



Application of Clayton Copula in Portfolio Optimization and its Comparison with Markowitz Mean-Variance Analysis

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ABSTRACT

With the aim of portfolio optimization and management, this article utilizes the Clayton-copula along with copula theory measures. Portfolio-Optimization is one of the activities in investment funds. Thus, it is essential to select an appropriate optimization method. In modern financial analyses, there is growing evidence indicating the distribution of proceeds of financial properties is not customary. However, in common risk management methods the main assumption is that the distribution of assets returns is normal. When the distribution of earnings isn't normal, the linear correlation coefficient isn't considered to be an appropriate measure to express the dependency structure. The investors are required to make use of methods that concentrate on the aggregated risks, considering the whole positions and the links between risk factors and assets. Therefore, we use copula as an alternative measure to model the dependency structure in this research. In this regard, given the weekly data pertaining to the early 2002 until the late 2013, we use Clayton-copula to generate an optimized portfolio for both copper and gold. Finally, the Sharpe ratio obtained through this method is compared with the one obtained through Markowitz mean-variance analysis to ascertain that Clayton-copula is more efficient in portfolio-optimization.

1 Introduction

In *Robust Portfolio Optimization and Management*, it is stated that “as the use of quantitative techniques has become more widespread in the financial industry, the issues of how to apply financial models most effectively and how to mitigate model and estimation errors have grown in importance [18]. There are different definitions of portfolio optimization and here is one example from Campbell [9]: “Determination of weights of securities in a portfolio such that it best suits a given objective, e.g. to maximize return for a given risk”.

Empirical results have shown that financial time series exhibit higher dependence on financial crises than in calm periods [15]. In their groundbreaking essay "Copula Concepts in Financial Mar-

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kets," Rachev and Stein [42] discuss some reasons of why the usual linear correlation is not a suitable measure for the dependency of different securities:

First, when the variance of returns in those securities tends to be infinite, that is, when extreme events are frequently observed, the linear correlation between these securities is undefined. Second, the correlation is a measure for linear dependence only. Third, the linear correlation is not invariant under nonlinear strictly increasing transformations, implying that returns might be uncorrelated whereas prices are correlated or vice versa. Fourth, linear correlation only measures the degree of dependence but does not clearly discover the structure of dependence. The last caveat has an especially important implication in light of the current crisis. It has been widely observed that market crashes or financial crises often occur in different markets and countries at about the same time period even when the correlation among those markets is fairly low.

Three evidences of the real financial markets act as the motivations for the formation of this paper. Firstly, financial return series are unequal and they cannot be estimated with a normal symmetrical distribution matching the market data. Secondly, there is instability in time since stable periods and very unstable periods come to happen alternately. And eventually, instead of simple linear correlation, a dependence structure of distribution with many variables is required. The dependence model must have adequate flexibility to explain numerous pragmatic phenomena perceived in the data.

Hoping to provide circumstances for more consumption in the future, each individual starts investing by cutting down on current consumption. Therefore, each individual seeks higher returns in comparison with lower risk tolerance so that the utility is maximized. He prepares a portfolio in order to achieve the above-mentioned objective. Some assets with different weights are placed together to prepare a portfolio. Optimization portfolio may be defined as "selecting the best combination of assets which guarantees the highest return at a specific risk level or the lowest risk tolerance at a specific return level is called optimization portfolio" [7].

Given the fact that one of the main activities done by investment funds, subsidiaries agencies and investment companies, so on is to organize and optimize portfolios, it is essential to select the appropriate optimization method for these companies. In fact, one of the popularities of these companies is their ability to organize portfolios [19]. Portfolio optimization has been taken from Markowitz [34] influential work which introduces a structure for controlling the risk of return/variance. Portfolio optimization is promoted and stirred through two rudimentary factors: "(1) adequate modeling of utility functions, risks, and constraints; (2) efficiency, i.e., ability to handle large numbers of instruments and scenarios. An important principle at work here is that of portfolio diversification" [6]. Portfolio diversification is also of much importance. Precariousness of the portfolio depends not only on the covariances of its components but on the average riskiness of its distinct assets. This principle was not ordinary in the analysis of classical financing which focused on the concept of the importance of single investments, that is, "the belief that investors should invest in those assets that offer the highest future value given their current price" [20]. In the seventh decade William Sharpe, Lintner [25] and Mossin [23] presented the first model concerning the capital asset pricing (CAPM) based on Markowitz's work [34]. This model is used as the measuring standard for the risk-adjusted performance of professional portfolio managers [46].

The question of what risk function should be used in the mean-risk approach has been examined extensively in the literature. Artzner et al. [5] commenced a new research in their paper "Coherent Risk Measures". They specified some mathematical properties that a meaningful risk measure has to

satisfy. It was argued that these axioms the interests of risk-averse investors. In another vein, Ogryczak and Ruszczyński [39,40,41] used stochastic dominance relations to compare portfolio returns [30]. They recognized a number of risk functions for which the top portfolio proceeds are non-dominated as the second order accidental ascendancy relation. Important examples include the semi deviation and weighted deviations from quantile [29,32]. Portfolio choice based on SD is critically more demanding compared to the conventional Mean-Variance (MV) analysis. Nevertheless, modern-day computer hardware and optimization software bring this approach within the reach of practical application [2]. For the common second-order SD (SSD) criterion and a discrete or discretized probability distribution, portfolio optimization is framed as a Linear Programming (LP) problem [48,51]. The problem is very large, but it remains tractable for realistic applications to security selection and asset allocation [20,49].

Using copula theory is traced back to researches by Sklar [52]; however, this tool has been a new financial theory which has been growing a lot for the recent years. Copula has many applications which includes risk management, time series' dependency, and pricing the financial derivatives. Copula has many applications in financial theories because it has expanded the hypothesis which states returns have normal distributions and provided financial models for each variable with every marginal distribution [29,32]. Modern monetary analyses indicate an increasing irregularity of the distribution of earnings on monetary assets. However, in common risk management methods such as minimum variance presented by Markowitz [34], the main assumption is that the distribution of returns on assets is normal. When there is no standard distribution of earnings, the linear correlation coefficient is not a proper measure for declaring the dependency structure [11]. Therefore, in this research copula is used as the substitute measure to model the structure of the dependence. First, the works done by copula are reviewed. Then copula and Clayton copula function is introduced, and financial portfolio optimization is stated in Clayton copula method. After that, the Shape measure obtained through this method is compared with the Sharpe measure obtained through Markowitz mean-variance analysis. Finally, the performances of these two methods are evaluated [8].

There are many documents which indicate that a lot of economic variables do not have a normal distribution, a fact which is traced back to the work done by some authors. Some cases which indicate that single-variable distributions are not normal are excessive strains in series and skewness. Recent studies have even indicated multi-variable distributions are not normal too. In other words, it is said that they have asymmetric dependency. One of the instances of asymmetric dependency is the case in which two assets have higher correlation coefficients at the time of descending market rather than ascending market. Ang and Chen [4] also have mentioned to this matter in their research works. They studied the correlation coefficient of stocks among G-7 countries. They realized that correlation coefficient in periods of recession is higher than in periods of economic growth. Using the extreme value theory, they modeled the series of multivariable distribution functions. In their research, the hypothesis pertaining to multivariable distribution of returns in negative series was refuted, while they could not refute the hypothesis zero in positive series.

Modeling the dependency is one of the key factors in creating the portfolio and risk management. Selecting an inappropriate model would result in the selection of a non-optimal portfolio and inaccurate risk measurement. Traditionally, the correlation coefficient has been used to explain the dependency between variables; however, the recent researches indicate the superiority of copulas for modeling dependency due to their higher flexibility rather than correlation coefficient approach. The works done by Embrechts et al. [17] can be assumed as an examples. The linear correlation coefficient has a major flaw. It is not based on nonlinear transformation invariant. However, the dependency measures which have been taken from copula can overcome this problem [44]. They have more extensive appli-

cations. The term copula was first used by Sklar [52]. It is now one of the important methods of generating multivariable distributions. They, studied the application of copula in pricing the derivatives. Also, Campbell et al. [9], and Ang and Chen [4] studied the use of copula in portfolio diversification [26]. Conditional copula was first introduced in the PhD thesis by Paton. They also studied the application of copula in pricing the options. Neslehova et al. used copula to investigate banking operations [43].

Wei and Xiong [55] made use of copula-GARCH with the intention of estimating the dependency structure between Stock Market of Shenzhen and that of Shanghai. Using the method of copula-GARCH, Ang and Chen [4] investigated portfolio risk in China's Stock Market. Archimedean copula functions have been applied by several authors to model the dependence structure between stock market returns [13,16,24,21,47,53] and the dependence structure between exchange rate returns. To model the dependence structure between stock market returns, several authors such as Chollete [12] and Tsafack [27] have utilized Archimedean copula functions and the dependence structure between exchange rate returns. Copulas have been used in various papers and books on portfolio optimization with different approaches and techniques. See for example; Alexander [1], Kakouris and Rustem [31] and Jahanbakhshi, [22].

Aleš Kresta [3] in a study entitled "Application of GARCH-Copula Model in Portfolio Optimization" states that Markowitz established the basis of modern portfolio theory in 1952. The portfolio optimization problem is a never-ending research topic for both academics and practitioners. In this problem, the future prediction of time series evolution plays an important role. However, it is rarely addressed in research. He analyses the applicability of the GARCH-copula model. To be more concrete he assumes the investor maximizing Sharpe ratio while the future evolution of the time series is simulated by means of the AR-GARCH model using the copula modelling approach. The bootstrapping technique is applied as a benchmark. From the empirical results, he found out that the GARCH-copula model provides better forecasts of future financial time series evolution than the bootstrapping method. Assuming the investor is maximizing the Sharpe ratio, both the final wealth increases and maximum drawdown decreases when he applies the GARCH-copula model compared to the application of bootstrapping technique.

2 Research Methodologies

2.1. Copula Functions

We can define a copula function as an instrument to join or "couple" a multivariate distribution function to its one-dimensional marginal distribution functions. Copula modeling describes multivariate distribution functions via their marginal distribution functions and a dependency function named copula. Nelsen [38] presents the theoretical and practical aspects. It is intended to set apart the dependency structure from the marginal distributions structure.

The copulas' characteristics simplify studying dependencies in financial markets. As some of these characteristics, we can note that first; copulas are invariant to constant transformations of random variables. Second; parameters of copulas and measures of concordance have a direct relation, which is widely used by Kendall's (tau). And Third; we can observe an asymptotic dependence treatment in the tails of the distributions by copulas.

In financial markets the marginal distribution of returns is not considered normal, and on the other hand the inter-return dependency should not be linear. Therefore, we can conclude that dependency cannot be measured by Pearson's linear correlation. Instead, copula functions which are very logical tools can be used to model joint distributions. Joint distribution functions can usually be modeled through copula functions in a better way than oval distributions.

To understand the concept of copula better, a short description pertaining to the simulation method based on a specific distribution function helps a lot. However, the basic concepts are dealt. We can consider X as a continuous random variable within the domain D .

The distribution function F for the random variable X is a monotonic ascending function from D to $[0, 1]$, so:

$$F(x) = P(X < x) \quad (1)$$

For each, therefore, the probability at which X is smaller than x is equal to the value of the distribution function X at point x .

Inverted distribution function X is defined as every other inverted function in the following way:

$$F^{-1}: [0,1] \rightarrow D \quad (2)$$

$$F^{-1}(F(x)) = x \quad (3)$$

If we assume that F is differentiable all over D ; then, the density function of X is equal to the derivative of distribution function F , and since F is a monotonic ascending function, then.

Quantile¹ is the continuous random variable X with the probability $\alpha \in [0,1]$ and the value of the random variable X in a way that:

$$P(X < X_\alpha) = \alpha$$

Quantile is used for simulation. First we simulate a random variable like u which indicates probability. Since this value is obtained from a standard uniform distribution, it is indicated with the symbol u . Now the inverted distribution function is used to find the relevant quantile. The u of quantile pertaining to the distribution X is as follows:

$$x = F^{-1}(u) \quad (4)$$

The variable, which has a uniform distribution, has a linear normal distribution. Especially, the standard uniform variable $U \sim U(0,1)$ has the following characteristics.

$$P(U < u) = u \quad (5)$$

$$P(F(x) < u) = P(X < F^{-1}(u)) = F(F^{-1}(u)) = u$$

Therefore, we conclude that:

¹. in statistics, a quintile for the case where the sample or population is divided into fifths or one of the four numbers (values) that divide a range of data into five equal parts, each being 1/5th (20 percent) of the range.

$$F(X) \sim U(0,1) \quad (6)$$

Equation (6) indicates that using the distribution function for the random variable X , would result in the new random variable $F(X)$. This variable has a standard uniform distribution. This action is called probability integral transform in mathematics. We can define probability integral transform as an instrument to transfer a continuous random variable to a uniform one. If, then we have:

$$P(U < F(x)) = F(x) \quad (7)$$

In other words, we have $F(x) = P(F^{-1}(U)) < x$. The above-mentioned equation indicates that the distribution of the variable X can be simulated by using the inverted distribution function and the standard uniform variable. Each time the random variable u is generated, a series of independent variables is transformed to a set of simulated values of distribution X by using the inverted distribution function.

In other words, whenever we want to simulate a value for the random variable, first we generate a random number which has a uniform distribution. Then we perform the simulation process by using the inverted distribution function. If we want to simulate some random variables, the procedure is the same. However, we should consider the mutual dependency of random variables here. If we go away from the assumption which states the joint distribution of variables is oval, the linear correlation coefficient loses its importance, and other dependency measures should be used as copula.

2.2. Definition of Copula

We can define Copulas as functions that join or link multivariate distribution functions to their one-dimensional marginal distribution function [38]. If two random variables such as X_1 and X_2 are considered with marginal continuous distribution functions such as $F_1(X_1)$ and $F_2(X_2)$, and $u_i = F_i(x_i)$, $i = 1, 2$, then the functions which have the following characteristics are considered to be 2-dimensional copula functions:

$$\begin{aligned} C: [0,1] \times [0,1] &\rightarrow [0,1] \\ C(u_1, 0) = C(0, u_2) &= 0 \\ C(u_1, 1) = u_1 \quad \text{and} \quad C(1, u_2) &= u_2 \\ C(v_1, u_2) - C(u_1, v_2) &\geq C(v_1, u_2) - C(u_1, u_2) \quad \text{If } u_1 \leq v_1 \text{ and } u_2 \leq v_2 \end{aligned} \quad (8)$$

The first condition requires that copula function be placed within the value limits of distribution function. According to the previous discussion, we know that the value of each distribution function is a standard uniform variable. Therefore, we can put $u_i = F_i(X_i)$, $i = 1, 2$.

The next three conditions indicate copula function as the joint distribution function of U_1 and U_2 . There are many functions which meet the above-mentioned conditions; therefore, copula functions in number is a lot.

In 1959, Esklar indicated that copula functions were uniform under certain circumstances. The double-variable format of Esklar's theory is as follows:

For each joint distribution function $F(x_1, x_2)$, there is a unique copula function.

Inversely, if C is a copula function, as $F_1(x_1)$ and $F_2(x_2)$ are distribution functions, then the copula function is a joint distribution function with marginal functions such as $F_1(x_1)$ and $F_2(x_2)$. If the copula function is differentiated on x_1 and x_2 , the joint density function $f(x_1, x_2)$ is indicated according to the marginal density functions $f_1(x_1)$ and $f_2(x_2)$ as follows:

$$f(x_1, x_2) = f_1(x_1)f_2(x_2)c(F_1(x_1), F_2(x_2)) \quad (9)$$

If we place in the above-mentioned equation, Equation (10) can be indicated as:

$$c(F_1(x_1), F_2(x_2)) = \frac{(\partial^2 C(F_1(x_1), F_2(x_2)))}{(\partial F_1(x_1) \partial F_2(x_2))} \quad (10)$$

If the above equation $F_i(x_i) = u_i$ placement, the equation (10) can be for $c(u_1, u_2)$ showed. When equation (10) is calculated according to (u_1, u_2) instead of (x_1, x_2) , it is called copula density.

The above-mentioned concepts can be more generalized. If n random variables such as are considered with specific marginal distribution

Functions such as, copula is a monotonic ascending function of provided that it meets all three conditions.

Eskelar's theory tells us that separate copulas for different density functions by using sets resulting from continuous suits of marginal functions. Using a specific joint density function such as, the unique copula functions C is obtained as Equation (11):

$$F(x_1, x_2, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)) \quad (11)$$

If this function exists, its copula density is as follows:

$$c(F_1(x_1), \dots, F_n(x_n)) \quad (12)$$

Now we can obtain the joint density function for variables by using copula density function and marginal density functions as follows:

$$f(x_1, \dots, x_n) = f_1(x_1) \dots f_n(x_n) c(F_1(x_1), \dots, F_n(x_n)) \quad (13)$$

Since the values of marginal density distribution functions are uniform, we can present copula functions by using the variables. If, each joint distribution function F is an implicit copula which is defined as follows:

$$C(u_1, \dots, u_n) = F(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)) \quad (14)$$

In the above-mentioned equation, is the quintile pertaining to marginal distribution functions. Therefore, there is one implicit copula function for each joint distribution function. Copula density function is indicated with the symbol as follows:

$$c(u_1, \dots, u_n) = \frac{\partial^n C(u_1, \dots, u_n)}{\partial u_1 \dots \partial u_n} \quad (15)$$

2.3 Archimedean Copulas

In contrast to implicit copulas, Archimedean copulas are often called explicit copulas since they can be constructed using an explicit formula. Within the Archimedean copulas this paper will focus on the one parameter Clayton copula. This means that there is only one parameter that forms the distribution and determines the strength of dependence.

The Archimedean copulas are constructed by using a generator function $\Psi(u)$ and its pseudo-inverse $\Psi^{[-1]}(u)$. A continuous, strictly decreasing convex function $\Psi : [0,1] \rightarrow [0, \infty]$ which satisfies $\Psi(1) = 0$, is define as an Archimedean copula generator. If $\Psi(0) = \infty$, it is also known as a strict generator. Let $\Psi(u)$ be an Archimedean generator function that fulfills $\Psi(0) \leq \infty$. Then the function Ψ has a pseudo inverse, $\Psi^{[-1]}$ defined by [1].

$$\Psi^{[-1]}(u) = \begin{cases} \Psi^{[-1]}(u), & 0 \leq u \leq \Psi(0). \\ 0, & \Psi(0) < u \leq \infty \end{cases}$$

Given any generator function $\Psi(u)$, an Archimedean copula is defined by:

$$C(u_1, \dots, u_n) = \Psi^{-1}(\Psi(u_1) + \dots + \Psi(u_n)) \quad (16)$$

Its density function is as follows:

$$c(u_1, \dots, u_n) = \Psi_{(n)}^{-1}(\Psi(u_1) + \dots + \Psi(u_n)) \prod_{i=1}^n \Psi'(u_i) \quad (17)$$

In above equation, $\Psi_{(n)}^{-1}$ is the n^{th} derivative of the inverse generator function and $\Psi'(u)$ denotes the derivative of $\Psi(u)$. Since there are many generator functions, different Archimedean copulas would be resulted. Only Nelson (2006) has introduced about 22 single-variable Archimedean copulas. Clayton copula is of importance for us because it presents the asymmetric dependence of the distribution sequence. Clayton Copula shows dependence in the sequence below:

2.4. Clayton Copula

The Clayton copula is one of the many proper subsets of Archimedean copulas and was first studied by Clayton (1978). This type, has become popular in financial risk modeling since it can handle asymmetric tail dependence and has a strong dependence in the lower tail of these modeling's. The Clayton copula is described by the parameter α and can be modeled in D dimensions, considering the generator function is strict and its inverse is entirely monotonic. The following function is of famous and commonly used generator functions in financial calculations:

$$\Psi(u) = \alpha^{-1}(u^{-\alpha} - 1), \alpha \neq 0 \quad (18)$$

Thus, the inverse of this function is as follows:

$$\Psi^{-1}(x) = (\alpha x + 1)^{-\frac{1}{\alpha}} \quad (19)$$

Using these two functions, the Archimedean Copula function is obtained as follows:
Equation (20)

$$C(u_1, \dots, u_n; \alpha) = (u_1^{-\alpha}, \dots, u_n^{-\alpha})^{-\frac{1}{\alpha}} \tag{20}$$

This function was firstly proposed by a person named Clayton in 1978, so it is called Clayton copula. If we take the first order derivative of the function Clayton copula, density function of Clayton copula will be obtained.

$$c(u_1, \dots, u_n) = (1 - n + \sum_{i=1}^n u_i^{-\alpha})^{-n - (\frac{1}{\alpha})} \prod_{j=1}^n (u_j^{-\alpha - 1} ((j - 1)\alpha + 1)) \tag{21}$$

While the parameter α varies, when $\alpha \rightarrow \infty$ we will notice the dependence of the Clayton copulas and complete positive dependence.

When there are only two variables, density function of Clayton copula is as follows:

$$c(u_1, u_2) = (\alpha + 1)(u_1^{-\alpha} + u_2^{-\alpha})^{-2 - (\frac{1}{\alpha})} u_1^{-\alpha - 1} u_2^{-\alpha - 1} \tag{22}$$

function is positive.

As noted above, Clayton copula presents the asymmetric dependence of the distribution sequence. This function has a sequence affinity above zero, but dependence coefficient of lower sequence of this. Dependence of lower sequence of this function is as follows:

$$\lambda^l = \begin{cases} 2^{-\frac{1}{\alpha}}, & \alpha > 0 \\ 0, & \alpha \leq 0 \end{cases} \tag{23}$$

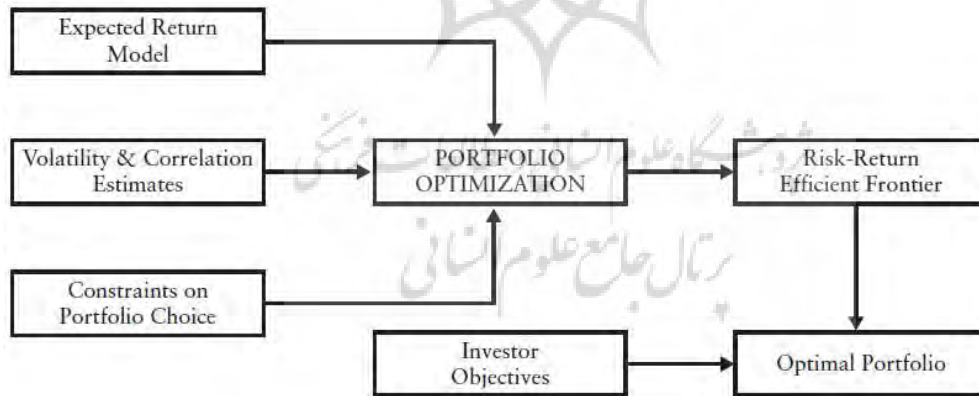


Fig. 1: Density function of Clayton copula with $\alpha=0.5$

2.5. Copula Function and Multi-Dimensional Simulation by Monte Carlo

Copula function can be defined as the joint cumulative distribution function (cdf) of variables such as x_1, x_2, \dots, x_n with the marginal cdf $F_1(x_1), F_2(x_2), \dots, F_n(x_n)$ respectively [38]. So, considering H

(x_1, x_2, \dots, x_n) as the joint cumulative distribution function, we can conclude that the copula function $C(u_1, u_2, \dots, u_n)$ satisfies $H(x_1, x_2, \dots, x_n) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n))$, where, $H(x_1, x_2, \dots, x_n) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n))$.

Now returning to how we defined the copula function, we notice that using of copula provide us the opportunity to estimate the joint cdf by two parts:

- I. Ascertaining the marginal distributions $F_1(x_1), F_2(x_2), \dots, F_n(x_n)$ which represent the distribution of each variable, and calculating their parameters. For the marginal cdf, we can use the traditional time series models like GARCH model, as well as the student-t distribution or empirical distribution;
- II. Ascertaining the dependence construction of the variables x_1, x_2, \dots, x_n , designating a proper copula function.

In this paper we have applied the empirical $\hat{F}_1(x) = \frac{1}{T} \sum (X_{it} \leq x)$ distribution for the marginal MLE, using the maximum likelihood estimation (MLE) for the Archimedean copula's parameters, and for the good-of-fit test of estimated copula function, we have applied Kolmogorov-Smirnov.

And regarding to maximum likelihood estimation, we have chosen the method of solving the extreme of the sample's joint pdf's multiplying value. The two-dimensional variable's joint pdf is $f(x_1, x_2) = c(u_1, u_2) \cdot f(x_1) \cdot f(x_2)$, where $f(x_1), f(x_2)$ are the cdfs of the variables x_1, x_2 ; and the function $c(u, v) = \partial^2 C(u, v) / \partial u \partial v$ is the copula's density function. So the maximum likelihood function $L((x_1, y_1), (x_2, y_2), \theta)$ can be stated as:

$$L = \prod f(x_1, x_2) = \prod c(u_1, u_2) f(x_1) f(x_2) \quad (24)$$

Now we can attain an estimate of copula's parameters by solving the extreme value. After designing the copula model, this model can describe the joint distribution involving all dependence structure, using it we can grasp a better conclusion for risk management and portfolio management, which is more accurate than the results given by normal hypothesis.

After perceiving all parameters of the copula, we have to generate series of variables x_1, x_2, \dots, x_n with the joint cdf of $C(u_1, u_2, \dots, u_n)$, where the uniformly distributed variables u_1, u_2, \dots, u_n equals the marginal cdf $F_1(x_1), F_2(x_2), \dots, F_n(x_n)$ of random variables x_1, x_2, \dots, x_n respectively.

Firstly, by using two-dimensional Clayton copula of $C(U_1, U_2)$, variables u_{1i} and u_{2i} are simulated and n denotes the number of simulated samples. Conditional distribution is used to achieve this goal. For this purpose, C_{u_1} is placed in two-dimensional Clayton copula of $C(U_1, U_2)$ as conditional distribution function of random variable u_2 in order to obtain the values of u_1 . Accordingly, the following equation will be resulted:

$$C_{u_1}(u_2) = \partial C(u_1, u_2) / \partial u_1$$

In addition, C_{u_1} is non-decreasing and always in the range $[0, 1]$.

2.6. Clayton Copula in Optimization of Portfolio

One of the most rewarding and useful areas in decision-making is the process of optimization of portfolio. The mean-variance formulation by Markowitz in 1950s was the cornerstone for modern portfolio selection analysis.

Tobin [54], using the works of Markowitz, added the risk-free asset to the portfolio. The main efficacy of using the variance for describing portfolio risk is because of easiness of the computation, but the symmetry problem makes the variance somehow unsatisfactory.

In the classical portfolio management theory, it is considered that the multivariate joint distribution of the return variables is joint normal distribution, but at the end it's not an ideal and useful instrument for the real financial return variables.

If we want to take care of the variables which are not distributed elliptically and to increase the accuracy of risk measuring, we can use the copula in the portfolio optimization, noting that we must express the variances without normal hypothesis. So for showing the variance of the portfolio using copula, we use the copula based simulation to calculate the variance in rundowns analytically.

The multivariate joint pdf can be indicated by the copula function, which is given by $f(x_1, x_2, \dots, x_n) = c(u_1, u_2, \dots, u_n) \cdot \prod_{j=1}^n f(x_j)$.

Where $c(u_1, u_2, \dots, u_n) = \frac{\partial C(u_1, u_2, \dots, u_n)}{\partial u_1 \partial u_2 \dots \partial u_n}$ is the cdf of copula in formula, the function $f(x_i)$ denotes the i^{th} variable's pdf. Let $f(r) = f(r_1, r_2, \dots, r_n)$, then the variance of the portfolio can be denoted by:

$$\sigma_p^2 = \int_{-\infty}^{+\infty} [r_p - E(r_p)]^2 f(r_p) dr_p = \int \left[\sum_{i=1}^n w_i r_i - E(r_p) \right]^2 f(r) dr_1 dr_2 \dots dr_n$$

To avoid the complex analytical calculation, we can use the distinct samples as follows:

$$\sigma_p^2 = \frac{1}{m} \sum_{j=1}^m \left[\sum_{i=1}^n w_i r_{ij} - E(r_p) \right]^2 = \frac{1}{m} \sum_{j=1}^m |w_1 r_{1j} + w_2 r_{2j} + \dots + w_n r_{nj} - E(r_p)|^2$$

Where n is the number of loss variables, and m is the number of discrete samples, which can be calculated by the copula based Monte Carlo simulation. This analytical calculating formulation, let us have the optimization of a portfolio manager using the copula method as follows;

$$\begin{aligned} \min \sigma_p^2 &= \frac{1}{m} \sum_{j=1}^m \left[\sum_{i=1}^n w_i r_{ij} - \sum_{i=1}^n w_i E(r_i) \right]^2 \\ \text{s. t. } E(r_p) &= \sum_{i=1}^n w_i E(r_i) = \kappa, \quad \sum_{i=1}^n w_i = 1 \end{aligned}$$

In which κ is the portfolio target return.

Now we can attain the mean-variance efficient frontier by giving series of target returns κ . noting the added risk-free asset, the optimal proportions can be calculated by the following programming formula.

$$\begin{aligned} \max \frac{E(r_p) - r_f}{\sigma_p} &= \frac{E(r_p) - r_f}{\sqrt{\frac{1}{m} \sum_{j=1}^m \left[\sum_{i=1}^n w_i r_{ij} - E(r_p) \right]^2}} \\ \text{S. t. } \sum_{i=1}^n w_i &= 1, \quad w_i > 0 \end{aligned}$$

2.7. Using Copula in Modeling the Marginal Distributions and the Dependence Structure

We considered the GARCH models for the series r_1, r_2 as initial models with normal and student-t distribution.

With analysis we find that there is no autocorrelation between the two series. And the simple GARCH (1,1) model we apply is in the following expression:

$$\begin{aligned} r_t &= \mu + a_t, a_t = \sqrt{h_t} \cdot \varepsilon_t \\ h_t &= c + \beta_1 \cdot a_{t-1}^2 + \beta_2 \cdot h_{t-1} \end{aligned}$$

Where ε_t is white noise processes with zero mean and unit variance, β_1 and β_2 follows the restriction $\beta_1 + \beta_2 < 1$. We can anticipate the distributions of the two return variables using the GARCH model. The copula function we have used here is Clayton copula which mainly defines the dependence of left tail. The multi-dimensional Clayton copula is indicated as follows:

$$C(u_1, u_2, \dots, u_n) = \left(\sum_{i=1}^n u_i^{-\theta} + 1 - n \right)^{-\frac{1}{\theta}}$$

After the maximum likelihood estimation, the parameter θ of the two-dimensional copula between Gold and Copper is 0.1186. Then the two-dimensional copula function can be expressed in the following formula.

$$C(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-\frac{1}{\theta}} = (u_1^{-0.1186} + u_2^{-0.1186} - 1)^{-\frac{1}{0.1186}}$$

As we have expressed, the conditional distribution $u_2 \rightarrow Cu_1(u_2) = \partial C(u_1, u_2) / \partial u_1$ is uniformly distributed on $[0, 1]$. Now using the Kolmogorov-Smirnov method, we can examine the values of $Cu_1(u_2)$ assessed by the sample data (r_{1i}, r_{2i}) . The result shows that the P value is 0.2869, which is higher than the confidence level 0.05. As a result, the two-dimensional Clayton copula puts in place the sample data properly.

Applying the copula based Monte Carlo simulation; we will have 10000 series of data for the following empirical analysis.

2.8. Traditional portfolio efficient frontier

We note ω_1, ω_2 , are the proportions of the portfolio. Based on the traditional mean-variance theory, we get the minimum standard variance with the optimal proportion under given target mean of the portfolio.

2.9. Copula based portfolio efficient frontier

Opposite to the traditional mean-variance theory, we used the GARCH based marginal distributions and copula based Monte Carlo simulation to calculate the efficient frontier. We have used MATLAB software.

2.10. Markowitz Mean-Variance Optimization Method

Mean-variance analysis has had the most prevalence in solving the problems of portfolio management. Mean-variance analysis presents the scheme to build and select portfolios, which is based on the expected performance of the investments and the investor tendency toward risks. Mean-variance analysis provided us with a comprehensive new vocabulary, which has become the standard in the area of investment management.

However, now and more than 50 years after Markowitz's seminal work, we see that only quantitative companies are using mean-variance portfolio optimization, where processes for automated forecast generation and risk control are already in place.

Still portfolio management remains an exclusively judgmental process based on qualitative, not quantitative, assessments. It seems that the quantitative efforts at most companies are meant to provide risk measures to portfolio managers.

These measures provide asset managers a view of the risk level in a particular portfolio, where risk is defined as underperformance relative to a mandate.

Markowitz believed that a wise investor, given different choices between portfolios, for any given level of expected return, would choose the portfolio with minimum variance. These different choices between portfolios is called the feasible set.

Minimum variance portfolios are called mean-variance efficient portfolios. The set of all mean-variance efficient portfolios, for different desired levels of expected return, is called the efficient frontier.

Mean-variance optimization model can be defined within the target function and the following constraints:

$$\min \Omega_p = \left[\sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_{ij} \right] \quad (26)$$

$$W^T \mathbf{1} = 1$$

$$W^T E(y) = \mu$$

$$W = [w_1, w_2, \dots, w_n] \quad W \geq 0$$

Where, $w, \sigma_{ij}, y, \mu,$ and $\mathbf{1}$ denote vector of portfolio weights, covariance matrix (conditional or unconditional) of return on asset (ROA), vector of ROA, expected return of optimized portfolio, and the vector whose all elements are equal to 1, respectively.

Since there is no short-selling assumption in portfolio optimization, the adverb which corresponds to the weights is considered equal to 1.

As the portfolio covariance is equal to correlation coefficient of assets (ρ) multiplied by standard deviation of the asset (σ), portfolio variance can be presented in the form of following model:

$$\sigma_{ij} = \sigma_i \sigma_j \rho_{ij}, \quad -1 \leq \rho_{ij} \quad (27)$$

$$\Omega_p = \left[\sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_i \rho_{ij} \sigma_j \right]$$

Standard deviation of portfolio is obtained by the following equation:

$$\Omega_p = \left[\sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_i \rho_{ij} \sigma_j \right]^{0.5} \tag{28}$$

As can be observed in the above equations, variance and correlations are decisive factors in determining the optimal weights of portfolio aimed at minimizing the risk. We can show a schematic view of the investment process as seen from the perspective of modern portfolio theory in the image below.

After the optimal portfolio was obtained by both methods, inter-subject performance of Clayton copula method, compared with Markowitz mean-variance method, was evaluated using Sharp index.

3 Data and Findings

Since copper and gold assets are traded in different markets, their trading hours are different. In order to eliminate false correlation in the present study, we attempted to use the data of the days when there was a price range for both assets.

Therefore, data of this study included 625 weekly observations of price of copper and gold in two-time series during the period from the early 2002 until the end of October 2013 and these data are extracted from the MetaTrader4 software. 615 observations were used for analysis and modeling. Then, 10 investment portfolios were fitted on the efficient border and Sharp index commensurate with these 10 portfolios were calculated. Then, by rolling method, a week was removed from the beginning of 2002 and was added to the first week in November 2013 and this process was repeated for 9 consecutive weeks. For investigating Clayton copula and mean-variance methods in optimization of portfolio, Sharp index was calculated for these two methods. The results are shown in the following tables.

Table 1: Sharpe Measure with Markowitz Mean-Variance Analysis

Sharpe Measure with Markowitz Mean-Variance Analysis								
First Week	Second Week	Third Week	Fourth Week	Fifth Week	Sixth Week	Seventh Week	Eighth Week	Ninth Week
0.043273	0.043968	0.042315	0.040234	0.040864	0.039866	0.040883	0.039647	0.040859
0.042919	0.043939	0.042294	0.040132	0.040644	0.039842	0.040859	0.039499	0.040796
0.041776	0.043839	0.042209	0.039294	0.039674	0.039075	0.040086	0.038626	0.039975
0.040021	0.043678	0.042056	0.037861	0.038123	0.037724	0.038709	0.037182	0.038553
0.037891	0.043437	0.041838	0.036043	0.036193	0.035989	0.03694	0.035361	0.036742
0.035594	0.043141	0.041558	0.034039	0.034087	0.03406	0.034968	0.033361	0.034741
0.033282	0.042785	0.041219	0.031992	0.031957	0.032082	0.032939	0.031324	0.032694
0.031053	0.042372	0.040827	0.030009	0.029898	0.030154	0.030957	0.029345	0.030702
0.028967	0.041909	0.040386	0.028142	0.027963	0.028333	0.029084	0.027482	0.028821
0.027038	0.041404	0.039902	0.026416	0.026179	0.026646	0.027347	0.025763	0.027081

Table 2: Sharpe Measure with Clayton Copula

Sharpe Measure with Clayton Copula and Normal Model for Variance Anisotropy GARCH (1.1)								
First Week	Second Week	Third Week	Fourth Week	Fifth Week	Sixth Week	Seventh Week	Eighth Week	Ninth Week
0.068343	0.055138	0.095888	0.073519	0.097714	0.08248	0.090781	0.098419	0.088538
0.068733	0.055718	0.095408	0.073653	0.099699	0.082966	0.093384	0.098549	0.090751
0.069039	0.056026	0.089307	0.073774	0.100275	0.083368	0.094309	0.098667	0.091588
0.069258	0.05606	0.081059	0.073883	0.099592	0.083683	0.093782	0.098772	0.091207
0.069393	0.055839	0.072831	0.073979	0.097857	0.083914	0.092111	0.098865	0.089827
0.069444	0.055378	0.065521	0.074061	0.095331	0.084058	0.089636	0.098945	0.087697
0.06941	0.054701	0.059314	0.07413	0.09228	0.084117	0.086677	0.099012	0.085072
0.069298	0.053843	0.054137	0.074187	0.088947	0.084092	0.083477	0.099066	0.082179
0.069108	0.052841	0.04983	0.074229	0.085522	0.083987	0.080231	0.099107	0.079166
0.068845	0.05173	0.046226	0.074258	0.082125	0.083806	0.077054	0.099135	0.07617

5 Conclusion and Discussion

This paper applies the Clayton copula theory for the optimization of portfolio management. Our empirical analysis in the first place uses two-dimensional copula for Monte Carlo simulation.

With the Clayton copula based multi-dimensional simulated scenarios, we attain the optimal investing proportions of the portfolio under the minimum of standard variance calculated by Clayton copula. Adding the risk-free asset, we also get the optimal proportion and the tangent of “the mean-standard variance” efficient frontier. Furthermore, we apply the Clayton copula into the portfolio optimization, which we call it Clayton copula method. Under the objective function of minimum of Clayton copula, we get another series of optimal investing proportions. And eventually, the “mean- variance” frontier is calculated with the tangent under risk-free asset. In order to study the significance of difference between the performance of Clayton copula and Markowitz mean-variance methods and for testing the hypotheses, t-student test was used for comparing the mean of two communities regarding the variance anisotropy. The results are shown in the following tables.

The null (H_0) and alternative (H_1) hypotheses were as follows:

H_0 : There is no significant difference between the mean community of Clayton copula and Markowitz mean-variance methods.

H_1 : There is a significant difference between the mean community of Clayton copula and Markowitz mean-variance methods.

According to Table 3, the obtained value of t-student is equal to -24, which it is in the critical range. This means that H_0 is rejected at a confidence level of 95% and it can be stated that there is a significant difference between the mean community of Clayton copula and Markowitz mean-variance methods. Since the mean value of Clayton copula method is more than that of Markowitz mean-variance method, it can be concluded that, at a confidence level of 95%, Clayton copula method is more accurate than Markowitz mean-variance method in the prediction of returns based on Sharp index. It should be also mentioned that with regard to the P-Value obtained from Excel, the results of the above-mentioned test are reliable at a confidence level of 99%.

The main objective of the present study was to identify the most appropriate method for optimization of portfolio between two methods of Clayton copula and Markowitz mean-variance. Hence, port-

folio optimization by Clayton copula function was introduced in previous parts. Although this method is commonly used in the world, it is less used in Iran.

Table 3: Clayton Copula Paired Comparison Test and Markowitz Mean-Variance Analysis

	Mean-Variance	Clayton Copula
Mean	0.0364633	0.0799480
Variance	2.97049E-	0.0002202
Observations	90	90
Pearson Correlation	-0.2917754	
Hypothesized Mean Difference	0	
Df	89	
t Stat	-23.931541	
P(T<=t) one-tail	0.0000000	
t Critical one-tail	1.6621553	
P(T<=t) two-tail	0.0000000	
t Critical two-tail	1.9869787	

From the empirical results we found out that COPULA- CVaR model provides better forecasts of future financial time series evolution than mean-variance method. Finally, the Sharpe ratio obtained through this method is compared with the one obtained through Markowitz mean-variance analysis to ascertain that Clayton-copula is more efficient in portfolio-optimization.

Given that, this study takes benefit from a research done by Bai and Sun [32] the results of this study are consistent with that research. The research of Bai and Sun Copula- CVaR model were compared with the mean-CVaR model. The results showed excellence copula- CVaR model than the mean-CVaR model. According to the results of this study, based on Sharpe index, Clayton copula method has a better performance than Markowitz mean-variance method. Also our research results are based on researches results of Jondeau & Rockinger [24], Aleš Kresta [3], Cherubini & Luciano & Vecchiato [11], Lujie Sun & Manying Bai [29] and Chollete et al. [12].

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