+\beta}{1-\beta}\left(\frac{1}{\big(1-\lambda(n-1)\big)}\right)^k|b_n|,\quad n=\;1,2,\dots,$$

Therefore, simply we have $f(Z)=\sum_{n=1}^{\infty}\big(u_nh_n(z)+w_ng_n(z)\big)$ and the proof is complete. □

The following theorem shows the condition for H to be a convex set.

Theorem 4. *If $f_j, j=1,2,\dots$ belongs to $H_{\lambda,k}(\alpha,\beta,t)$, then the function $F(z)=\sum_{j=1}^{\infty}u_jf_j(z)$ is also in $H_{\lambda,k}(\alpha,\beta,t)$, where $\sum_{j=1}^{\infty}u_j=1$ and $f_j(z)$ defined by*

$$F_j(z)=z-\sum_{n=2}^{\infty}a_{nj}z^n+\sum_{n=1}^{\infty}a_{nj}(\overline{z})^n\quad,\quad j=1,2,\dots$$

Proof.

Since $f_j(z)\in H_{\lambda,k}(\alpha,\beta,t)$, then according to the relation (9), for every $j=1,2,\dots$, we obtain the following result

$$\sum_{n=1}^{+\infty}\left[\frac{k(n-1)+n-\beta}{1-\beta}|a_{nj}|+\frac{k(n+1)+n+\beta}{1-\beta}|b_{nj}|\right]\frac{1}{\big(1-\lambda(n-1)\big)^k}\le 2.$$

Additionally, we have

$$F(z)=\sum_{j=2}^{\infty}u_jf_j(z)=z-\sum_{n=2}^{\infty}\left[\sum_{j=1}^{\infty}u_ja_{n,j}\right]z^n+\sum_{n=1}^{\infty}\left[\sum_{j=1}^{\infty}u_jb_{n,j}\right](\overline{z})^n$$

Despite of them and with regard to the theorem (7) we conclude that

$$\sum_{n=1}^{+\infty}\left[\frac{k(n-1)+n-\beta}{1-\beta}\left(\sum_{j=1}^{\infty}u_j a_{n,j}\right)\right.$$
$$\left.+\frac{k(n+1)+n+\beta}{1-\beta}\left(\sum_{j=1}^{\infty}u_j b_{n,j}\right)\right]\frac{1}{\left(1-\lambda(n-1)\right)^k}$$
$$+\sum_{j=1}^{\infty}\left\{\left[\sum_{n=1}^{\infty}\frac{k(n-1)+n-\beta}{1-\beta}a_{n,j}\right.\right.$$
$$\left.\left.+\frac{k(n-1)+n-\beta}{1-\beta}b_{n,j}\right]\frac{1}{\left(1-\lambda(n-1)\right)^k}\right\}$$
$$\leq 2\sum_{j=1}^{\infty}u_j=2.$$

The latter relation completes the proof. Hence $H_{\lambda,k}(\alpha,\beta,t)$ is a convex set. □

The following theorem gives the distortion bounds which yields a covering result for the class.

Theorem 5. *Suppose* $f(z)\in H_{\lambda,k}(\alpha,\beta,t)$, *then for* $|z|=r<1$, we have

$$|f(z)|\leq(1-|b_1|)r+(1+\lambda)^k\left(\frac{1-\beta}{k+2-\beta}-\frac{2k+1+\beta}{k+2-\beta}|b_1|\right)r^2 \tag{15}$$

And, also we have

$$|f(z)|\geq(1-|b_1|)r-(1-\lambda)^k\left(\frac{1-\beta}{k+2-\beta}-\frac{2k+1+\beta}{k+2-\beta}|b_1|\right)r^2 \tag{16}$$

Proof.

Since $f(z)\in H_{\lambda,k}(\alpha,\beta,t)$, therefore we have

$$|f(z)|=\left|z-\sum_{n=2}^{\infty}a_nz^n+\sum_{n=2}^{\infty}b_n(\overline{z})^n\right|=\left|z-\sum_{n=2}^{\infty}a_nz^n+b_1\overline{z}+\sum_{n=2}^{\infty}b_n(\overline{z})^n\right|$$
$$\geq(1-|b_1|)r-\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,r^n\geq(1-|b_1|)r-\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,r^2$$
$$\geq(1-|b_1|)r-\frac{(1-\beta)(1+\lambda)^k}{k+2-\beta}\sum_{n=2}^{\infty}\left[\frac{(1-\beta)}{k+2-\beta}\times\frac{1}{(1+\lambda)^k}(|a_n|+|b_n|)\right]r^n$$
$$\geq(1-|b_1|)r-\frac{(1-\beta)(1+\lambda)^k}{k+2-\beta}\sum_{n=2}^{\infty}\left[\frac{(1+\lambda)^k(k+2-\beta)}{(1-\beta)}|a_n|+\frac{(1+\lambda)^k(2k+1+\beta)}{(1-\beta)}|b_n|\right]r^2$$
$$\geq(1-|b_1|)r-\frac{(1-\beta)(1+\lambda)^k}{k+2-\beta}\sum_{n=2}^{\infty}\left[1+\frac{(k+2+\beta)}{(1-\beta)}|b_1|\right]r^2$$
$$\geq(1-|b_1|)r-(1+\lambda)^k\left[\frac{(1-\beta)}{k+2-\beta}-\frac{(2k+1+\beta)}{k+2-\beta}|b_1|\right]r^2,$$

And, accordingly we obtain

$$|f(z)| \leq (1-|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)\, r^n \leq (1-|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)\, r^2$$
$$= (1+|b_1|)r - \frac{(1-\beta)(1+\lambda)^k}{k+2-\beta}\sum_{n=2}^{\infty}\left[\frac{(k+2-\beta)}{(1-\beta)}\left(\frac{1}{1+\lambda}\right)^k (|a_n|+|b_n|)\right] r^2$$
$$\leq (1+|b_1|)r - \frac{(1-\beta)(1+\lambda)^k}{k+2-\beta}\sum_{n=2}^{\infty}\left[\frac{(k+2-\beta)}{(1-\beta)((1+\lambda)^k)}|a_n| + \frac{(2k+1+\beta)}{(1-\beta)((1+\lambda)^k)}|b_n|\right] r^2$$
$$\leq (1+|b_1|)r - \frac{(1-\beta)(1+\lambda)^k}{k+2-\beta}\sum_{n=2}^{\infty}\left[1 - \frac{(2k+1+\beta)}{k+2-\beta}|b_1|\right] r^2$$
$$\geq (1+|b_1|)r - (1+\lambda)^k\left[\frac{(1-\beta)}{k+2-\beta} - \frac{(2k+1+\beta)}{k+2-\beta}|b_1|\right] r^2$$

That completes the proof. □

The following important theorem shows that the functions in the class $H_{\lambda,k}(\alpha,\beta,t)$ are closed under convolution.

Theorem 6. *If* $f(z) \in \overline{H}_{\lambda,k}(\alpha,\beta,t)$ and for $0 \leq \gamma \leq \beta < 1$, $g(z) \in \overline{H}_{\lambda,k}(\alpha,\gamma,t)$, then $H_{\lambda,k}(\alpha,\beta,t) \subseteq H_{\lambda,k}(\alpha,\gamma,t)$ and $(f * g)(z) \in H_{\lambda,k}(\alpha,\beta,t)$, where

$$(g * g)(z) = \left[z - \sum_{n=2}^{\infty}|a_n|z^n + \sum_{n=1}^{\infty}|b_n|(\overline{z})^n\right] * \left[z - \sum_{n=2}^{\infty}|c_n|z^n + \sum_{n=1}^{\infty}|d_n|(\overline{z})^n\right] \tag{17}$$
$$= z - \sum_{n=2}^{\infty}|a_n c_n|z^n + \sum_{n=1}^{\infty}|b_n d_n|(\overline{z})^n \quad , \quad (|c_n| \leq 1, |d_n| \leq 1).$$

Proof.

We assume that, $f(z) \in \overline{H}_{\lambda,k}(\alpha,\beta,t)$, then according to the definition and the hypothesis of the theorem, we have

$$Re\left\{\left(1+te^{i\alpha}\right)\left\{\frac{z\left(T_\lambda^k h(z)\right)' - \overline{z\left(T_\lambda^k g(z)\right)'}}{T_\lambda^k h(z) + \overline{T_\lambda^k g(z)}} - te^{i\alpha}\right\}\right\} \geq \beta \geq y,$$

That leads to the $f(z) \in H_{\lambda,k}(\alpha,\beta,t)$. Additionally, according to the definition f $*g$ given by (17) and regard to the $|c_n| \leq 1$ and $d_n \leq 1$, we concluded that $f * g \in \overline{H}_{\lambda,k}(\alpha,\beta,t)$ and the proof is complete. □

Our motivation came from mathematical finance, more precisely from establishing a subclass of harmonic univalent functions that have important role in finance.

5 Conclusion

One of the most active areas of research in the finance area is mathematical modeling of financial phenomenon. Hence, in this paper we carried out a new version of subclass of harmonic univalent functions that are useful in mathematical finance [42]. As a result, we defined and verified a novel subclass of harmonic univalent functions involving the argument of complex-value functions of the form $f = h + \overline{g}$. Furthermore, we investigated some properties of the proposed subclass such as necessary and sufficient coefficient bounds, extreme points, distortion bounds and Hadamard product.

References

[1] Black, F., Scholes, M., *The pricing of options and corporate liabilities*, J. Political Econom., 1973, **81**, P. 637-654.

[2] Merton, R.C., *Theory of rational option pricing*, Bell J. Econom. Manag. Sci., 1973, **4**, P. 141-183.

[3] Delbaen, F., Schachermayer, W., *Handbook of the Geometry of Banach* Spaces, vol. 1, Edited by William B. Johnson and Joram Lindenstrauss, 2001, **9**, P. 367-392.

[4] Harms, P., Stefanovits, D., *Affine representations of fractional processes with applications in mathematical finance*, Stochastic Processes and their Applications, 2019, **129** (4), P. 1185-1228. Doi: 10.1016/j.spa.2018.04.010.

[5] Choquet, G., *Sur un type de transformation analytique generalisant la representation conforme et definie au moyen de fonctions harmoniques*, Bull. Sci. Math., 1945, **89** (2), P. 156-165.

[6] Kneser, H., *Losung der Aufgabe 41*, Jahresber. Deutsch. Math.-Verein., 1926, **36**, P. 123-124.

[7] Lewy, H., *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Sot., 1936, **42**, P. 689-692.

[8] Rado, T., *Aufgabe 41*, Jahresber. Deutsch. Math.-Vermin., 1926, **35**, P. 49-62.

[9] Clunie, J., and Sheil-Small, T., *Harmonic univalent functions*, Ann. Acad. Aci. Penn. Ser. A I Math., 1984, **9**, P. 3-25.

[10] Karapinar, E., Mosai, S., Taherinejad, F., *A Corporate Perspective on Effect of Asymmetric Verifiability on Investors' Expectation Differences*. Advances in Mathematical Finance and Applications, 2019, **4**(4), P. 1-18. Doi: 10.22034/amfa.2019.1869165.1227

[11] Farshadfar, Z., Prokopczuk, M., *Improving Stock Return Forecasting by Deep Learning Algorithm*, Advances in Mathematical Finance and Applications, 2019, **4**(3), P. 1-13. Doi: 10.22034/amfa.2019.584494.1173

[12] Izadikhah, M., *Improving the Banks Shareholder Long Term Values by Using Data Envelopment Analysis Model*, Advances in Mathematical Finance and Applications, 2018, **3**(2), P. 27-41. Doi: 10.22034/amfa.2018.540829

[13] Tripathi, S., *Application of Mathematics in Financial Management*. Advances in Mathematical Finance and Applications, 2019, **4**(2), P. 1-14. Doi: 10.22034/amfa.2019.583576.1169

[14] Ahuja, O.P., Jahangiri, J.M., Silverman, H., *Convolutions for Special Classes of Harmonic Univalent Functions*, Applied Mathematics Letters, 2003, **16**, P. 905-909. Doi: 10.1016/S0893-9659(03)90015-2

[15] Silverman, H., *Harmonic Univalent Functions with Negative Coefficients*, Journal of Mathematical Analysis and Applications, 1998, **220**, P. 283, 289. Doi: 10.1006/jmaa.1997.5882

[16] Ahuja, O.P., Jahangiri, J.M., *Certain multipliers of univalent harmonic functions*, Applied Mathematics Letters, 2005, **18**, P. 1319–1324. Doi: 10.1016/j.aml.2005.02.003

[17] Yalçin, S., *A new class of Salagean-type harmonic univalent functions*, Applied Mathematics Letters, 2005, **18**, P. 191–198. Doi: 10.1016/j.aml.2004.05.003

[18] Muir, S., *Weak subordination for convex univalent harmonic functions*, J. Math. Anal. Appl., 2008, **348**, P. 862–871. Doi: 10.1016/j.jmaa.2008.08.015

[19] Wang, Z-G., Liu, Z-H., Li, Y-C., *On the linear combinations of harmonic univalent mappings*, J. Math. Anal. Appl., 2013, **400**, P. 452–459. Doi: 10.1016/j.jmaa.2012.09.011

[20] Li, L., Ponnusamy, S., *Disk of convexity of sections of univalent harmonic functions*, J. Math. Anal. Appl., 2013, **408**, P. 589–596. Doi: 10.1016/j.jmaa.2013.06.021

[21] Ho, K-P., *Integral operators on BMO and Campanato spaces*, Indagationes Mathematicae, 2019, **30** (6), P. 1023–1035. Doi: 10.1016/j.indag.2019.05.007

[22] Berra, F., Carena, M., Pradolini, G., *Mixed weak estimates of Sawyer type for fractional integrals and some related operators*, J. Math. Anal. Appl., 2019, **479** (2), P. 1490–1505. Doi: 10.1016/j.jmaa.2019.07.008

[23] Li, B., He, B., Zhou, D., *Approximation on variable exponent spaces by linear integral operators*, Journal of Approximation Theory, 2017, **223**, P. 29-51, Doi:10.1016/j.jat.2017.07.009

[24] Bernstein, S.N., *D´emonstration du t´eor´eme de Weirerstrass, fond´ee sur le calcul des probabilit´es*, Commun. Soc. Math. Kharkow, 1912-1913, **13**, P. 1-2.

[25] Ruzhansky, M., Sugimoto, M., *A local-to-global boundedness argument and Fourier integral operators*, J. Math. Anal. Appl., 2019, **473** (2), P. 892-904, Doi: 10.1016/j.jmaa.2018.12.074

[26] Grinshpan, A.Z., *Estimating the Argument of Some Analytic Functions*, journal of approximation theory, 1997, **88** (1), P. 135-138. Doi: 10.1006/jath.1996.3084

[27] Cho, N.E., Srivastava, H.M., *Argument Estimates of Certain Analytic Functions Defined by a Class of Multiplier Transformations*, Mathematical and Computer Modelling, 2003, **37** (1-2), P. 39-49. Doi: 10.1016/S0895-7177(03)80004-3

[28] Aouf, M.K., *Argument estimates of certain meromorphically multivalent functions associated with generalized hypergeometric function*, Applied Mathematics and Computation, 2008, **206** (2), P. 772–780. Doi: 10.1016/j.amc.2008.09.046

[29] Cho, N.E., Owa, S., *Argument estimates of meromorphically multivalent functions*, J. Inequal. Appl., 2000, **5**, P. 49–432.

[30] Dziok, J., Srivastava, H.M., *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput., 1999, **103** (1), P. 1–13. Doi: 10.1016/S0096-3003(98)10042-5

[31] Brigo, D., Hanzon, B., *On some filtering problems arising in mathematical finance*, Insurance: Mathematics and Economics, 1998, **22** (1), P. 53-64. Doi: 10.1016/S0167-6687(98)00008-0

[32] Al-Oboudi, F.M., *Convolution properties of harmonic univalent functions preserved by some integral operator*, Acta Universitatis Apolensis, 2010, **23**, P. 139–145.

[33] Al-Oboudi, F.M., Al-Amoudi, K.A., *Subordination results for classes of analytic functions related to Conic domains defined by a fractional operator*, J. Math. Anal. Appl., 2009, **354** (2), P. 412–420. Doi: 10.1016/j.jmaa.2008.10.025

[34] Al-Oboudi, F.M., Al-Qahtani, Z.M., *On a subclass of analytic functions defined by a new multiplier integral operator*, Far East J. Math. Sci., 2007, **25**(1), P. 59–72.

[35] Izadikhah, M., Saen, RF., Ahmadi, K., *How to assess sustainability of suppliers in the presence of dual-role factor and volume discounts? A data envelopment analysis approach*, Asia-Pacific Journal of Operational Research, 2017, **34** (03), 1740016, Doi: 10.1142/S0217595917400164

[36] Salagean, G.S., *Subclasses of analytic functions*, Proc. 5th Rom. Finn. Semi. Bucharest, Part 1, Lect. Notes Math., 1983, **1013**, P. 362–372.

[37] Shuai, L., Peide, L., *A new class of harmonic univalent functions by the generalized Salagean operator*, WUJNS, 2007, **12**(6), P. 965–970. Doi: 10.1007/s11859-007-0044-6

[38] Hernández, I., Mateos, C., Núñez, J., Tenorio, A.F., *Lie Theory: Applications to problems in Mathematical Finance and Economics*, Applied Mathematics and Computation, 2009, **208** (2), P. 446–452. Doi: 10.1016/j.amc.2008.12.025

[39] Jahangiri, J.M., *Harmonic functions star like in the unit Disk*, J. Math. Anal. Appl., 1999, **235** (2), P. 470–477. Doi: 10.1006/jmaa.1999.6377

[40] Izadikhah, M., Saen, RF., Roostaee, R., *How to assess sustainability of suppliers in the presence of volume discount and negative data in data envelopment analysis?,* Annals of Operations Research, 2018, **269** (1-2), 241-267. Doi: 10.1016/j.eswa.2014.08.019

[41] Rosy, T., Stephen, B.A., Subramanian, K.G., Jahangiri, J.M., *Goodman-Rnning-type harmonic univalent functions*, Kyangpook Math. J., 2001, **4**(1), P. 45–54.

[42] Khalique, C.M., Motsepa, T., *Lie symmetries, group-invariant solutions and conservation laws of the Vasicek pricing equation of mathematical finance*, Physica A, 2018, Doi: 10.1016/j.physa.2018.03.053