Advances in Mathematical Finance
& Applications, 5 (1), (2020), 29-51
DOI: 10.22034/amfa.2019.1864620.1200

Published by IA University of
Arak, Iran
Homepage: www.amfa.iau-arak.ac.ir

Using MODEA and MODM with Different Risk Measures for Portfolio Optimization

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ARTICLE INFO

Article history:
Received 13 March 2019
Accepted 12 July 2019

Keywords:
Portfolio optimization
Data Envelopment Analysis
Multi-Objective Decision Making
Negative data
MeanSharp-βRisk
Multi-Objective MeanSharp-βRisk
Value at Risk
Conditional Value at Risk.

ABSTRACT

The purpose of this study is to develop portfolio optimization and assets allocation using our proposed models. The study is based on a non-parametric efficiency analysis tool, namely Data Envelopment Analysis (DEA). Conventional DEA models assume non-negative data for inputs and outputs. However, many of these data take the negative value, therefore we propose the MeanSharp-βRisk (MShβR) model and the Multi-Objective MeanSharp-βRisk (MOMShβR) model base on Range Directional Measure (RDM) that can take positive and negative values. We utilize different risk measures in these models consist of variance, semivariance, Value at Risk (VaR) and Conditional Value at Risk (CVaR) to find the best one as input. After using our proposed models, the efficient stock companies will be selected for making the portfolio. Then, by using Multi-Objective Decision Making (MODM) model we specified the capital allocation to the stock companies that selected for the portfolio. Finally, a numerical example of the Iranian stock companies is presented to demonstrate the usefulness and effectiveness of our models, and compare different risk measures together in our models and allocate assets.

1 Introduction

Portfolio selection and portfolio management are the most important problems from the past that has attracted the attention of investors. To solve these problems, Markowitz [19] proposed his model that was named Markowitz or mean-variance (MV) model. He believed that all investors want a maximum return and minimum risk in their investment. So, he presented his model that expresses investors want minimum risk for each level of expected return. Markowitz results in an area with an efficient frontier of return and risk. For which point along an efficient frontier, there is no point with higher return and less risk. Sharpe [38] expressed that risk is only depended to the expected return of a company and the expected return of the market. So, Sharpe [39] proposed his model for solving the portfolio selection problem (β-coefficient and Sharp ratio). Beta is a measure of the risk arising from exposure to general market movements as opposed to idiosyncratic factors. The Sharpe ratio is a way to examine the performance of an investment by adjusting for its risk. At first, the risk was defined as uncertainty to gain the expected return. One of the usual risk measure for this definition is Variance that Markowitz [19]

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used this in his MV model. Today the definition of risk is more accurate and it is better than a measure of risk is coherent risk measure. Risk can be generally divided into two categories upside and downside. Upside risk brings the increase in returns, it is suitable for those who are interested in risk for higher returns, but downside risk represents the risk of loss. By this definition, the variance is included upside and downside risk. So, the semivariance was defined. Markowitz at al. [20] proposed the mean-semi-variance model as an alternative to a mean-variance model. One of the other risk measure for manage and control risk is Value at Risk (VaR) that proposed by Baumol [6] and known as quantile in the literature. This risk measure focuses on returns come with high risk. A portfolio's VaR is the maximal loss one expects to endure at the confidence level by holding that portfolio over the time horizon. The goal is to measure the loss of return on the left side of the portfolio's return repartition by reporting a number. Duffie and Pan [10] used VaR to measure the risk of firms. Silvapulle and Granger [41] by using regular statistics and nonparametric kernel approximation of density function, estimated VaR. Glasserman et al. [13] use the Monte Carlo method along with quadratic estimation to measure the portfolio's VaR. Chen and Tang [8] verified other nonparametric approximation of VaR for related financial returns. A nonparametric estimation of dynamic VaR is developed by Jeong and Kang [17] based on the adaptive fluctuations estimation and the nonparametric quantiles estimation.

Schaumburg [37] used the nonparametric quantile regression, along with the extreme value theory for predict VaR. Despite VaR a very popular risk measure but it is not a coherent risk measure, it has an undesirable mathematical characteristic such as a lack of sub-additivity and convexity (Artzner et al. [1], [2]). VaR is coherent risk measure only when it is based on the standard deviation of the normal distribution. Therefore, Rockafellar and Uryasev ([32], [33]), expressed another risk measure which was named Conditional Value at Risk (CVaR). CVaR is also called Expected shortfall (ES), Average Value at Risk (AVaR) and expected tail loss (ETL). Pflug [29] proved that CVaR is a coherent risk measure having the following properties such as monotonicity, sub-additivity, positive homogeneity, translation invariance, and convexity. CVaR is defined as the average of more losses than VaR. CVaR became so popular for its advantages like convexity (Pflug [29], Ogryczak and Ruszczynski [24]) and researcher use CVaR as a risk measure for portfolio and financial problems (John and Hafize [18], Huang et al. [16], Zhu and Fukushima [47], Yau et al. [44], Sawik [34], Claro and Pinho de Sousa [9]). Scaillet ([35], [36]), considered a nonparametric estimation of CVaR by using kernel estimator. The group of fully non-parametric estimators based on the empirical conditional quantile function are considered in Peracchi and Tanase [25]. Hong and Liu [14] used the Monte Carlo simulation method to calculate CVaR for portfolio optimization. Another nonparametric estimation of CVaR is proposed by Yu et al. [45] based on the kernel quantile estimation approach. Navidi et al. [22] proposed their method by using CVaR for portfolio optimization.

Data Envelopment Analysis (DEA) models used to estimate the performance of Decision Making Units (DMUs) by measuring the relative efficiency. Farrell [12] was the first one who used the linear programming for evaluating the relative efficiency of DMUs. For using DEA models, must defined inputs and outputs (For example risk can be considered as input and return as output). Majority of DEA models cannot be used for the case in which DMUs include both negative and positive inputs/outputs. For example, CCR model (Charnes, Cooper, Rhodes [7]) and BCC model (Banker, Charnes, Coopper, [5]). Portela et al. [30] represented a DEA model which can be used in cases where input/output data take positive and negative values. Moreover, there are many models can be used for negative data such as Modified slacks-based measure model (MSBM), Sharp et.al. [40], semi oriented radial measure (SORM), Emrouznejad [11]. Some of the researchers use BCC model by normalizing the data. Normalization of data is not practically useful and it may change the real solution. Therefore, it is better to use the model that take the negative data instead of normalizing data. So, we propose the MeanSharp-βRisk (MShβR) model and the Multi-Objective MeanSharp-βRisk (MOMShβR) model base on Range Directional Measure (RDM) that can take positive and negative values. Some of the researchers believed that the data that use for their paper is not exact and they should be Fuzzy (Peykani et al. [26], [27], [28]). Rahmani et al. [31], represented their method for portfolio optimization.

Investors have different attitudes, but always the main concern of investors is the return and risk. Maximizing return and minimizing risk are two opposite targets that the investor wants to focus on both of them at the same time. To achieve compromise solutions in this context, the Multi-Objective Decision Making (MODM) models are used. MODM model is a suitable method for supporting and helping decision makers in the situations in which multiple opposite decision factors, must be considered concurrently. The solution of MODM problem is an actually adaptive solution, not an optimal one. Markowitz [19] was the first one who expressed the portfolio management as a MODM problem with two objectives (return and risk). Zeleny [46] and Zopounidis ([48], [49]) were researchers who have noted the multidimensional nature of financial decisions and considered all relevant factors involved. Ogryczak [23] proposed a multiple criteria linear programming model, which is based on the work of Sharpe [39]. Steuer and Paul [42] augmented a general review of MODM for financial problems. Subbu et al. [43] proposed a model that maximizes the return and minimizes the variance and VaR of the portfolio. Huang C.Y et al. [15] used MODM to determine the capital allocation in the portfolio optimization problem. Banihashemi et al. ([3], [4]) and Miryekemami et al. [21] used MODM in their represented method. In this paper, we propose the MeanSharp-βRisk (MShβR) model and the Multi-Objective MeanSharp-βRisk (MOMShβR) model by different risk measures such as variance, semivariance, Value at Risk and Conditional Value at Risk as input. After using our proposed models, the efficient stock companies will be selected for making the portfolio. We use MODM two times. At the first time, for considering the expected return and Sharpe ratio maximization and β-coefficient and risk measure minimization. At the second time, to determine the capital allocation to the stock companies in the portfolio.

The remainder of this paper is organized as follows. Mathematical definitions and formulations such as expected return, variance, semivariance, VaR, CVaR, MV model and RDM model are explained in section 2. Our proposed models are described in section 3. The MODM model is described in section 4. The experimental testing of our proposed models and comparing the different results of different risk measures in Iran stock companies are represented and are drawn in section 5. The conclusion is represented in section 6.

2 Definition and formulation

In this section, we lay out some mathematical models and definitions and explain some risk measures that are used in this paper.

2.1 Mean-Variance model

Assume that, n is the number of total assets, $r_{R_iR_j}\ \sigma_{R_i}\sigma_{R_j}$ is the covariance between returns of asset i and j, μ_i is the expected return of asset i and R_f is the riskless return. The decision variable λ_i represents the proportion of capital to be invested in asset i. The MV model is a description as follow:

$$Min\ \ z\ = \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\,\lambda_j r_{R_iR_j}\ \sigma_{R_i}\sigma_{R_j}$$

$$s.t. \quad \sum_{i=1}^{n} \lambda_i \mu_i \geq R_f \tag{1}$$
$$\sum_{i=1}^{n} \lambda_i = 1$$
$$0 \leq \lambda_i \leq 1 \quad , \quad i = 1,2,\dots,n$$

The objective is finding a portfolio with the minimum risk under the situation that the corresponding expected return must be greater than riskless return (R_f). The sum of the proportions of capital allocated to all stocks must be equal to 1 and they should be in the range of [0, 1].

2.2 Rang Directional Measure model

The RDM model is a description as follow:

$$Max \quad \theta$$
$$s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} - \theta R_{io} \qquad i = 1,\dots,m \tag{2}$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} + \theta R_{ro} \qquad r = 1,\dots,s$$
$$\sum_{j=1}^{n} \lambda_j = 1$$
$$\lambda_j \geq 0 \qquad j = 1,\dots,n$$

where

$$R_{io} = x_{io} - \min_j \{x_{ij} : j = 1,\dots,n\} \ , \qquad i = 1,\dots,m \tag{3}$$

$$R_{ro} = \max_j \{y_{rj} : j = 1,\dots,n\} - y_{ro} \ , \qquad r = 1,\dots,s \tag{4}$$

Ideal point (I) within the attendance of negative data is:

$$\mathrm{I} = \left(\max_j \{y_{rj} : r = 1,\dots,s\}, \min_j \{x_{ij} : i = 1,\dots,m\}\right) \tag{5}$$

and the purpose is to project each under evaluation asset's point to this ideal point.

Definition 1. Assume that a portfolio is going to be selected from n financial assets, λ_i is the proportion of invested money in asset i. The set of our acceptable portfolios is:

$$\phi = \{\lambda_i \in \mathbb{R}^n \ ; \ \sum_{i=1}^{n} \lambda_i = 1 \ , \ \lambda_i \geq 0\} \tag{6}$$

Return of portfolio $r(\lambda)$ is:

$$r(\lambda) = \sum_{i=1}^{n} \lambda_i r_i \tag{7}$$

The expected return of this portfolio is:

$$E(r(\lambda)) = \sum_{i=1}^{n} \lambda_i E(r_i) \tag{8}$$

The Variance of this portfolio is:

$$Var(r(\lambda)) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \Omega_{ij} \lambda_j \tag{9}$$

(Ω_{ij} is the variance-covariance matrix)

2.3 Semivariance

If portfolio's return is below the expected return, semivariance tries to minimize the scattering of the portfolio returns from the expected return.

Let

$$(R-E)^{-}=\begin{cases} R-E\,, if\ (R-E)\leq 0 \\ 0\,,\ if\ (R-E)>0 \end{cases} \tag{10}$$

Then semivariance is the expected value of $[(R-E)^{-}]^2$.

2.4 Value at Risk

VaR is defined as the maximum quantity of invest that one may lose in a specified time interval. In the other words, VaR can answer this question: how much one can expect to lose in the specified time (a day, weak, month, …). VaR defined as the quantile of a distribution. Suppose that P_t is the initial wealth and P_{t+k} is the Secondary wealth after k period time, the probability of loss is:

$$p(-\Delta_k P_t < VaR)=\alpha \tag{11}$$

where $\Delta_k P_t = P_{t+k} - P_t$ and $1-\alpha$ is the margin of error so α is the confidence level.

There are different methods for computing the VaR, such as Variance-Covariance method, Historical simulation, and Monte Carlo simulation. The variance-covariance method only uses for normal distribution data. Since the price of the stock has not a normal distribution, so we cannot use this method for calculating the VaR. There is no need for normal distribution data in Historical simulation and Monte Carlo simulation methods, thus we can use these methods for computing the VaR. One of the nonparametric methods for calculating the VaR is the historical simulation. In this method, there is no need to know the distribution of data. In fact, VaR is computed by the attention of an assumptive time series of returns and supposition that changes of future data are based on historical changes. The convenience of this method is no variance and covariance need to calculate. This method believes that behavior of returns is the same as before. Another nonparametric method for calculating the VaR is Monte Carlo simulation. This method is based on stronger supposition about the distribution of returns in comparison with historical simulation method. This method specifies possibility distribution of returns. First distribution most determines, then a lot of samples of returns will simulate and parameters will calculate based on those samples. For using Monte Carlo method to calculate the VaR, distribution of stock companies must be known. Because of the fluctuations of stock price, it is hard to obtain distributions. Thus, we used sampling methods. First, we specified the margin of error and number of needed samples that is shown the whole population. Then, we used bootstrapping method. We repeat sampling procedure for 1000 times and calculate VaR of each stock companies. At the end, the $\overline{VaR}$ is:

$$\overline{VaR}=\frac{1}{1000}\sum_{i=1}^{1000} VaR_i \tag{12}$$

Where VaR_i is the Value at Risk of stock company i and $\overline{VaR}$ is the estimate of Value at Risk of the population.

2.5 Conditional Value at Risk

Let $\lambda \epsilon \phi \subset \mathbb{R}^n$ be a decision vector, $r \epsilon \mathbb{R}^n$ be the random vector representing the value of underlying risk factors, and $f(\lambda, r)$ be the corresponding loss. For simplicity, we assume that $r \epsilon \mathbb{R}^n$ is a continuous random vector. For a given portfolio λ, the probability of the loss not exceeding a threshold Γ is given by the probability function $\mathbb{P}(\cdot)$

$$\psi(\lambda, \Gamma) \coloneqq \mathbb{P}(f(\lambda, r) \leq \Gamma) \tag{13}$$

The VaR associated with a portfolio λ and a specified confidence level α $(0 < \alpha < 1)$ is the minimal Γ satisfying $(\lambda, \Gamma) \geq \alpha$, that is:

$$VaR_\alpha(\lambda) \coloneqq \inf\{\Gamma \epsilon \, \mathbb{R} \,, \psi(\lambda, \Gamma) \geq \alpha\} \tag{14}$$

Since $\psi(\lambda, \Gamma)$ is continuous by assumption, we have:

$$\mathbb{P}\big(f(\lambda, r) \leq VaR_\alpha(\lambda)\big) = \psi\big(\lambda, VaR_\alpha(\lambda)\big) = \alpha \tag{15}$$

CVaR is defined as the conditional expectation of the portfolio loss exceeding or equal to VaR

$$CVaR_\alpha(\lambda) \coloneqq E[f(\lambda, r) \mid f(\lambda, r) \geq VaR_\alpha(\lambda) = \frac{1}{1-\alpha} \int_{VaR_\alpha(\lambda)}^{+\infty} xp(x)dx \tag{16}$$

where E is the expectation operator and $p(x)$ is the probability density function of the loss $f(\lambda, r)$.

Rockafellar and Uryasev ([24], [25]) prove that CVaR has an equivalent definition as follows:

$$CVaR_\alpha(\lambda) = \min_{\Gamma} F_\alpha(\lambda, \Gamma) \tag{17}$$

where $F_\alpha(\lambda, \Gamma)$ is defined as:

$$F_\alpha(\lambda, \Gamma) \coloneqq \Gamma + \frac{1}{1-\alpha} E[(f(\lambda, r) - \Gamma)^+] \tag{18}$$

with $(x)^+ = \max\{x, 0\}$. They also show that minimizing CVaR over $\lambda \epsilon \phi \subset \mathbb{R}^n$ is equivalent to minimizing $F_\alpha(\lambda, \Gamma)$ over $(\lambda, \Gamma) \epsilon \phi \times \mathbb{R}$. i.e.,

$$\min_{\lambda \epsilon \phi} CVaR_\alpha(\lambda) = \min_{(\lambda, \Gamma) \epsilon \phi \times \mathbb{R}} F_\alpha(\lambda, \Gamma). \tag{19}$$

Furthermore, when ϕ is a convex set and $f(\lambda, r)$ is convex with respect to λ, the problem is a convex programming problem.

Definition 2. β-coefficient of an investment indicates whether the investment is more or less volatile than the market. In general, a β less than 1 indicates that the investment is less volatile than the market, while a β more than 1 indicates that the investment is more volatile than the market. β-coefficient is:

$$\beta_i = \frac{Cov(R_i, R_m)}{Var(R_m)} \tag{20}$$

where R_i is the return of stock i and R_m is the return of the market.

Definition 3. The Sharpe ratio also known as Reward to Variability Ratio (RVAR) is a way to examine the performance of an investment by adjusting for its risk. The ratio measures the risk premium ($\mu_P - R_f$) per unit of deviation in an investment asset. Sharpe ratio is:

$$RVAR = \frac{\mu_P - R_f}{\sigma_p} \tag{21}$$

Where μ_P is the expected return and σ_p is the standard deviation of the portfolio, R_f is the riskless return.

Definition 4. Weakly efficient frontier described as:

$$\Delta^w(\phi) = \{(\mu, RVAR, \beta, F) \in S\,; (-\mu', -RVAR', \beta', F') < (-\mu, -RVAR, \beta, F) \Rightarrow (\mu', RVAR', \beta', F') \notin S\} \tag{22}$$

This frontier is a part of the boundary of the disposal region set (S). The weakly frontier can contain points that are not reachable by real portfolios.

Definition 5. Strongly efficient frontier described as:

$$\Delta^s(\phi) = \{(\mu, RVAR, \beta, F) \in S\,; (-\mu', -RVAR', \beta', F') \leq (-\mu, -RVAR, \beta, F) \text{ and } (-\mu', -RVAR', \beta', F') \neq (-\mu, -RVAR, \beta, F) \Rightarrow (\mu', RVAR', \beta', F') \notin S\} \tag{23}$$

In Definition 4 and 5, μ, $RVAR, \beta,$ and F are expected return (mean), Sharpe ratio, β-coefficient and risk measure of a point in disposal region. Similarly, μ', $RVAR', \beta'$ and F' are expected return (mean), Sharpe ratio, β-coefficient and risk measure of an optional point in MeanSharp-βRisk space. As we know, the strongly efficient frontier is included in the weakly efficient frontier

3 Proposed models

Based on the RDM model provided by Portela et al. [23], we propose the MeanSharp-βRisk (MShβR) model and the Multi-Objective MeanSharp-βRisk (MOMShβR) model. After using our proposed models, the efficient stock companies will select for making the portfolio. Let

$$g = (R_{\mu_o}, R_{RVAR_o}, R_{\beta_o}, R_{F_o}) \in [0, +\infty) \times [0, +\infty) \times [0, +\infty) \times [0, +\infty) \tag{24}$$

be a vector shows the direction in which $\theta$is going to be maximized. MShβR model defines as:

$$\xi : \mathbb{R}^4 \to (0,1],$$

$$\xi(y) = \sup\{\theta; y + \theta g \in S | \theta \in \mathbb{R}_+\}. \tag{25}$$

Based on vector $\boldsymbol{g}$, definition and mentioned set ofθ, it is obvious that the aim is to simultaneously increase mean of return and Sharp ratio and to reduce β coefficient and risk of a portfolio in direction of vector $\boldsymbol{g}$. One should care about directions in an interpretation of model while directions affect

MShβR model. For instance proportional interpretation is suitable, if vector of direction is chosen as

$$\boldsymbol{g} = \begin{pmatrix} \left(\max_j(\mu_j : j = 1, \dots, n) - \mu_o\right) = R_{\mu_o}, & \left(\max_j(RVAR_j : j = 1, \dots, n) - RVAR_o\right) = R_{RVAR_o}, \\ \left([\beta_o - \min(\beta_j : j = 1, \dots, n)]\right) = R_{\beta_o} & \left([F_o - \min(F_j : j = 1, \dots, n)]\right) = R_{F_o} \end{pmatrix}.$$

Definition 6. Consider a vector with specified direction$g = (R_{\mu_o}, R_{RVAR_o}, R_{\beta_o}, R_{F_o})$ and an under-evaluation asset $y = (\mu_o, RVAR_o, \beta_o, F_o)$, the linear MSh$\beta$R model is the description as follow:

$$\begin{aligned} & Max \quad \theta \\ & s.t. \quad E(r(\lambda)) \geq \mu_o + \theta R_{\mu_o} \qquad (26) \\ & \quad RVAR(r(\lambda)) \geq RVAR_o + \theta R_{RVAR_o} \\ & \quad \beta(r(\lambda)) \leq \beta_o + \theta R_{\beta_o} \\ & \quad F(r(\lambda)) \leq F_o + \theta R_{F_o} \\ & \quad \sum_{i=1}^{n} \lambda_i = 1 \\ & \quad \theta \geq 0 \quad , \quad 0 \leq \lambda_i \leq 1 \qquad i = 1, \dots, n \end{aligned}$$

The efficient projected point in the direction of vector g is the point in MShβR space with coordinates determined by the right- hand sides of the inequality constraints of above model evaluated at the optimal solution (i.e.,($\mu_o + \theta^* R_{\mu_o}, RVAR_o + \theta^* R_{RVAR_o}, \beta_o + \theta^* R_{\beta_o}, F_o + \theta^* R_{F_o}$)). Mechanism of the MSh$\beta$R model is just like the RDM model. When the amount of θ for under evaluation asset equal to zero, means that this asset is efficient and MShβR point is part of the weakly efficient frontier. Otherwise, as can be seen from the right-hand-sides of the inequality constraints the above model, the optimal θ indicates a change in the mean of return, sharp ratio, β coefficient and risk measure that results in a projection of the evaluated MShβR point onto the weakly efficient frontier. In the other words, $1 - \theta$ is the amount of the efficiency. The MeanSharp-βRisk (MShβR) model seeks simultaneously to improve mean of return and Sharp ratio and to reduce β coefficient and risk in the direction of the vector g. The use of this model guarantees that a projected MShβR point is part of the weakly efficient subset. To ensure that the projection of an MShβR point is part of the strongly efficient subset, one should change proportionally in all dimension. Therefore, we should introduce another model that project point proportionally.

Definition 7. Consider a vector with specified direction$g = (R_{\mu_o}, R_{RVAR_o}, R_{\beta_o}, R_{F_o})$ and an under-evaluation asset y= $(\mu_o, RVAR_o, \beta_o, F_o)$, by using the multi-objective function for the MShβR model, the MOMSβR function is the description as follow:

$$MF : \mathbb{R}^4 \to (0,1]$$

$$MF(y) = \sup\left\{\frac{1}{4}\sum_i \theta_i \, ; y + \theta \boldsymbol{g} \in S\right\}. \qquad (27)$$

This function tries to maximize θ in directions of the mean of return and Sharp ratio and β coefficient and risk separately. Because of having more than one parameter to maximize, based on rules of optimization of multi-objective functions, the average of objects is tried to be maximized. Note that θ and

$\boldsymbol{g}$ are both vectors. This function evaluates arithmetic average proportional changes in each direction, which makes interpretations more complicated. MOMSβR model is computed through the following model:

$$\begin{aligned}
Max\quad & \frac{1}{4}\theta_1 + \frac{1}{4}\theta_2 + \frac{1}{4}\theta_3 + \frac{1}{4}\theta_4 \\
s.t.\quad & E\big(r(\lambda)\big) \geq \mu_o + \theta_1 R_{\mu_o} \\
& RVAR(r(\lambda)) \geq RVAR_o + \theta_2 R_{RVAR_o} \\
& \beta(r(\lambda)) \leq \beta_o + \theta_3 R_{\beta_o} \\
& F(r(\lambda)) \leq F_o + \theta_4 R_{F_o} \\
& \textstyle\sum_{i=1}^{n} \lambda_i = 1 \\
& \theta_1,\ \theta_2,\ \theta_3,\ \theta_4\ \geq 0 \qquad , \qquad 0 \leq \lambda_i \leq 1 \qquad i = 1, \dots, n
\end{aligned} \tag{28}$$

If the above model equals zero, then MOMSβR point is part of the strongly efficient frontier. If it is nonzero, then optimal θ_i indicate the proportional change per mean of return, Sharp ratio, β coefficient and risk dimension that guarantees a projection of the evaluated MOMSβR point on to the strongly efficient frontier. As a consequence, by this model, the weakly and strongly efficient frontiers always coincide. Also, as can be seen using MOMSβR model leads to clustered projection points. This clustering occurs while MOMSβR model is a more flexible model than MShβR model in a determination of optimal directions. It is well-known that the multi-objective models (like MOMShβR model) always result in larger or equal optimal values than single objective models (like MShβR model). Therefore, MOMShβR models' efficiencies are always less than or equal to the MShβR models' efficiencies. Multi-objective functions try to maximize the average of objects (because of having more than one parameter to maximize). Multi-objective functions in here try to maximize θ in directions of mean, Sharpe ratio, β-coefficient and risk measure proportionality. Mechanism of the MOMShβR model is just like the MShβR model. When the amount of θ for under evaluation asset equal to zero, means that this asset is efficient. In the other words, $1-\theta$ is the amount of the efficiency. We want to compare different results of our models by using different risk measures as input in our models. The risk measures are variance, semivariance, VaR, and CVaR. Section 5 includes the practical work and comparing the results.

4 MODM Model

Return and risk are the most important objectives for investors in the portfolio selection. Investors want a portfolio with maximum return and minimum risk together, this solution named Positive Ideal Solution (PIS). Vice versa, the solution for a portfolio with maximum risk and minimum return together, named Negative Ideal Solution (NIS). Zeleny [38] proposed the compromise Programming. In the compromise programming distance of the solution will be counted from PIS and NIS. Each solution that is closer to PIS and farther from NIS, is better. By using the MODM model, investors can allocate different weights for return and risk objectives, according to the degree of their risk hatred.

W1: weight allocated to decision return,

W2: weight allocated to decision risk, and $W_1 + W_2 = 1$.

The MODM model is description as fallow:

$$Min\ \ z = W_1\left(\frac{f_1^* - f_1(x)}{f_1^* - f_1^-}\right) + W_2\left(\frac{f_2(x) - f_2^*}{f_2^- - f_2^*}\right)$$

$$s.t. \qquad f_1(x) \geq R_f \tag{29}$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$0 \leq \lambda_i \leq 1\ ,\ i = 1,2,\dots,n$$

where

$$f_1(x) = \sum_{i=1}^{n} \lambda_i\, \mu_i \begin{cases} f_1^* = Max \sum_{i=1}^{n} \lambda_i\, \mu_i \\ f_1^- = Min \sum_{i=1}^{n} \lambda_i \mu_i \end{cases} \tag{30}$$

$$f_2(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j r_{R_i R_j}\, \sigma_{R_i} \sigma_{R_j} \begin{cases} f_2^* = Min \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j r_{R_i R_j}\, \sigma_{R_i} \sigma_{R_j} \\ f_2^- = Max \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j r_{R_i R_j}\, \sigma_{R_i} \sigma_{R_j} \end{cases} \tag{31}$$

(f^* is PIS and f^- is NIS).

The objective function represents the distance of both objectives (return and risk) from PIS, which is searching for the closest solution to the PIS. This solution is the best portfolio that investor can select. The first section of the objective function calculates the distance to the PIS of the return objective and the second section of the objective function calculates the distance to the PIS of the risk objective. By allocating different weights to these two sections of the objective function, investors can represent their preference for return or risk. Also, it should be noted that:

The expected return of the selected portfolio must be better than riskless return (R_f);

Sum of the proportions of the capital allocated to all stocks equal to 1;

The proportions of the capital allocated to each stock must are in the range of [0, 1].

(The riskless return (R_f) were chosen from Iranian bank profit during our study period.)

5 Empirical Discussion

This process involves:

I. Calculating the efficiency of stock companies and making the portfolio

II. Allocating the capital to the stocks of companies that make the portfolio.

5.1 Data Collection

The dataset was randomly collected from the stock's price of the 15 Iranian stock companies, from 25/04/2015 to 25/04/2016. The dataset was obtained from http://www.irvex.ir/index. All of the stock companies are shown by company symbol in Table 1. Also, the price volatility of the stock companies is shown in Fig. 1.

Table 1 Symbol of the stock companies that were used

company symbol	company symbol	company symbol	company symbol	company symbol
CONT1	NAFT1	TRIR1	RENA1	PSIR1
DJBR1	SHND1	TRNS1	GHAT1	KRTI1
DSIN1	KHAZ1	AZAB1	IPAR1	PASH1

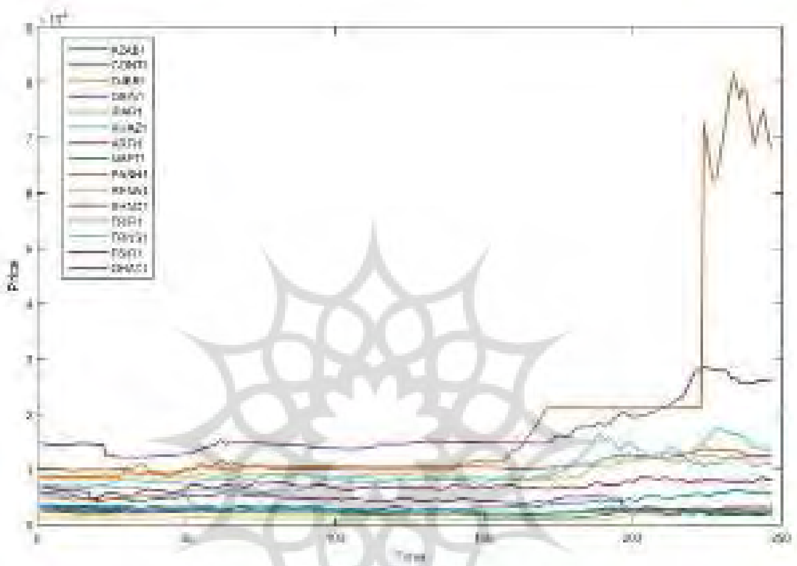

Fig 1: The price volatility of the selected stock companies

5.2 Constructing the Portfolio

Table 2 reveals constant data for inputs and outputs. Input includes β-coefficient, outputs include expected return and Sharpe ratio.

Table 2: Constant input and outputs

Stock companies	Input	Outputs	
	β-coefficient	**expected return**	**Sharpe ratio**
AZAB1	1.3417	0.0026	0.0902
CONT1	0.1089	0.0085	0.0990
DJBR1	0.6384	0.0013	0.0539

Table 2: Continue

Stock companies	Input	Outputs	
	β-coefficient	Expected return	Sharpe ratio
DSIN1	0.4935	0.0023	0.1160
IPAR1	0.4902	0.0019	0.0854
KHAZ1	0.8071	0.0017	0.0487
KRTI1	1.4303	-0.0003	-0.0175
NAFT1	0.9958	-0.0006	-0.0398
PASH1	0.3434	0.0009	0.0433
RENA1	1.5404	0.0030	0.0544
SHND1	-1.3693	-0.0029	-0.0544
TRIR1	-0.1302	-0.0035	-0.0853
TRNS1	0.6126	0.0027	0.1085
PSIR1	0.7381	0.0011	0.0241
GHAT1	1.0476	-0.0023	-0.0000

The return volatility of the stock companies is shown in Fig. 2. Table 3 reveals changeable data for risk measure as one of the other inputs. Risk measure includes variance, semivariance, Value at Risk which has calculated by historical simulation and Monte Carlo simulation methods, Conditional Value at Risk.

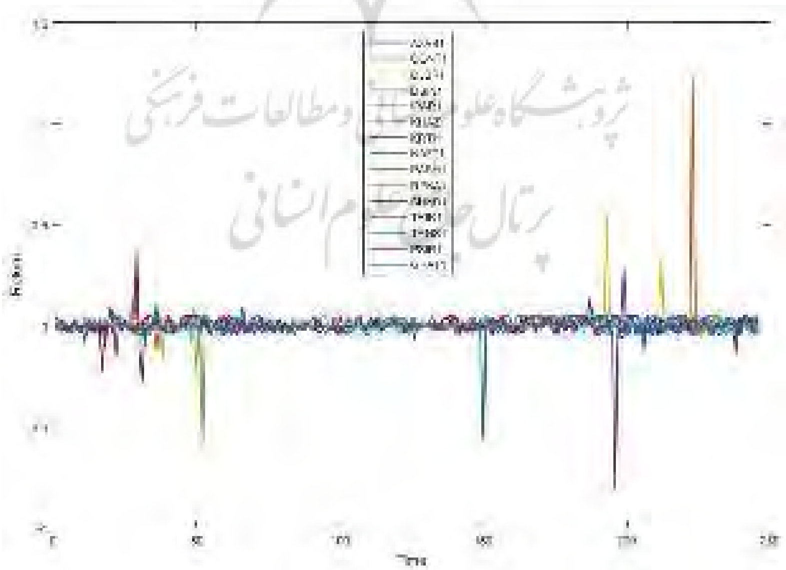

Fig 2: The Return volatility of the selected stock companies

Table 3: Changeable input

Stock companies	Variance	Semivariance	VaR Historical simulation			VaR Monte Carlo simulation			Conditional Value at Risk		
			90 %	95 %	99 %	90 %	95 %	99 %	90 %	95 %	99 %
AZAB1	0.0006	0.0168	0.0315	0.0392	0.0469	0.0285	0.0352	0.0374	0.0392	0.0430	0.0476
CONT1	0.0067	0.0170	0.0225	0.0364	0.0513	0.0195	0.0310	0.0348	0.0361	0.0452	0.0513
DJBR1	0.0003	0.0129	0.0074	0.0137	0.0315	0.0088	0.0222	0.0267	0.0231	0.0348	0.0901
DSIN1	0.0003	0.0122	0.0046	0.0080	0.0500	0.0066	0.0217	0.0267	0.0195	0.0328	0.0941
IPAR1	0.0003	0.0113	0.0101	0.0197	0.0470	0.0115	0.0233	0.0273	0.0265	0.0396	0.0681
KHAZ1	0.0007	0.0183	0.0385	0.0469	0.0503	0.0321	0.0418	0.0451	0.0471	0.0516	0.0653
KRTI1	0.0016	0.0260	0.0303	0.0442	0.1302	0.0299	0.0557	0.0643	0.0586	0.0802	0.1945
NAFT1	0.0006	0.0176	0.0364	0.0453	0.0512	0.0321	0.0402	0.0429	0.0455	0.0499	0.0574
PASH1	0.0001	0.0093	0.0040	0.0077	0.0243	0.0046	0.0135	0.0165	0.0150	0.0245	0.0749
RENA1	0.0023	0.0189	0.0339	0.0447	0.0497	0.0320	0.0392	0.0416	0.0433	0.0471	0.0506
SHND1	0.0037	0.0544	0.0304	0.0393	0.0734	0.0296	0.0723	0.0866	0.0755	0.1163	0.3914
TRIR1	0.0021	0.0429	0.0220	0.0397	0.0570	0.0260	0.0686	0.0828	0.0680	0.1062	0.3497
TRNS1	0.0005	0.0138	0.0216	0.0343	0.0466	0.0212	0.0302	0.0332	0.0343	0.0422	0.0476
PSIR1	0.0008	0.0218	0.0322	0.0397	0.0499	0.0300	0.0442	0.0489	0.0481	0.599	0.1227
GHAT1	0.0021	0.0406	0.0353	0.0456	0.1074	0.0326	0.0683	0.0802	0.0717	0.1021	0.3059

5.3 Calculating the efficiency

As mentioned before, since we have negative data such as expected return, Sharpe ratio, and β-coefficient, we must use the DEA model that can take positive and negative values, so we used the MShβR model and the MOMShβR model to calculate the efficiency of the stock companies. The software GAMS was used to measure the relative efficiency of selected stock companies. In the MShβR model and the MOMShβR model, θ shows the amount of inefficiency. Therefore, when the amount of θ for the stock company equal to zero, means that the stock company is efficient.

Table 4 reveals the amount of inefficiency of the stock companies by using MShβR model.

Table 4: Inefficiency of the stock companies by using the MShβR model

stock companies	θ with Variance	θ with SemiVariance	θ with VaR Historical simulation			θ with VaR Monte Carlo simulation			θ with CVaR		
			90 %	95 %	99 %	90 %	95 %	99 %	90 %	95 %	99 %
AZAB1	0.00	0.28	0.45	0.36	0.19	0.46	0.35	0.31	0.34	0.20	0.00

Table 4: Continue

stock companies	θ with Variance	θ with Semi-Variance	θ with VaR Historical simulation			θ with VaR Monte Carlo simulation			θ with CVaR		
			90 %	95 %	99 %	90 %	95 %	99 %	90 %	95 %	99 %
CONT1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
DJBR1	0.04	0.14	0.10	0.13	0.00	0.10	0.11	0.15	0.11	0.11	0.32
DSIN1	0.00	0.08	0.00	0.00	0.26	0.00	0.09	0.14	0.00	0.04	0.28
IPAR1	0.00	0.00	0.12	0.16	0.22	0.12	0.12	0.14	0.12	0.16	0.23
KHAZ1	0.00	0.30	0.47	0.40	0.23	0.45	0.36	0.35	0.36	0.32	0.34
KRTI1	0.12	0.49	0.36	0.31	0.55	0.44	0.51	0.52	0.49	0.52	0.54
NAFT1	0.00	0.32	0.48	0.41	0.27	0.47	0.39	0.38	0.38	0.35	0.38
PASH1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.18
RENA1	0.00	0.25	0.39	0.28	0.06	0.45	0.34	0.30	0.29	0.16	0.08
SHND1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
TRIR1	0.00	0.31	0.05	0.17	0.09	0.20	0.35	0.36	0.30	0.35	0.39
TRNS1	0.00	0.12	0.29	0.30	0.20	0.30	0.23	0.22	0.23	0.22	0.00
PSIR1	0.00	0.32	0.40	0.31	0.20	0.41	0.36	0.35	0.35	0.52	0.39
GHAT1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 5 reveals the amount of inefficiency of the stock companies by using MOMShβR model.

Table 5: Inefficiency of the stock companies by using the MOMShβR model

stock companies	θ with Variance	θ with Semi-Variance	θ with VaR Historical simulation			θ with VaR Monte Carlo simulation			θ with CVaR		
			90 %	95 %	99 %	90 %	95 %	99 %	90 %	95 %	99 %
AZAB1	0.04	0.35	0.52	0.40	0.26	0.55	0.46	0.42	0.42	0.30	0.06
CONT1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
DJBR1	0.05	0.16	0.10	0.14	0.00	0.10	0.12	0.15	0.12	0.12	0.56
DSIN1	0.00	0.09	0.00	0.00	0.38	0.00	0.10	0.15	0.00	0.06	0.54
IPAR1	0.00	0.00	0.12	0.18	0.31	0.15	0.12	0.14	0.16	0.22	0.51
KHAZ1	0.00	0.42	0.55	0.49	0.33	0.54	0.52	0.51	0.51	0.47	0.51
KRTI1	0.14	0.52	0.43	0.41	0.57	0.48	0.52	0.53	0.52	0.53	0.54
NAFT1	0.02	0.40	0.56	0.48	0.36	0.57	0.53	0.51	0.51	0.45	0.55

Table 5: Continue

stock companies	θ with Variance	θ with Semi-Variance	θ with VaR Historical simulation			θ with VaR Monte Carlo simulation			θ with CVaR		
			90 %	95 %	99 %	90 %	95 %	99 %	90 %	95 %	99 %
PASH1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.53
RENA1	0.00	0.32	0.43	0.37	0.19	0.46	0.38	0.36	0.35	0.27	0.11
SHND1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
TRIR1	0.04	0.35	0.20	0.27	0.25	0.30	0.37	0.37	0.35	0.37	0.40
TRNS1	0.00	0.12	0.35	0.34	0.27	0.43	0.35	0.33	0.33	0.29	0.00
PSIR1	0.00	0.49	0.50	0.39	0.31	0.51	0.50	0.50	0.49	0.55	0.52
GHAT1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Here we used the same inputs and outputs for the MShβR model and the MOMShβR model. By comparing these tables, we figure out:

a. In calculating VaR, the results of Monte Carlo simulation method are much more accurate than historical simulation method.
b. In calculating VaR (by Monte Carlo simulation method) and CVaR, the higher confidence levels are more accurate than lower levels.
c. CVaR is the most accurate risk measure.
d. The results of the MOMShβR model for variance and CVaR (99%) is better than results of the MShβR model. Also, we can derive that results of the MOMShβR model for other risk measures is generally better and more accurate than results of the MShβR model.

5.4 Allocating the capital

Here we describe how an investor allocates his/her capital to the stocks of the portfolio. The MODM model that described in Section 4 was used to specify the capital allocation. The weights allocated to the objectives of return and risk (W_1, W_2), rely on investor privilege. Here, we calculated nine sets of weights combination that is, (return, risk) = (0.1,0.9), (0.2,0.8), (0.3,0.7), …, (0.9,0.1). The software GAMS was used to calculate the capital allocation of the efficient stock companies. Since each risk measure has produced different results of efficiency, therefore it should be different results of capital allocation. As the results of the MOMShβR model is more accurate than results of the MShβR model, we considered the results of Table 5 for the next step.

Table 6: Capital allocation for variance

stock companies	(0.1,0.9)	(0.2,0.8)	(0.3,0.7)	(0.4,0.6)	(0.5,0.5)	(0.6,0.4)	(0.7,0.3)	(0.8,0.2)	(0.9,0.1)
CONT1	0.03	0.07	0.12	0.18	0.27	0.40	0.61	0.95	1
DSIN1	0.29	0.34	0.31	0.27	0.18	0	0	0	0

Table 6: Continue

stock companies	(0.1,0.9)	(0.2,0.8)	(0.3,0.7)	(0.4,0.6)	(0.5,0.5)	(0.6,0.4)	(0.7,0.3)	(0.8,0.2)	(0.9,0.1)
IPAR1	0.20	0.19	0.13	0.04	0	0	0	0	0
KHAZ1	0.09	0.10	0.08	0.05	0	0	0	0	0
PASH1	0.18	0	0	0	0	0	0	0	0
RENA1	0.04	0.07	0.08	0.11	0.14	0.18	0.17	0.05	0
SHND1	0	0	0	0	0	0	0	0	0
TRNS1	0.14	0.23	0.28	0.35	0.41	0.42	0.22	0	0
PSIR1	0.03	0	0	0	0	0	0	0	0
GHAT1	0	0	0	0	0	0	0	0	0

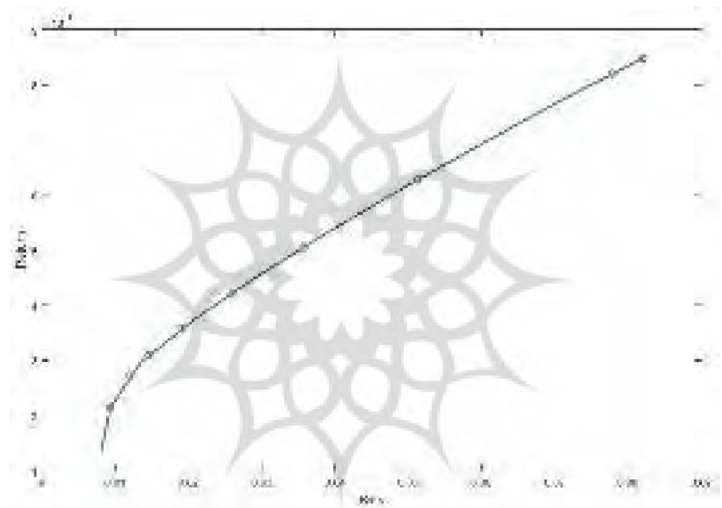

Fig 3: Obtained weights from table 6 on the mean-variance frontier

Table 7: Capital allocation for semivariance

stock companies	(0.1,0.9)	(0.2,0.8)	(0.3,0.7)	(0.4,0.6)	(0.5,0.5)	(0.6,0.4)	(0.7,0.3)	(0.8,0.2)	(0.9,0.1)
CONT1	0.04	0.09	0.14	0.22	0.31	0.45	0.68	1	1
IPAR1	0.36	0.44	0.54	0.68	0.69	0.55	0.32	0	0
PASH1	0.60	0.47	0.32	0.10	0	0	0	0	0
SHND1	0	0	0	0	0	0	0	0	0
GHAT1	0	0	0	0	0	0	0	0	0

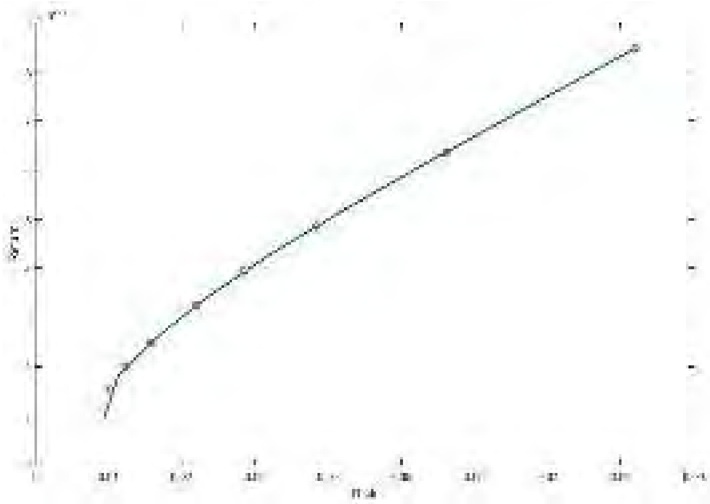

Fig 4: Obtained weights from table 7 on the mean-variance frontier

Table 8: Capital allocation for VaR H 90, H 95, M 90, CVaR 90

Stock companies	(0.1,0.9)	(0.2,0.8)	(0.3,0.7)	(0.4,0.6)	(0.5,0.5)	(0.6,0.4)	(0.7,0.3)	(0.8,0.2)	(0.9,0.1)
CONT1	0.04	0.08	0.13	0.20	0.29	0.42	0.64	1	1
DSIN1	0.41	0.55	0.73	0.80	0.71	0.58	0.36	0	0
PASH1	0.55	0.37	0.14	0	0	0	0	0	0
SHND1	0	0	0	0	0	0	0	0	0
GHAT1	0	0	0	0	0	0	0	0	0

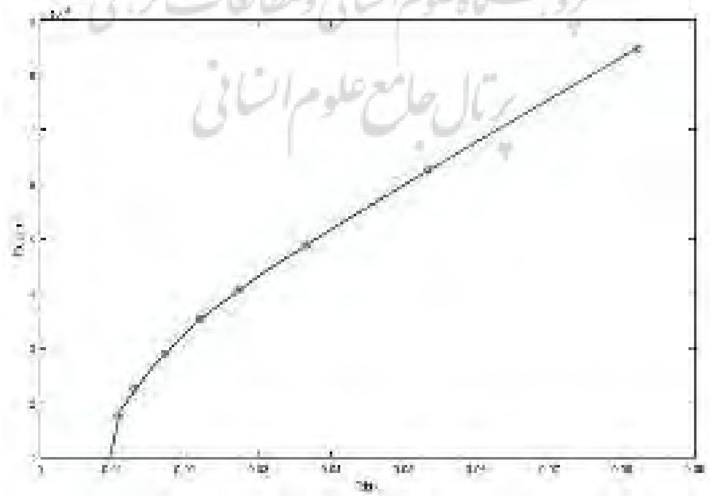

Fig 5: Obtained weights from table 8 on the mean-variance frontier

Table 9: Capital allocation for VaR H 99

stock companies	(0.1,0.9)	(0.2,0.8)	(0.3,0.7)	(0.4,0.6)	(0.5,0.5)	(0.6,0.4)	(0.7,0.3)	(0.8,0.2)	(0.9,0.1)
CONT1	0.05	0.09	0.15	0.23	0.33	0.49	0.73	1	1
DJBR1	0.33	0.37	0.42	0.47	0.56	0.51	0.27	0	0
PASH1	0.62	0.54	0.43	0.30	0.11	0	0	0	0
SHND1	0	0	0	0	0	0	0	0	0
GHAT1	0	0	0	0	0	0	0	0	0

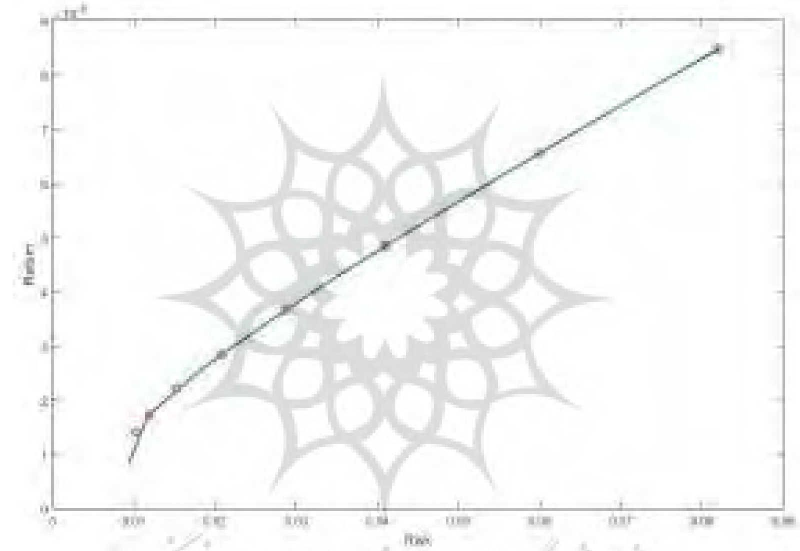

Fig 6: Obtained weights from table 9 on the mean-variance frontier

Table 10: Capital allocation for VaR M 95, M 99, CVaR 95

stock companies	(0.1,0.9)	(0.2,0.8)	(0.3,0.7)	(0.4,0.6)	(0.5,0.5)	(0.6,0.4)	(0.7,0.3)	(0.8,0.2)	(0.9,0.1)
CONT1	0.05	0.09	0.15	0.23	0.34	0.50	0.77	1	1
PASH1	0.93	0.91	0.85	0.77	0.66	0.50	0.23	0	0
SHND1	0	0	0	0	0	0	0	0	0
GHAT1	0.02	0	0	0	0	0	0	0	0

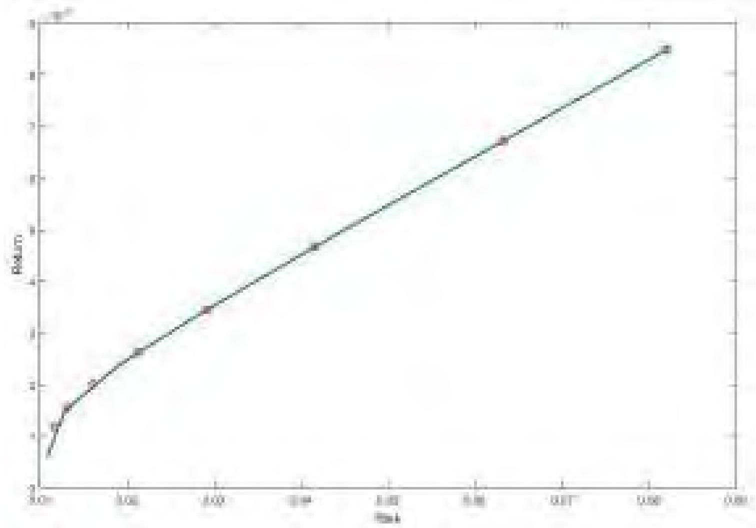

Figure 7: Obtained weights from table 10 on the mean-variance frontier

Table 11: Capital allocation for CVaR 99

stock companies	(0.1,0.9)	(0.2,0.8)	(0.3,0.7)	(0.4,0.6)	(0.5,0.5)	(0.6,0.4)	(0.7,0.3)	(0.8,0.2)	(0.9,0.1)
CONT1	0.06	0.10	0.14	0.20	0.27	0.39	0.58	0.96	1
SHND1	0.08	0.04	0	0	0	0	0	0	0
TRNS1	0.81	0.86	0.86	0.80	0.73	0.61	0.42	0.04	0
GHAT1	0.05	0	0	0	0	0	0	0	0

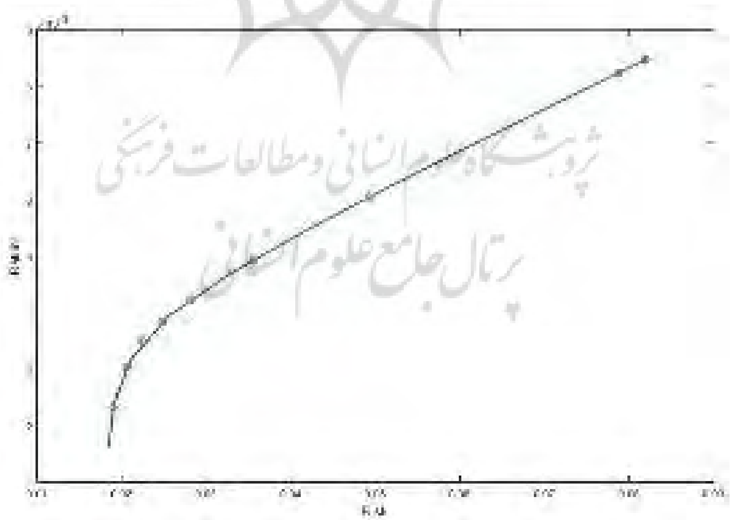

Fig 8: Obtained weights from table 11 on the mean-variance frontier

As you see in figure 3, 4, 5, 6, 7 and 8, the obtained weights from the table 6, 7, 8, 9, 10 and 11 are on the mean-variance frontier. It is mean that the weights which obtained by described MODM model in section 4, are the best portfolios. As you see in table 6, 7, 8, 9, 10 and 11, the MODM model help the investor to allocate his/her capital, as he/she likes. For example, risk avoiders are more worry about risk than return, so they try to apportion their capital among more stock companies. Vice versa, risk takers are more worry about return than risk, so they are ready for risk and they allocate their capital to fewer stock companies (Investors who choose (0.9,0.1) weight, chose just one company from all). As mentioned before, CVaR is the most accurate risk measure. So, the mean-CVaR frontier is more accurate than mean-variance frontier. In figure 8 that shows mean-variance frontier, all of the obtained weights from table 11 is on the frontier. But in figure 9 that shows mean-CVaR frontier, just 3 of the obtained weights from table 11 is on the frontier. It is mean that CVaR is the best risk measure for portfolio optimization.

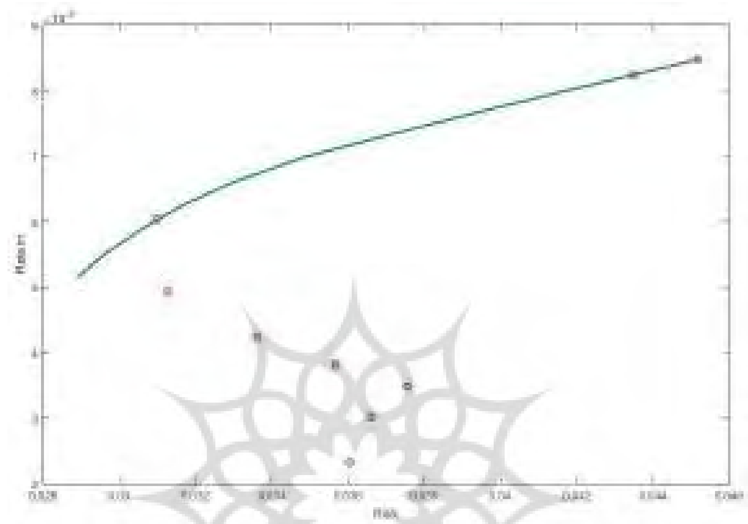

Fig 9: Obtained weights from table 11 on the mean-CVaR frontier

6 Conclusion

In this paper, we compared different risk measures such as variance, semivariance, Value at Risk (Historical simulation and Monte Carlo simulation) and Conditional Value at Risk to find the best one for portfolio optimization. We figure out CVaR is the most accurate risk measure and the higher confidence levels are more accurate than lower levels. For calculating the efficiency of the stock companies, we must use DEA models. Because of the negative data, we proposed the MShβR model and the MOMShβR model to calculate the relative efficiency of the stock companies. Multi-objective functions are more accurate, so the general results of the MOMShβR model are generally better than results of the MShβR model. The stock companies which are relatively efficient with the MOMShβR model were selected for the portfolio. Also, we used MODM to specified the capital allocation to the stock companies in the portfolio. By using MODM model, investors with different preferences of risk and return can make their portfolio as they like. Finally, the proposed method was applied to the 15 Iranian stock companies and the results were shown in the tables and figures. For future studies, other risk measures can be compared to find the best one.

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