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Numerical Solution of Multidimensional Exponential Levy Equation by Block Pulse Function

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ABSTRACT
The multidimensional exponential Levy equations are used to describe many stochastic phenomena such as market fluctuations. Unfortunately in practice an exact solution does not exist for these equations. This motivates us to propose a numerical solution for *n*-dimensional exponential Levy equations by block pulse functions. We compute the jump integral of each block pulse function and present a Poisson operational matrix. Then we reduce our equation to a linear lower triangular system by constant, Wiener and Poisson operational matrices. Finally using the forward substitution method, we obtain an approximate answer with the convergence rate of *O(h)*. Moreover, we illustrate the accuracy of the proposed method with a 95% confidence interval by some numerical examples.

1 Introduction

The multidimensional exponential Levy equation is a powerful tool for modelling of market fluctuations, both for hedging and option pricing goals. There are a considerable volume of research articles related to this topic in different financial mathematics journals; see [1-6]. Unfortunately, these stochastic differential equations have not an exact solution and we cannot solve explicitly them. So we have to approximate the answer by numerical methods. In this regard, during recent years various tools such as Block Pulse Functions (BPFs), Fourier series, Chebyshev polynomials, etc., were applied to find an approximate solution for such stochastic systems, see [7-20]. Orthogonal functions block pulse has been used for solving many problems. Specially, in stochastic systems Maleknejad et al. [10] applied BPF for an m-dimensional linear stochastic Itô–Volterra integral equation and presented the approximate solution with the convergence rate $O(h)$. Also Maleknejad et al. [11] estimated numerical solutions of linear stochastic Volterra integral equations by BPF and stochastic operational matrix. Khodabin et al. [12] employed BPF for estimating an approximate answer for linear stochastic Volterra–Fredholm integral equation. In all of aforementioned papers authors reduced stochastic equations to the triangle system by introducing stochastic operational matrices. In current text we extend previous works and employ BPF for multidimensional exponential Levy equations. We propose a Poisson operational matrix for the jump integral of each BPF and convert our problem to a linear lower triangular system by operational matrices and then solve it by the forward substitution method. The advantage of BPF approach compared to other methods is that due to some properties such as orthogonality, disjointness, piecewise constant trajectories, etc., calculations are simple and effective. We consider the complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, P)$, where P is the probability measure and $\{\mathcal{F}_t\}_{t\geq 0}$ is the filtration produced by the stock price process $\{S_t\}_{t\geq 0}$. We suppose

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that the dynamic of the stock price process is modeled by the multidimensional exponential Levy equation, as follows:

$$\boldsymbol{S}(\boldsymbol{t})=\boldsymbol{r}(\boldsymbol{t})+\int_0^t S(u^-)\mu(u,t)du+\sum_{i=1}^{p}\int_0^t S(u^-)\sigma_i(u,t)dW_i(u)+\sum_{j=1}^{q}\int_0^t\int_{\mathbb{R}^+} S(u^-)\eta(z)\zeta_j(u,t)\tilde{N}_j(du,dz) \tag{1}$$

where

- $W(t)$ is the Wiener process.
- $\tilde{N}(dt,dz)=N(dt,dz)-\upsilon(dz)dt$ is the compensated Poisson random measure with intensity measure $\upsilon(.)$ on $\mathbb{R}^+$.
- Components of p-dimensional Wiener process $W=(W_1,\ldots,W_p)$ and q-dimensional Poisson process $N=(N_1,\ldots,N_q)$ are mutually independent as well as we set $p=q=n$.
- $\mu(u,t)$, $\sigma_i(u,t)$ and $\zeta_j(u,t), i, j=1,\ldots,\mathrm{n}$, are measurable functions.

The current study proceeds as follows: In section 2, we summarize some basic features of BPFs. In Section 3, we review the operational matrix [8] and the stochastic one [10]. Also, we calculate the jump integral of each BPF and present the Poisson operational matrix. In Section 4, we simplify our equation to a triangle system by operational matrices and then we solve it by the forward substitution method. In section 5, we show that the convergence rate of the proposed method is $O(h)$. Finally, we provide some numerical examples in section 6 and verify the accuracy of our approach with a 95% confidence interval.

2 Block Pulse Functions (BPFs)

In this section we recall some key definitions and features of BPFs, for more details see [8, 9]. Using these relations we can simplify computations and solve our problem.

Definition 1. *A block pulse function $\psi_i(t), i=1,2,\ldots,n,$ is defined by*

$$\psi_i(t)=\begin{cases}1 & (i-1)h\le t<ih\\ 0 & otherwise,\end{cases} \tag{2}$$

where $h=\dfrac{T}{n}$ in the interval $[0,T)$.

From above definition, we derive BFSs are disjoint, i.e.,

$$\psi_i(t)\psi_j(t)=\begin{cases}\psi_i(t) & i=j\\ 0 & i\ne j,\end{cases} \tag{3}$$

orthogonal with each other in [0,T], i.e.,

$$\int_0^T \psi_i(t)\psi_j(t)dt=\begin{cases}h & i=j\\ 0 & i\ne j,\end{cases} \tag{4}$$

and for an arbitrary function $g\in L^2([0,T])$, as $n\to\infty$, the Parseval's identity is satisfied, i.e.,

$$\int_0^T g^2(t)dt=\sum_{i=1}^{\infty} g_i^2\parallel\psi_i(t)\parallel^2,$$

Where

$$g_i = \frac{1}{h}\int_0^T g(t)\psi_i(t)dt, \tag{5}$$

for $i, j = 1, 2, \ldots, n$.

Then we can write

$$\Psi(t)\Psi^T(t) = \begin{pmatrix} \psi_1(t) & 0 & \cdots & 0 \\ 0 & \psi_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_n(t) \end{pmatrix}_{n\times n}, \tag{6}$$

$$\Psi^T(t)\Psi(t) = 1,$$

and

$$\Psi(t)\Psi^T(t)G^T = \begin{pmatrix} g_1 & 0 & \ldots & 0 \\ 0 & g_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & g_n \end{pmatrix}_{n\times n} \Psi(t), \tag{7}$$

where $\Psi(t) = (\psi_1(t), \psi_2(t), \ldots, \psi_n(t))^T$, $G = (g_1, g_2, \ldots, g_n)^T$ and g_i s are obtained by (5).

Also by block pulse series representation, we can approximate any arbitrary real bounded function $g \in L^2([0,T])$ as follows:

$$g(t) \simeq \sum_{i=1}^{n} g_i\, \psi_i(t) = G^T\Psi(t) = \Psi^T(t)G \tag{8}$$

As well as for any two variables function $f(u,t) \in L^2\big([0,T_1)\times[0,T_2)\big)$, we have

$$f(u,t) = \Phi^T(u)F\Psi(t) = \Psi^T(t)F^T\Phi(u), \tag{9}$$

where $\Phi(u)$ and $\Psi(t)$ are n_1 and n_2 dimensional vectors of BFSs, respectively, and F is a $n_1 \times n_2$ matrix with entries

$$f_{ij} = \frac{1}{h_1 h_2}\int_0^{T_1}\int_0^{T_2} f(u,t)\phi_i(u)\psi_j(t)dtdu, \tag{10}$$

where $h_1 = \frac{T_1}{n_1}$, $h_2 = \frac{T_2}{n_2}$. Also we set $n_1 = n_2 = n$.

3 Operational Matrices

Operational matrices have the special role in solving deterministic and stochastic integrals of BPF. Here, first in two subsections we review calculations of two operational matrices [8] and [10], for convenience name them constant and Wiener operational matrices, respectively. Then in the third subsection we solve the jump integral of BPF and present the Poisson operational matrix.

3.1 The Constant Operational Matrix

From [7], we can write

$$\int_0^t \psi_i(u)du = \begin{cases} 0 & 0 \le t < (i-1)h, \\ t-(i-1)h & (i-1)h \le t < ih, \\ h & ih \le t < T. \end{cases}$$

With assumption $t-(i-1)h = \frac{h}{2}$ at the mid-point of $[(i-1)h, ih)$, we can write

$$\int_0^t \psi_i(u)du \simeq (0,\dots,0,\frac{h}{2},h,\dots,h)\Psi(t),$$

where $\frac{h}{2}$ inserts in i th place of vector. Therefore,

$$\int_0^t \Psi(u)du \simeq M_c\big(\Psi(t)\big), \tag{11}$$

where M_c is a constant operational matrix, as follows:

$$M_c = \frac{h}{2}\begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n}.$$

So, for every function $g(t)$

$$\int_0^t g(u)du \simeq \int_0^t G^T\Psi(u)du \simeq G^T P\Psi(t) \tag{12}$$

3.2 The Wiener Operational Matrix

For computing the Ito integral of BPFs we proceed as follows, [10]:

$$\int_0^t \psi_i(u)dW(u) = \begin{cases} 0 & 0 \le t < (i-1)h \\ W(t)-W((i-1)h) & (i-1)h \le t < ih \\ W(ih)-W((i-1)h). & ih \le t < T \end{cases}$$

At the mid-point of $[(i-1)h, ih)$, we set $W(t)-W\big((i-1)h\big) = W\big((i-0.5)h\big)-W\big((i-1)h\big)$. Then

$$\int_0^t \psi_i(u)dW(u) \simeq \Big(0,\dots,0,W\big((i-0.5)h\big)-W\big((i-1)h\big),$$
$$W\big(ih\big)-W\big((i-1)h\big),\dots,W\big(ih\big)-W\big((i-1)h\big)\Big)\Psi(t),$$

where $W\big((i-0.5)h\big)-W\big((i-1)h\big)$ is i th component. Therefore,

$$\int_0^t \Psi(u)dW(u) \simeq M_w\Psi(t) \tag{13}$$

where M_w is the Wiener operational matrix, as follows:

$$M_w = \begin{pmatrix} W(h/2) & W(h) & W(h) & \dots & W(h) \\ 0 & W(3h/2)-W(h) & W(2h)-W(h) & \dots & W(2h)-W(h) \\ 0 & 0 & W(5h/2)-W(2h) & \dots & W(3h)-W(2h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & W\left(((2n-1)h)/2\right)-W\left((n-1)h\right) \end{pmatrix}_{n\times n}.$$

Thus,

$$\int_0^t g(u)dW(u) \simeq \int_0^t G^T \Psi(u)dW(u) \simeq G^T M_w \Psi(t), \tag{14}$$

for every function $g(t)$.

3.3 The Poisson Operational Matrix

We compute the jump integral of BFSs, as follows:

$$\int^t \int_{\mathbb{R}^+} \psi_i(u)\eta(z)\tilde{N}(du,dz) = \begin{cases} 0 & 0 \le t < (i-1)h \\ \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(t,dz) - \tilde{N}((i-1)h,dz)) & (i-1)h \le t < ih \\ \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(ih,dz) - \tilde{N}((i-1)h,dz)) & ih \le t < T \end{cases}$$

Assuming

$$\int_{\mathbb{R}^+} \eta(z)(\tilde{N}(t,dz) - \tilde{N}((i-1)h,dz)) \simeq \int_{\mathbb{R}^+} \eta(z)(\tilde{N}((i-0.5)h,dz) - \tilde{N}((i-1)h,dz))$$

at the mid-point of $[(i-1)h, ih)$ we have

$$\int_0^t \int_{\mathbb{R}^+} \psi_i(u)\eta(z)\tilde{N}(du,dz) \simeq (0,\dots,0, \int_{\mathbb{R}^+} \eta(z)(\tilde{N}((i-0.5)h,dz) - \tilde{N}((i-1)h,dz)), \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(ih,dz) - \tilde{N}((i-1)h,dz)), \dots, \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(ih,dz) - \tilde{N}((i-1)h,dz)))\Psi(t),$$

where $\int_{\mathbb{R}^+} \eta(z)(\tilde{N}((i-0.5)h,dz) - \tilde{N}((i-1)h,dz))$ is i th member of the vector. Then

$$\int_0^t \int_{\mathbb{R}^+} \Psi(u)\eta(z)\tilde{N}(du,dz) \simeq M_p \Psi(t), \tag{15}$$

where M_p is the Poisson operational matrix, as follows:

$$M_p = \begin{pmatrix} \int_{\mathbb{R}^+} \eta(z)\tilde{N}(h/2,dz) & \int_{\mathbb{R}^+} \eta(z)\tilde{N}(h,dz) & \int_{\mathbb{R}^+} \eta(z)\tilde{N}(h,dz) & \dots & \int_{\mathbb{R}^+} \eta(z)\tilde{N}(h,dz) \\ 0 & \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(3h/2,dz) - \tilde{N}(h,dz)) & \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(2h,dz) - \tilde{N}(h,dz)) & \dots & \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(2h,dz) - \tilde{N}(h,dz)) \\ 0 & 0 & \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(5h/2,dz) - \tilde{N}(2h,dz)) & \dots & \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(3h,dz) - \tilde{N}(2h,dz)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(((2n-1)h)/2,dz) - \tilde{N}((n-1)h,dz)) \end{pmatrix}_{n\times n}$$

So, for every function $g(t)$

$$\int_0^t \int_{\mathbb{R}^+} g\,(u)\eta(z)\widetilde{N}(du,dz) \simeq \int_0^t \int_{\mathbb{R}^+} G^T\,\Psi(u)\eta(z)\widetilde{N}(du,dz) \simeq G^T M_p\Psi(t).$$

4 An Approximate Answer for Exponential Levy Equation

We apply results of previous sections to find an approximate answer for equation (1).
Using (8) and (9), we can write

$$\begin{aligned}
&S(t) \simeq S^T\Psi(t) = \Psi^T(t)S,\\
&r(t) \simeq R^T\Psi(t) = \Psi^T(t)R,\\
&\mu(u,t) \simeq \Phi^T(u)\Gamma\Psi(t) = \Psi^T(t)\Gamma^T\Phi(u),\\
&\sigma_i(u,t) \simeq \Phi^T(u)\Sigma_i\Psi(t) = \Psi^T(t)\Sigma_i^T\Phi(u), i = 1,\dots,n,\\
&\zeta_i(u,t) \simeq \Phi^T(u)\Xi_i\Psi(t) = \Psi^T(t)\Xi_i^T\Phi(u), i = 1,\dots,n,
\end{aligned} \tag{16}$$

where S and R are block pulse coefficient vectors. Also Γ, Σ_i and Ξ_i, $i = 1,\dots,n$, are block pulse coefficient matrices. Substituting (16) in equation (1), we have

$$S^T\Psi(t) \simeq R^T\Psi(t) + S^T(\textstyle\int_0^t \Phi\,(u)\Phi^T(u)du)\Gamma\Psi(t) + S^T(\displaystyle\sum_{i=1}^n(\textstyle\int_0^t \Phi\,(u)\Phi^T(u)dW_i(u))\,\Sigma_i)\Psi(t)$$
$$+S^T(\sum_{i=1}^n(\textstyle\int_0^t\int_{\mathbb{R}^+} \Phi\,(u)\Phi^T(u)\eta(z)\widetilde{N}_i(du,dz))\,\Xi_i)\Psi(t).$$

From (6) and (11), we get

$$\begin{aligned}
(\int_0^t \Phi(u)\Phi^T(u)du)\Gamma\Psi(t) &= (\int_0^t \Psi(u)\Psi^T(u)du)\Gamma\Psi(t) = \begin{pmatrix} M_c^1\Psi(t)\Gamma^1\Psi(t)\\ M_c^2\Psi(t)\Gamma^2\Psi(t)\\ \vdots\\ M_c^n\Psi(t)\Gamma^n\Psi(t)\end{pmatrix}\\
&= \begin{pmatrix} M_c^1 D_{\Gamma^1}\\ M_c^2 D_{\Gamma^2}\\ \vdots\\ M_c^n D_{\Gamma^n}\end{pmatrix}\Psi(t) = \mathcal{C}\Psi(t),
\end{aligned} \tag{17}$$

where Γ^j and M_c^j, $j = 1,2,\dots,n$, are the j th row of matrices Γ and M_c, respectively. D_{Γ^j} is a diagonal matrix which Γ^j s are its diagonal entries and

$$\mathcal{C} = \frac{h}{2}\begin{pmatrix} \gamma_{11} & 2\gamma_{12} & 2\gamma_{13} & \dots & 2\gamma_{1n}\\ 0 & \gamma_{22} & 2\gamma_{23} & \dots & 2\gamma_{2n}\\ 0 & 0 & \gamma_{33} & \dots & 2\gamma_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & \gamma_{nn}\end{pmatrix}_{n\times n},$$

as well as γ_{ij} s are entries of the matrix Γ.

For the Ito integrals, we obtain

$$\left(\int_0^t \Phi(u)\Phi^T(u)dW(u)\right)\Sigma_i \Psi(t) = \left(\int_0^t \Psi(u)\Psi^T(u)dW(u)\right)\Sigma_i \Psi(t)$$

$$= \begin{pmatrix} M_w^1 \Psi(t)\Sigma_i^1 \Psi(t) \\ M_w^2 \Psi(t)\Sigma_i^2 \Psi(t) \\ \vdots \\ M_w^n \Psi(t)\Sigma_i^n \Psi(t) \end{pmatrix} = \begin{pmatrix} M_w^1 D_{\Sigma_i^1} \\ M_w^2 D_{\Sigma_i^2} \\ \vdots \\ M_w^n D_{\Sigma_i^n} \end{pmatrix} \Psi(t) = \mathcal{W}_i \Psi(t), \qquad (18)$$

where M_w^j and $\Sigma_i^j, j = 1,2,\ldots,n,$ are the j th row of matrices M_w and Σ_i, respectively. $D_{\Sigma_i^j}$ is a diagonal matrix which Σ_i^j s are its diagonal entries and

$$\mathcal{W}_i = \begin{pmatrix} \sigma_i^{11}W(h/2) & \sigma_i^{12}W(h) & \sigma_i^{13}W(h) & \ldots & \sigma_i^{1n}W(h) \\ 0 & \sigma_i^{22}\big(W(3h/2)-W(h)\big) & \sigma_i^{23}\big(W(2h)-W(h)\big) & \ldots & \sigma_i^{2n}\big(W(2h)-W(h)\big) \\ 0 & 0 & \sigma_i^{33}\big(W(5h/2)-W(2h)\big) & \ldots & \sigma_i^{3n}\big(W(3h)-W(2h)\big) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \sigma_i^{nn}\Big(W\big(((2n-1)h)/2\big)-W((n-1)h)\Big) \end{pmatrix},$$

as well as σ_{ij} s are entries of the matrix Σ.

Also for the jump integrals, we can write

$$\left(\int_0^t \int_{\mathbb{R}^+} \Phi(u)\Phi^T(u)\eta(z)\tilde{N}(du,dz)\right)\Xi_i \Psi(t) = \left(\int_0^t \int_{\mathbb{R}^+} \Psi(u)\Psi^T(u)\eta(z)\tilde{N}(du,dz)\right)\Xi_i \Psi(t)$$

$$= \begin{pmatrix} M_p^1 \Psi(t)\Xi_i^1 \Psi(t) \\ M_p^2 \Psi(t)\Xi_i^2 \Psi(t) \\ \vdots \\ M_p^n \Psi(t)\Xi_i^n \Psi(t) \end{pmatrix} = \begin{pmatrix} M_p^1 D_{\Xi_i^1} \\ M_p^2 D_{\Xi_i^2} \\ \vdots \\ M_p^n D_{\Xi_i^n} \end{pmatrix} \Psi(t)$$

$$= \mathcal{P}_i \Psi(t), \qquad (19)$$

where M_p^j and $\Xi_i^j, j = 1,2,\ldots,n,$ are the j th row of matrices M_p and Ξ_i, respectively. $D_{\Xi_i^j}$ is the diagonal matrix that Ξ_i^j s are its diagonal components and

$$\mathcal{P}_i = \begin{pmatrix} \xi_i^{11}\int_{\mathbb{R}^+}\eta(z)\tilde{N}(h/2,dz) & \xi_i^{12}\int_{\mathbb{R}^+}\eta(z)\tilde{N}(h,dz) & \xi_i^{13}\int_{\mathbb{R}^+}\eta(z)\tilde{N}(h,dz) & \ldots & \xi_i^{1n}\int_{\mathbb{R}^+}\eta(z)\tilde{N}(h,dz) \\ 0 & \xi_i^{22}\int_{\mathbb{R}^+}\eta(z)(\tilde{N}(3h/2,dz)-\tilde{N}(h,dz)) & \xi_i^{23}\int_{\mathbb{R}^+}\eta(z)(\tilde{N}(2h,dz)-\tilde{N}(h,dz)) & \ldots & \xi_i^{2n}\int_{\mathbb{R}^+}\eta(z)(\tilde{N}(2h,dz)-\tilde{N}(h,dz)) \\ 0 & 0 & \xi_i^{33}\int_{\mathbb{R}^+}\eta(z)(\tilde{N}(5h/2,dz)-\tilde{N}(2h,dz)) & \ldots & \xi_i^{3n}\int_{\mathbb{R}^+}\eta(z)(\tilde{N}(3h,dz)-\tilde{N}(2h,dz)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \xi_i^{nn}\int_{\mathbb{R}^+}\eta(z)(\tilde{N}(((2n-1)h)/2,dz)-\tilde{N}((n-1)h,dz)) \end{pmatrix},$$

as well as ξ_{ij} s are components of the matrix Ξ. Thus

$$S^T\Psi(t) \simeq R^T\Psi(t) + S^T\mathcal{C}\Psi(t) + S^T\Big(\sum_{i=1}^n \mathcal{W}_i\Big)\Psi(t) + S^T\Big(\sum_{i=1}^n \mathcal{P}_i\Big)\Psi(t),$$

and

$$S^T\Big(I - \mathcal{C} - \sum_{i=1}^n \mathcal{W}_i - \sum_{i=1}^n \mathcal{P}_i\Big) \simeq R^T.$$

Assuming $A = \left(I - \mathcal{C} - \sum_{i=1}^n \mathcal{W}_i - \sum_{i=1}^n \mathcal{P}_i\right)^T$, we obtain the following linear lower triangular system

$$AS \simeq R.$$

Finally, by the forward substitution method, we can easily solve the latter equation.

5 The Order of Convergence

The current section confirms that the convergence rate of the proposed method is $O(h)$.

Theorem 1. [10]: *Suppose that* $\hat{g}(t)=\sum_{i=1}^{n} g_i \psi_i(t)$ *is the block pulse representation for* $g(t)\in L^2[0,1)$. *Then* $\| E(t) \| = \| g(t)-\hat{g}(t) \| \le \frac{h}{2\sqrt{3}} \left(\sup_{t\in[0,1]} \|g'(t)\| \right)$.

Theorem 2. [10]: *Suppose that* $\hat{f}(u,t)=\sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}\phi_i(u)\psi_j(t)$ *is the block pulse representation for* $f(u,t)\in L^2\big([0,1)\times[0,1)\big)$. *Then*

$$\| E(u,t) \| = \| f(u,t)-\hat{f}(u,t) \| \le \frac{h}{2\sqrt{3}} \left(\sup_{(x,y)\in T} \|f_u'(x,y)\|^2 + \sup_{(x,y)\in T} \|f_t'(x,y)\|^2 \right)^{1/2},$$

where $T=[0,1)\times[0,1)$.

Theorem 3: *Suppose that* $S(t)$ *and* $\hat{S}(t)$ *are exact and approximate solution of (1), respectively, as well as*

$$\| S(t) \| \le \nu,$$
$$\| \mu(u,t) \| \le \rho,$$
$$\| \sigma_i(u,t) \| \le \chi_i, i=1,\dots,n,$$
$$\| \zeta_i(u,t) \| \le \delta_i, i=1,\dots,n,$$
$$\Lambda < 1,$$

for $t\in[0,1)$ *and* $(u,t)\in T=[0,1)\times[0,1)$, *where*

$$\Lambda=\rho+V(h)+\sum_{i=1}^{n}\big(\chi_i+K_i(h)\big)\sup_{t\in[0,1)}\|W_i(t)\| + \sum_{i=1}^{n}\big(\delta_i+Q_i(h)\big)\sup_{t\in[0,1)}\left\|\int_{\mathbb{R}^+}\widetilde{N}_i(t,dz)\right\|$$

Then

$$\| S(t)-\hat{S}_{(t)} \| \le \frac{\Theta(h)}{1-\Lambda}, \qquad t\in[0,1)$$

where

$$\| S(t) \| = (E[S^2(t)])^{1/2},$$
$$N(h)=\frac{h}{2\sqrt{3}}\left(\sup_{t\in[0,1)}\|r'(t)\|\right),$$
$$V(h)=\frac{h}{2\sqrt{3}}\left(\sup_{(x,y)\in T}\|\mu_u'(x,y)\|^2+\sup_{(x,y)\in T}\|\mu_t'(x,y)\|^2\right)^{1/2},$$
$$K_i(h)=\frac{h}{2\sqrt{3}}\left(\sup_{(x,y)\in T}\|\sigma_{iu}'(x,y)\|^2+\sup_{(x,y)\in T}\|\sigma_{it}'(x,y)\|^2\right)^{1/2}, i=1,\dots,n,$$
$$Q_i(h)=\frac{h}{2\sqrt{3}}\left(\sup_{(x,y)\in T}\|\zeta_{iu}'(x,y)\|^2+\sup_{(x,y)\in T}\|\zeta_{it}'(x,y)\|^2\right)^{1/2}, i=1,\dots,n,$$
$$\Theta(h)=N(h)+V(h)\nu+\nu\sum_{i=1}^{n}K_i(h)\sup_{t\in[0,1)}\|W_i(t)\|+\nu\sum_{i=1}^{n}Q_i(h)\sup_{t\in[0,1)}\left\|\int_{\mathbb{R}^+}\widetilde{N}_i(t,dz)\right\|.$$

Proof.
Regarding to theorem 3 of [10], we can obtain $\| S(t) - \hat{S}(t) \| = O(h)$.

6 Numerical Examples

In this section, we check the accuracy of the presented approach by some numerical examples. The solution of SDE (1) is:

$$S(t) = r(t)\exp\Big(\int_0^t [\mu(u,t) - \frac{1}{2}\sum_{i=1}^{k}\sigma_i^2(u,t)]du + \sum_{i=1}^{k}\int_0^t \sigma_i(u,t)dW_i(u)$$
$$+\sum_{i=1}^{k}\int_0^t\int_{\mathbb{R}^+}\big(\ln(1+\eta(z)\zeta_i(u,t)) - \eta(z)\zeta_i(u,t)\big)\upsilon_i(dz)du$$
$$+\sum_{i=1}^{k}\int_0^t\int_{\mathbb{R}^+}\ln\big(1+\eta(z)\zeta_i(u,t)\big)\tilde{N}_i(du,dz)\Big). \qquad u,t \in [0,1)$$

Since there is not an exact solution for Ito and jump integrals, we approximate them by the Simpson method. Then, we compare the final answer of the Simpson method with the solution of the presented method. Furthermore, we suppose that the jump component has the Gamma distribution and simulate it by the Poisson weighted algorithm of [21] in Matlab software.

Example 1. Let $k=1,\ \ r(t)=0.005,\ \ \mu(u,t)=0.04,\ \ \sigma(u,t)=0.02,\ \ \eta(z)=z,\ \ \zeta(u,t)=0.02.$ Table 1 shows the results of this example. M_e and S_e show the mean and the standard deviation of the error, respectively. Also, Fig. 1 depicts paths of approximate solutions of the stock price process by Simpson and the proposed methods.

Table 1: Numerical Solutions of Example 1 with $m=1000$ Iteration

n	M_e	S_e	$0.95CI_E$	
			Lower	Upper
10	3.09040e−04	7.48017e−05	3.04404e−04	3.13676e−04
25	5.09588e−04	8.19198e−05	5.04511e−04	5.14666e−04
40	6.45761e−04	8.65358e−05	6.40398e−04	6.51125e−04
55	7.54360e−04	8.44626e−05	7.49125e−04	7.59595e−04
70	8.54824e−04	8.51983e−05	8.49544e−04	8.60105e−04
85	9.44408e−04	9.33951e−05	9.38619e−04	9.50196e−04

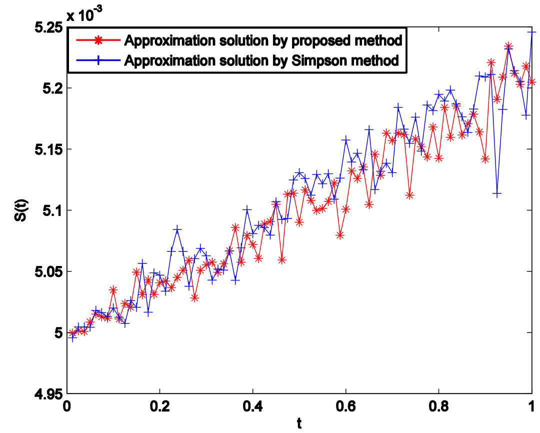

Fig. 1: Numerical Results of Example 1 with $n = 80$ and $m = 500$

Example 2. Let

$k = 2,\ \ r(t) = 0.01,\ \ \mu(u,t) = u^2,\ \ \sigma_1(u,t) = 0.1u^2,\ \ \sigma_2(u,t) = 0.12u^3,$

$\eta_1(z) = z^2,\ \ \eta_2(z) = z^4,\ \ \zeta_1(u,t) = 0.02,\ \ \zeta_2(u,t) = 0.04.$

In two dimensions, the results are shown in Table 2 and Fig. 2.

Table 2: Numerical solutions of Example 2 with $m = 1000$ iteration

n	M_e	S_e	$0.95CI_E$	
			Lower	Upper
10	1.69490e−03	2.26317e−03	1.55463e−03	1.83517e−03
25	2.35791e−03	4.32536e−03	2.08982e−03	2.62600e−03
40	2.84706e−03	2.97147e−03	2.66288e−03	3.03123e−03
55	3.50646e−03	5.07232e−03	3.19207e−03	3.82084e−03
70	4.08130e−03	9.79987e−03	3.47389e−03	4.68870e−03
85	7.84944e−03	2.63014e−02	6.21926e−03	9.47962e−03

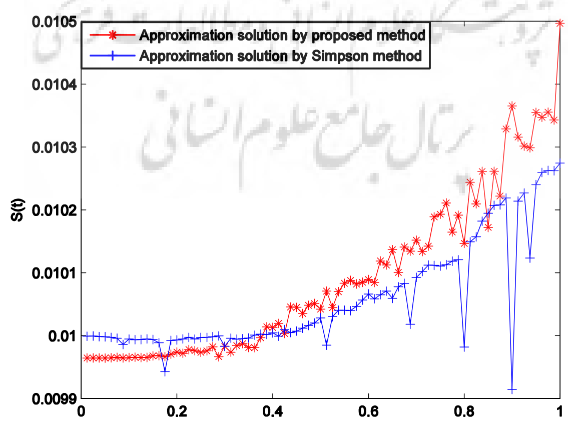

Fig. 2: Numerical Results of Example 2 with $n = 80$ and $m = 500$

Example 3. Let

$k = 3,\ \ r(t) = 0.01,\ \ \mu(u,t) = 0.1u^2,\ \ \sigma_1(u,t) = 0.2u^2,\ \ \sigma_2(u,t) = 0.15u^3,$
$\sigma_3(u,t) = 0.25u,\ \ \eta_1(z) = z^2,\ \ \eta_2(z) = z,\ \ \eta_3(z) = z^4,\ \ \zeta_1(u,t) = 0.02,$
$\zeta_2(u,t) = 0.04,\ \ \zeta_3(u,t) = 0.07.$

Table 3 and Fig. 3 reveal the numerical results of the current example in three dimensions.

Table 3: Numerical Solutions of Example 3 with $m = 1000$ Iteration

n	M_e	S_e	$0.95CI_E$	
			Lower	Upper
10	3.53089e−03	1.77218e−03	3.42105e−03	3.64073e−03
25	5.73899e−03	1.68289e−03	5.63468e−03	5.84329e−03
40	7.20607e−03	1.98405e−03	7.08309e−03	7.32904e−03
55	8.57532e−03	2.38263e−03	8.42765e−03	8.72300e−03
70	9.85402e−03	2.61912e−03	9.69169e−03	1.00163e−02
85	1.09557e−02	3.10986e−03	1.07630e−02	1.11485e−02

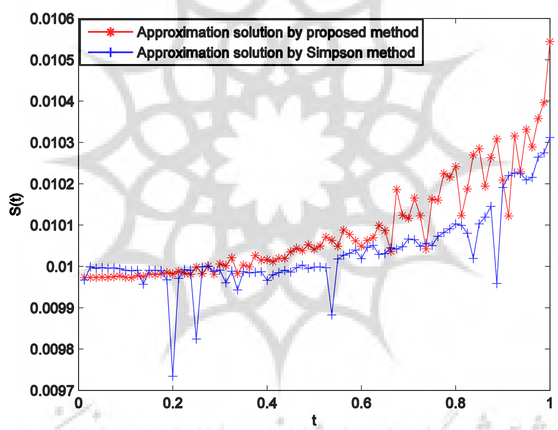

Fig. 3: Numerical Results of Example 3 with $n = 80$ and $m = 500$

7 Conclusions

Multidimensional exponential Levy equations arise in many hedging and pricing problems. Unfortunately, there is not an exact solution for such stochastic equations and also their computational rate is relatively high. Thus, we need numerical methods to approximate the answer. For this purpose, we apply block pulse functions (BPFs) as basic functions and obtain the Poisson operational matrix for the jump integral of each BPF. We show that these functions can convert our equation to a linear lower triangular system by operational matrices. Then via the forward substitution method, we get an approximate answer with the convergence rate of $O(h)$. Finally, we check the accuracy of our results on some examples.

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