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Control Theory and Economic Policy Optimization: The Origin, Achievements and the Fading Optimism from a Historical Standpoint

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Abstract

Economists were interested in economic stabilization policies as early as the 1930's but the formal applications of stability theory from the classical control theory to economic analysis appeared in the early 1950's when a number of control engineers actively collaborated with economists on economic stability and feedback mechanisms. The theory of optimal control resulting from the contributions of mathematicians Lev SemenovichPontryagin and Richard Ernest Bellman in the late 1950's was first applied successfully to models of economic growth in the 1960's by the economists who were interested in discovering the optimality properties of economic growth trajectories. It is shown that the collaborations of control engineers with econometricians in the 1970's on the computation of optimal state and control trajectories in econometric models were the earliest attempts to demonstrate the possibility of applying deterministic, stochastic and adaptive optimal control to the numerical solution of optimal economic policies. We have explained why the collaborations of control engineers with econometricians on formulating and computing optimum system design in macroeconomic optimal planning models failed and why the economic applications of optimal control theory have proved to be more productive in the analysis of optimality conditions in mathematical economics and not in the computation of optimal trajectories in econometric models.

Keywords: Optimal Control Theory, Stochastic Control, Adaptive Control, Economic Applications.

JEL Classification: C54, C61, E61

1. Introduction

Our objective in this paper is to examine the origin of the application of optimal control theory to economic policy optimization from a historical standpoint and to explain where and why this application has been partially successful and/or proved to be a failure. We do not intend to provide a literature review on this subject, hence the earliest applications are examined to explore how economic optimization problems within a control theoretic framework were

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conceptualized, mathematically formulated and econometrically estimated. Moreover, this paper attempts to examine the question that to what extent the unwillingness of control engineers, mathematical economists and econometricians to identify the structural differences between economic and physical systems was responsible for the failure of the application of optimal control theory to optimum system design and estimation in economic modeling. Since optimal control is an advanced topic in mathematical optimization, this paper can be considered as an essay in the limitations of mathematical approach to economic policy formulations.

Most definitions of economics share the idea of allocating given *means* for the optimum *satisfaction* of given ends. In this sense, an economic system can be regarded as a *closed system* with given means being defined as a *bounded control space* and the satisfaction of given ends being represented by a *performance* (or *objective*) function. From a mathematical point of view, optimal control can, in principle, solve this problem. Moreover, economic growth theories and stabilization policies possess the characteristics which may facilitate the application of optimal control theory.

Optimal growth theory is concerned with the optimal choice among alternative trajectories along which an economic system can be transformed from a given initial position to a desired state at the end of a specified horizon, where each trajectory is generated by applying a set of feasible economic controls. According to optimal control theory, an admissible control should possess an optimizing character. This has made the application of optimal control theory to economic growth and planning models more productive due to the fact that an economic stabilization program with no optimality condition may not guarantee an optimum design of an economic system.

Section 2 deals with the applications of classical control theory to economic stabilization policies. In section 3, we examine the factors which were historically conducive in the application of optimal control theory to economic optimization problems. Section 4 deals with the early applications of the dynamic programming, developed by Richard Ernest Bellman, to the optimum economic policy design. The early applications of the Maximum Principle developed by Lev SemenovichPontryagin to the optimality conditions in models of economic growth are the subject matter of section 5. Section 6 considers the contributions of control engineers and their collaborations with econometricians on the computation of optimal state and control trajectories in econometric models. Stochastic and adaptive optimal control applications to econometric models are also discussed in this section. Section 7 considers the nature of the unfulfilled expectations in the application of optimal control theory to economic policy optimization and attempts to provide an answer to the question that why the application of optimal control theory to economic policy optimization did not provide results of value particularly at macroeconomic level. And finally the summary and concluding remarks are the subject matter of section 8.

2. The Early Collaborations of Control Engineers with Economists: Applications of the Classical Control Theory to Stabilization Policies

In the 1930's the underlying theory of servo-mechanism, and particularly the self-regulating systems and automatic stabilizations, was being established in engineering sciences and applied mathematics. This motivated a number of economists to study cyclical behavior and oscillations in economic variables in the context of self-regulating systems. The work of Ragnar Frisch (1933) and Michal Kalecki (1935) fall in this category. Most writers in this era, for example John Maynard Keynes (1936), were of the opinion that there is no tendency inherent in the economic system to generate stability and full employment. Hence, it was concluded that control actions in the form of government economic policies were necessary.

The formal economic applications of servo-mechanism did not take place until the 1950's when a number of control engineers became interested in economic stabilization policies. We may refer to a historical significant event which marks the symbolic partnership of economists and control engineers; a partnership which is characterized by a happy beginning, instability in the course of its development and the unfulfilled expectations.

On 18 July 1951, an informal evening session took place during the Conference on Automatic Control at the College of Aeronautics, Cranfield. The purpose of this session was to bring to the attention of the conference the analogy between problems arising in stabilizing economic systems and those of physical systems, with the implication that economists and control engineers could possibly benefit from their respective specializations (Arnold Tustin, 1952).

Richard Stone from Cambridge opened the session with a lecture in which he referred to the use of electrical analogues in interpreting the Leontief transaction matrices and demonstrated the similarity of Kirchhoff's first law to the accountancy relationships for a basic unit. Arnold Tustin, Professor of Electrical Engineering at the University of Birmingham, showed how dynamic economic models being used by econometricians corresponded precisely with the engineer's *scheme of dependence*. He interpreted the Keynesian economic system in terms of a closed sequence with the multiplier as the effect of a feedback. The session concluded that economists might profit from the work of control engineers in making an economic system work as a regulator to maintain full employment without inflation; and that the cooperation of control engineers and economists would be both practical and useful.

Twenty-seven years later, i.e., in 1978, and after the publication of about 1400 research work on the applications of systems and control theory to economic analysis[1], the *Committee on Policy Optimization* chaired by Professor Robert James Ball of the London Business School, published their report in March 1978, the purpose of which was "...to consider the present state of the development of optimal control techniques as applied to macro-economic policy. To make recommendations concerning the feasibility and value of applying these techniques within Her Majesty's Treasury." (Ball, 1978, p. 1).The Committee concluded that "the application of optimal control to the analysis of economic policy is feasible and, applied at working level to the generation of simulations and as a means of testing the properties of economic models, it is likely to be of value. We are not, however, able to say that this is the single most important priority in the development of modeling and forecasting practice." (Ball, 1978, p. 113). We will show that how and why the application of control theory to economic analysis since then has confirmed the prediction made by the Ball's report but at the same time has opened new avenues of research work in this field.

Contributions of the engineers Richard M. Goodwin (1951a, 1951b), William W. Cooper (1951), Herbert A. Simon (1952), Arnold Tustin (1952) and the economist Alban William Phillips (1954) are the early works on the direct applications of *classical control theory* to economic analysis. Goodwin demonstrated that a servo-mechanism system regulates its behavior by its own behavior in the light of the defined objectives. This explains why, for example, a human being usually succeeds in a complicated operation of picking up an object by minimizing the distance between hand and the object (a *tracking* problem), and showed how this idea can be used in economic analysis. Goodwin (1951b) is of special importance for it is the earliest attempt in which an *error activated feedback* is applied to the analysis of market behavior and business cycles.

The applicability of servo-mechanism to the theory of firm has been discussed by Cooper (1951). Simon (1952) studied very carefully the problem of controlling the rate of production on a single product in terms of servo-mechanism theory. He used the Laplace transformation method to examine the stability and the steady-state behavior of the production control system. Tustin analyzed the Keynesian model by control system theory and used the Nyquist criterion, Fourier analysis and Laplace transformation from control theory to demonstrate the possibility of stabilizing the economy.

The work of Phillips (1954) was also concerned with the stabilization of a closed economy. The government was seen as the main stabilizer, and three types of stabilization policies were used. These policies, taken from control theory, were the proportional, integral and proportional plus integral techniques. He was specifically concerned with the question that to what extent can government

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[1]. See MasoudDerakhshan (1978) for a bibliography on economic applications of systems and control theory published as books, chapters in books, conference papers, papers published in engineering and mathematical journals and also as Ph.D. dissertations in economics, mathematics and control engineering.

expenditures be used as a controller to drive the economy along a desired trajectory, and in particular, to offset a deficiency in private demand while avoiding the undesired fluctuations in output. Using the principles of servo-mechanism and feedback control theory, he demonstrated that in the usual multiplier-accelerator model, the time-path stability of the stabilizer (government expenditures in his example) differs for different types of economic policies. In Phillips' analysis, the full employment level of aggregate output is taken as the desired target, the deviations from which are penalized by public expenditures in the form of addition or subtraction of government demand from aggregate private consumption and investment. This work played a significant role in demonstrating the importance of the concept of stability in classical control theory in finding the conditions under which unwanted oscillations in an economic system, like those existing in the great depression of 1933, could be avoided.

3. Collaborations of Control Engineers with Economists on Economic Applications of Optimal Control Theory: The Background

From a mathematical point of view, optimum regulation of an economic system for attaining a desired objective can be defined as a problem in dynamic optimization. The solution to this problem involves finding an optimum trajectory for the admissible control variables (equivalently known as instruments or policy variables) and applying them to the system dynamics (equivalently known as equations of motion) to derive the optimum trajectory of state variables. The optimality criterion is defined by maximizing (or minimizing) an objective function (equivalently known as performance or cost functions). Hence, optimal control was considered to be the most advanced available method in optimizing system's behavior, including economic systems.

Historically, optimal regulation and control of physical systems received considerable attention from mathematicians and control engineers following the World War II. In fact, further advancement in the calculus of variations in the 1960's was largely the result of rapid progress in space technology and the strong competition between the US and the ex-USSR in the field of space engineering. The work of Pontryagin and his associates, the Russian mathematicians, known as the *Maximum Principle*, which appeared in the period of 1955-59 and in an English translation in 1962, was a major breakthrough in the calculus of variation towards the new discipline of optimal control theory. The work of Bellman, the American mathematician, in 1953 and 1957 in the mathematical formulation of multi-stage decision processes known as the *Dynamic Programming*, which is based on the appealing concept of the *principle of optimality*[1], solved many control and system optimization problems in the late fifties and the early sixties. These contributions together with the advances in systems theory, particularly the *state-space representation* of systems developed by LotfaliAskar-Zadeh known as Lotfi A. Zadeh and Charles A. Desoer (1963), established the discipline known as the modern (or optimal) control theory.

When optimal control theory as a new and powerful mathematical tool in pure and applied mathematics and its wide range of applications in space technology and other industries emerged in the 1950’s and the early 1960’s, the foundations of modern mathematical economics had been firmly established. This provided a solid background for the economic applications of optimal control theory. It is interesting to note that Richard Bellman, as an inventor of optimal control theory, elaborated carefully for the first time the potentiality of dynamic programming in designing optimum economic decisions. However, as we will show in section 5, the rapid development of mathematical theories of economic growth in the 1950’s and 1960’s by economists motivated the application of Pontryagin’s maximum principle in deriving the optimality conditions in optimal growth theories. Let us first, briefly examine the contribution of Bellman’s dynamic programming in optimality of economic decision processes.

4. The Unfavorable Circumstances in the 1960’s for Economic Applications of Dynamic Programming

In the *Preface* to his *Dynamic Programming* (1957), Richard Bellman has plainly explained his method and its importance to economic analysis. After defining the subject matter of dynamic programming as a mathematical theory of multi-stage decision processes, he maintains that "The point we wish to make is that ... in economic, industrial, scientific and even political spheres, we are continually surrounded by multi-stage decision processes ... Unfortunately for the peace of mind of the economist, industrialist and engineer, the problems that have arisen in recent years in economic, industrial and engineering fields are too vast in portent and extent to be treated in the haphazard fashion that was permissible in a more leisurely bygone era ... .” He continues that “whether [the multi-stage decision processes] arise in the study of optimal inventory or stock control, or in an input-output analysis of a complex of interdependent industries ... or in the study of logistics or investment policies ... they possess certain common thorny features which stretch the confines of conventional mathematical theory. It follows that new methods must be devised to meet the challenge of these new problems and to a mathematician nothing could be more pleased."

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[1]. “An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.” (Bellman, 1957, Chap. III. p. 3)

Bellman is the first mathematician who has viewed economics from a *state-space* point of view in system theory. In the second paragraph of the *Preface* to his *Dynamic Programming* he stated the followings: "Let us suppose that we have a physical system *S* whose state at any time *t* is specified by a vector *p*. If we are in an optimistic frame of mind we can visualize the components of *p* to be quite definite quantities, such as Cartesian coordinates or position and momentum coordinates ... or if we are considering an economic system, supply and demand or stockpile and production capacities."

Bellman, as a mathematician, had a clear idea of the wide range of applications of his method, hence addressed the following five groups: mathematicians, economists, statisticians, engineers and operations analysts. While he recommended Chapters 1, 2, 3 and 9 of his book to engineers, Chapters 1, 2, 3, 5 and 9 were recommended to economists (*ibid*, *preface*, p. *xvii*). However, he had forgotten to recommend Chapter 7 to economists as well since this chapter, entitled *Bottleneck Problems,* is concerned with a multi-stage production process involving auto, steel and tool industries.

Despite Bellman's attempts to signify the importance and relevance of dynamic programming in solving economic decision problems, his method did not receive a warm welcome by the community of economists at the time. This was mainly due to the nature of the dynamic programming which is heavily dependent on computational algorithms and digital computers: "If we do not wish to suffer the usual atrophy of armchair philosophers, we must occasionally roll up our sleeves and do some spade-work. With the aid of dynamic programming and digital computers, we can methodically engage in mathematical experimentations." (Richard Bellman, and Stuart E. Dreyfus, 1962, p. *ix*).This argument is essentially similar to what AugustinCournot (1838, p. 1) wrote nearly 125 years before Bellman about the importance of mathematical experimentations in economic reasoning: "... now the demand is for so-called 'positive' matter ... such as will throw the light of experience on the important questions which are being agitated before the country."

Bellman's dynamic programming was basically fit and proper for the econometric models designed for computations of optimal economic policies. However, in the 1960's, econometricians were not enthusiastic about applying the dynamic programming for the estimation and computation of optimal economic trajectories. The following factors were at work: *i*) The inherent computational difficulties associated with the dynamic programming known as the *curse of dimensionality,* made the computations of optimal state and control variable excessively involved for medium to large scale econometric models. *ii*) High-speed computers were not available, and *iii*) Econometric models of any practical value were usually medium to large scale, hence demanding an excessive amount of computation which was not technically viable.

However, a number of mathematical economists were successful in applying the dynamic programming to mathematically formulate economic optimization problems and to examine the properties of optimal paths for economic variables. In this regard, the two earliest interesting works were Herbert Simon (1956) and Roy Radner (1967). Simon (1956) demonstrated, for the first time, that in the case of a quadratic objective function and a linear model with uncertainty, the determination of optimal strategies is quite possible. In this class of problems, the "uncertain" future values of variables can be replaced by their unconditional expectations, thus reducing the stochastic problem to a deterministic one.

Radner (1967) successfully formulated the optimal economic growth in terms of the functional equation approach in dynamic programming. Using a welfare function measuring the maximum total discounted utility that can be achieved starting from a given initial state of the economy, he obtained the properties of its continuity and concavity and showed how the application of the dynamic programming is superior to the maximum principle with regard to the number of constraints involved. Radner used the concept of *production correspondence* which gives for each state of the economy a set of alternative states to which the economy can move in the next period. This is exactly the *production possibilities* of an economy, which is expressed in terms of the dynamic programming formulations.

Paul A. Samuelson (1969) and Robert C. Merton (1969) are in fact the most successful early applications of the dynamic programming to optimum economic policy making problems. Using a welfare function with discounted utility, Samuelson (for discrete case) and Merton (for continuous case) demonstrated how the dynamic programming can be utilized in determining the optimal consumption behavior of an individual who is facing a portfolio selection. Using the recurrence functional equation in dynamic programming, Samuelson derived the optimality condition for the problem of portfolio selections.

It should be mentioned however that the subsequent majority of research work in dynamic programming applications to models of economic growth in the 1960’s and the 1970’s were mainly directed towards demonstrating that the results obtained by applying the maximum principle were equally obtainable by applying dynamic programming technique.

5. Economists’ Enthusiasms for Mathematical Economic Policy Optimization: The Role of Favorable Circumstances in the 1960’s

The contributions of Joseph Louis Lagrange (1736-1813), Leonhard Euler (1708-1783) and to some extent, Johann Bernoullis (1667-1748), known as the calculus of variations[1], were used extensively in the 19th Century to develop a systematic solution to optimization problems in physical systems. Moreover, the

[1]. For a history of the calculus of variations see Herman H. Goldstine (1980).

formulation of *Hamiltonian* function[1] in variational problems in the nineteenth century was a significant contribution in the optimum system design in control engineering. Pontryagin's maximum principle, which is based on the Hamiltonian dynamical system, is a profound contribution in optimal control theory.

From a mathematical point of view, the classical calculus of variations which deals with the problem of finding a function (and not a point as in the differential calculus) that optimizes an objective function, has wide applications in economics particularly in determining the optimal decisions over time. The following research work are known as the most successful applications of classical calculus of variations to problems of economic optimization, which provided an appropriate background for using Pontryagin's maximum principle to derive the optimality conditions in models of economic growth: Harold Hotelling (1925) on economic depreciation, Griffith C. Evans (1928a and 1928b) on general problems arising in economic applications of the classical calculus of variations, Frank P. Ramsey (1928) on optimal savings and Robert M. Solow (1956) and Trevor Winchester Swan (1956) on the development of early models of economic growth.

In the 1950's, mathematical economists were fully aware of Ramsey's work (1928) and *Ramsey-type* models. The necessary conditions of optimality given by Pontryagin's maximum principle were an ideal technique for generating the optimal paths for economic state and control variables in Ramsey and similar models in economic growth theory. Mathematical economists were well acquainted with Pontryagin's maximum principle in the early 1960's. Hence, with a well-developed literature in growth economics, the 1960's can be regarded as the decade of applications of the maximum principle to theories of economic growth and stabilization. Within this framework, the objectives of most research work in this period were to prove the existence of an optimal plan, (Menachem E. Yaari, 1964), to provide economic interpretation of the maximum principle (Robert Dorfman, 1969), and a careful examination of optimality conditions in models of economic growth using control techniques.[2]

The analytical framework was usually assumed to be the maximization of a utility function subject to certain constraints. For mathematical convenience, just a few control variables were used such as government investment expenditures and the rate of interest. By applying Pontryagin's maximum principle, optimal trajectories for state variables were derived. For the most part, only the

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[1]. The Hamiltonian function and the Hamiltonian dynamical system are named after their inventor, the Irish mathematician, William Rowan Hamilton (1805-1865). Paul A. Samuelson and Robert M. Solow (1956) are the first economists to use Hamiltonian function in their work.

[2]. See, for example, Karl Shell (1967), Edwin Burmeister, A. Rodney Dobell (1970) , Duncan K. Foley, and Miguel Sidrauski (1971).

qualitative properties of these trajectories were analyzed and no attempts were made at the computation of optimal trajectories.

To demonstrate the perfect compatibility of the maximum principle to theories of optimal economic growth, Robert Dorfman (1969) maintained that "optimal control theory is formally identical with capital theory, and that its main insights can be attained by strictly economic reasoning." To justify this, Dorfman started with a well-known problem in capital theory, i.e., a firm that wishes to maximize its total profit over some period of time with a given initial stock of capital. The rate of profit per unit of time depends on the initial condition as well as the decisions taken by the firm. Maximization of the total profit earned from initial date to some terminal date will be a function of the entire time path of decision variables. Although the firm is almost free to choose the time path of policy variables, it cannot arbitrarily select the amount of capital at each period since the latter is a function of the policies taken earlier. The firm is thus facing a policy formulation problem in a dynamic context whose solution is best provided by the maximum principle. Using this economic example, Dorfman obtained the necessary optimality conditions in the maximum principle, which provided an interesting economic interpretation of optimal control theory. This provides an excellent example of the contribution of the maximum principle to economic optimization problems.

It follows therefore that in the 1960’s and the 1970’s optimal growth theory, as a discipline in mathematical economics, could not have been developed without the direct application of the maximum principle. This clearly demonstrates the fact that optimal control as a *mathematical method* has significantly contributed to pure economic analysis.

6. The New Phase of Collaborations of Control Engineers with Econometricians: Optimal Estimation and Control of Economic State and Control Trajectories

As discussed in the previous section, economic applications of optimal control theory in the 1950’s and the 1960’s were mainly confined to optimality conditions in models of optimal economic growth. By the early 1970's, efforts were made towards numerical computations of optimal economic state and control trajectories in more complex multi-state and multi-control econometric models. This necessarily required further collaborations of control engineers with econometricians on applying more advanced topics in optimal control theory to economic optimization problems.

Fortunately, a number of eminent optimal control engineers became interested in mathematical economics and econometrics. Masanao Aoki from the Department of System Science, School of Engineering and Applied Science, University of California, Los Angeles and John H. Westcott from the Department of Electrical and Electronic Engineering, Imperial College of Science and

Technology, University of London, who were both Professors of Control Systems in the 1960's, actively participated in the application of optimal control techniques to econometric models. Moreover, by the late 1960's and the early 1970's, a number of Ph.D. dissertations in the departments of electrical and electronic engineering were directed towards economic applications of optimal control theory. In what follows, we refer to two Ph.D. dissertations in the application of optimal control theory to economic optimization policies, which can be regarded as the first serious attempts by students with engineering backgrounds.

i) David Leif Erickson from the School of Engineering, University of California, Los Angeles, completed his Ph.D. dissertation in 1968 entitled *Sensitivity Constrained Optimal Control Policies for a Dynamic Model of the U.S. National Economy*. In the conventional approach, optimal control policies are usually formulated subject to the dynamics of the state of the economic system only. In this dissertation, Erickson for the first time formulated the optimal control policies as being constrained by the sensitivity of economic state trajectory to parameter deviations from the nominal values determined by the policy makers.

ii). Robert S. Pindyck who received his B.Sc. degree in Electrical Engineering and Physics from M.I.T. in 1966 and his M.Sc. degree in Electrical Engineering from M.I.T. in 1967 completed his Ph.D. dissertation at M.I.T. in 1971entitled *Optimal Economic Stabilization Policies.* In this dissertation, he provided for the first time a complete demonstration of the application of discrete-time Pontryagin's maximum principle in the computation of optimal state and control trajectories for a small deterministic linear model of the post-Korean US economy. Based on his Ph.D. dissertation, Pindyck published a paper in 1972 in the *IEEE Transactions on Automatic Control,* a well-known engineering journal on control theory, which attracted the attention of more control engineers to economic applications of optimal control theory. He published his dissertation as a book in 1973 entitled *Optimal Planning for Economic Stabilization,* which made the topic more popular with the community of system and control engineers.

Pindyck's work significantly influenced the direction of subsequent research in econometric applications of optimal control theory. He treated economic stabilization policies as a tracking problem in optimal control theory in which the objective of optimization was to track the desired (nominal) state and the desired control trajectories. The model included several basic macroeconomic state variables such as consumption, investment, GNP, interest rate, price level, wages and unemployment. The policy or control variables were the money supply, government spending and taxes. Fiscal policies were provided for through exogenous government expenditures as well as surtax and the monetary policy was realized in the money supply. By defining new state variables to replace those with lags greater than one period and adding their definitional equations to the model, he represented his model in terms of linear difference equations in the state space format with a quadratic performance function. He then applied the discrete Pontryagin's maximum principle to obtain optimal economic policies.

Optimal control approach to economic policy optimization is based on the assumption that the desired (or nominal) state and control vectors are known over the entire planning horizon. The question in the simplest case, i.e., the optimal control of a linear econometric model with quadratic objective function, is to track the desired state and control trajectories defined in the objective (or cost) function subject to the linear system dynamics and a set of constraints imposed on control variables. The cost function usually includes two matrices whose entries enable policy makers to penalize the deviations of state and control variables from their desired or nominal values.

The problem is to find an optimal control sequence which minimizes the objective function subject to the linear econometric model and the constraints imposed on control variables. For mathematical convenience, both matrices in the objective function are usually assumed to be diagonal and their elements give the *relative costs* of deviating from the nominal values of each state and control variables over the entire planning horizon. As an example, two diagonal elements in the matrix related to state variables may measure the cost of deviating from the desired unemployment level relative to the cost of deviating from the desired value of inflation. Similarly, the elements of matrix related to control variables measure the relative cost of deviating from the desired values of control variables; for instance, the costs involved in manipulating tax rates as compared to that of the money supply. These matrices can be time-varying to allow the ranking variations of policy-makers on the relative importance of deviations over time.

There are a number of problems with quadratic objective functions[1] but its significance is the mathematical property known as the *certainty equivalence*, i.e., the linear systems with quadratic performance functions produce control laws which are linear and thus computationally tractable. Using this property, Pindyck did not consider the effects of additive random terms in his work.

Pindyck performed several experiments using different objective functions with different weighting matrices. Changing the elements in weighting matrices

[1]. The very arbitrariness of weighting matrices in objective functions can be regarded as the main shortcoming of optimal control of linear economic systems with quadratic objective functions (LQP). It may be impossible to reduce the complex process of ranking economic priorities into a relatively straight-forward exercise of determining elements of weighting matrices. Moreover, the penalization mechanism of deviating from the nominal values of state and control variables in quadratic objective functions cannot differentiate the direction of deviations from the desired values. This is not a serious problem in an engineering application, but in an economic optimization problem it is a decisive issue whether, for example, unemployment or inflation targets are over-or under-reached. (See, for example, David A. Livesey, 1973). Non-quadratic objective functions are better alternatives although the computational complexities involved in this class of objective functions are a major hindrance.

may provide more insight into the trade-offs inherent in policy formulations. These experiments demonstrated that optimal control of economic models as a tracking problem is valuable both as a tool for policy planning and as a method of analyzing the dynamic properties of economic models.

Following Pindyck, a number of researchers with engineering and mathematical backgrounds worked on economic applications of stochastic and adaptive controls. In the following sections we will refer to the earliest applications in order to historically demonstrate the contributions of engineers and mathematicians in economic applications of advanced topics in optimal control theory.

6.1 Computations of Stochastic Optimal Control in Economic Policy Optimization

Stochastic optimal control of an economic system arises when there are uncertainties in the system's dynamics. It is usually assumed that there are two sources of uncertainties, namely, additive noise in the state space representation of the economic system and additive measurement noise in the measurement sub-system. The certainty equivalence theorem of Herber A. Simon (1956) and Henri Theil (1957, 1964) can easily be applied to the class of linear stochastic optimal control with quadratic objective functions.

The earliest successful attempts to apply stochastic control to the computation of optimal trajectories in econometric models are KioumarsParyani (1972), Gregory C. Chow (1972), DogobertBrito and Donald D. Hester (1974), Jeremy Bray (1974) and Edmund S. Phelps and John B. Taylor (1975). The contribution of KioumarsParyani was significant because this was his Ph.D. dissertation in the Department of Electrical Engineering and System Science, Michigan State University, hence motivated further interests amongst control engineers to apply more advanced control techniques to economic optimization problems. A systematic analysis of non-linear economic models with additive noise and quadratic performance measure is discussed for the first time by Kenneth D. Garbade (1975).

The contribution of John H. Westcott and Kent Dell Wall (1976) in stochastic policy optimization of economic systems, published in *Automatica*, one of the leading engineering journals in optimal control theory, significantly influenced the subsequent direction of research work in this area, particularly due to the fact that Westcott was Professor of Control Systems at Imperial College of Science and Technology, University of London. They used a linear stochastic control model with a quadratic performance function and a Gaussian distribution of disturbances. The model was designed to obtain optimal economic strategies for the four major problems confronting the UK policy-makers, i.e. unemployment, inflation, balance of payments and economic growth. There were thirteen behavioral equations in the model which were estimated using quarterly data over the period 1955-1973 specifying the behavior of unemployment, employment, private investment, stock building, private consumption, consumer price index at factor cost, wages, profits, real exports, real imports, export price index, import price index and factor cost adjustments. There were thirteen definitional equations in the model. The parameters were estimated by a dynamic generalization of simultaneous multivariate maximum likelihood estimators in conjunction with residual correlation diagnosis. The estimated model is then converted to an equivalent state-space format to apply the technique of stochastic optimal control. A minimal realization procedure, developed by Edward C. Prescott, and Kent Dell Wall (1973) was used to obtain the minimal state-space dimension. The behavioral equations were expressed in terms of growth rates which is equivalent to the first difference in natural logarithms.

The main objective was to demonstrate which instruments were most effective in achieving a given target or policy goal. For example, should only the foreign exchange rate be used to balance the trade, or should fiscal policies be applied instead or as well. The equivalently important problem was how to coordinate the instruments in order to achieve simultaneously a given combination of policy goals, which might involve problems in optimization under conflicting objectives. With regard to the first problem, the elements of weighting matrices in the objective function could be manipulated to identify which control variables were the most suitable in reaching a specific economic objective. Westcott and Wall's experiments demonstrated that fiscal policies, such as public investment and social expenditures were effective instruments in both balancing the trade and stimulating growth.

6.2 Computations of Adaptive Optimal Control in Economic Policy Optimization

A new extension within the framework of stochastic optimal control of dynamic systems was achieved by introducing uncertainties in system's parameters. This branch of stochastic control is also known as adaptive, self-organizing, self-optimizing, self-regulating and learning system. It can also be assumed that some or all of the endogenous variables in the system dynamics are not available for exact measurement. A sub-system for measurements in which the observations are assumed to be linear functions of endogenous variables with additive random disturbances could also be formulated.

The adaptive control of a linear economic model with unknown parameters is to find the optimal economic decision sequence which yields the minimum value of the objective function subject to the economic model (system dynamics) and the measurement sub-system. In this class of problems, the accuracy of the estimation is a function of the control action while the quality of control will depend upon the degree of accuracy by which the econometric model is estimated. The controller must, therefore, compromise between estimation and control. This problem is usually referred to as the *dual control problem.* The uncertain parameters are usually regarded as additional state variables.

The above-mentioned treatment of parametric uncertainties transforms most linear econometric models into essentially a problem in non-linear stochastic control theory and thus takes the econometricians into the realm of non-linear estimation theory. The Bayesian approach has the potential to solve this class of problems where there exists *a priori* knowledge about the probabilities of unknown parameters.

Amongst the most successful early applications of adaptive control to economic policy design problems are Gregory C. Chow (1973), Triveni N. Upadhyay (1973), David Andrew Kendrick and Joe Majors (1974) and Stephen J. Turnovsky (1975). Again, the Ph.D. dissertations in engineering departments played their significant role in the application of adaptive control to economic policy optimization in econometric models. TreveniUpadhyay completed his Ph.D. dissertation entitled *Adaptive Control of Linear Stochastic Systems* in the Department of Electrical Engineering at Texas University in 1973, in which he applied the method of adaptive control to the recursive linear difference equation model of the US economy developed by Robert Pindyck. As discussed in section 6, the Pindyck model was a deterministic optimal control of a linear system with a quadratic objective function. Upadhyay extended Pindyck's model by making the parameters of the model as random variables with assumed statistics. He formulated the simultaneous estimation and control of Pindyck's model and showed that the unknown parameters in the model can be identified while simultaneously controlling the economy. His results indicated the advantages of applying adaptive control techniques to economic modeling and control. Using the average value of the objective function as a measure of comparison, he demonstrated that the adaptive control scheme yields smaller value for the objective function as compared to the optimal deterministic control approach.

7. Underlying Causes of the Fading Optimism in Economic Applications of Optimal Control Theory

Advances in optimal control theory during the 1950's and the 1960’s had been motivated largely by the rapid progress in automatic control of physical systems in general and space technology in particular. The fact that economic systems are in sharp contrast to physical systems underlies the limitations, shortcomings and failures of optimal control applications to economic policy optimization.

We may broadly classify the literature on economic applications of optimal control into the following two categories. *i*) Economic applications of *mathematical* optimal control theory with the aim of identifying the optimality conditions for state and policy variables as well as probing into the dynamics of an economic system. *ii*) Economic applications of *engineering* optimal control theory with the aim of numerical computations of optimal trajectory for policy variables in order to drive the economy along a desired path while satisfying an objective function. The economic results of value are not expected to come from the latter since automatic control of economic systems always fails due to the structural differences between economic and physical systems.

An examination of the literature in economic applications of optimal control reveals the fact that the engineering approach aimed at automatic design and computations of optimal trajectories have been progressively replaced by mathematical approach which aims at identifying the economic system dynamics and the optimality conditions in models of economic growth. Let us examine this point further from a historical standpoint.

Applications of optimal control theory to models of economic growth as well as to econometric models had their heyday in the 1960's and the 1970's. MasoudDerakhshan (1978) reports that prior to 1978 the number of papers on economic applications of optimal control published in *engineering* and *mathematical* journals were 347 as compared with 400 papers which appeared in *economic* journals. The prime objective of the considerable number of papers published in engineering journals was to apply the idea of automatic control in modeling the economy at micro and macro levels with serious attempts at computing the optimal state and control trajectories. Although most of the control engineering journals were not widely read by economists, there is no doubt that the research work of high quality published in them during that period motivated control engineers to apply the more advanced optimal control techniques to economic optimization problems.

The active role of engineering and mathematical institutions in furthering the economic applications of optimal control cannot be overlooked. In the early 1970's, the well-known control engineering societies such as the Institute of Electrical and Electronic Engineers (*IEEE*), and the International Federation of Automatic Control (*IFAC*), became interested in the control of economic systems and published research work on this topic in their journals and conference proceedings. Examples of these conferences in the 1970's are the followings: the *IEEE* Conferences on Decision and Control; Joint Automatic Control Conferences; Conferences on Dynamic Modeling and Control of National Economies; and *IEEE* Conferences on Systems, Man and Cybernetics.

During the 1970's, conferences organized by the National Bureau of Economic Research (1972, 1973, 1974, 1975, 1976) on *Stochastic Control and Economic Systems*, the Social Science Research Council (1972) on *Modeling of the UK Economy* and the USSR Academy of Sciences Central Mathematical Economics Institute (1971, 1974) on *Optimal Planning and Control of the National Economy*, have all been particularly useful in fostering a hospitable environment for the collaboration of control engineers with economists on applying advanced control techniques to economic policy optimization. Furthermore, the publication of *Economic Computations and Economic Cybernetics Studies and Research* and the *Journal of Economic Dynamics and Control,* the two specialized journals on economic applications of optimal control theory,has also been enriching.

Nevertheless, a number of leading engineering journals, among the 38 journals in engineering and mathematical sciences, reviewed in MasoudDerakhshan (1978), which were quite active in publishing papers on economic applications of control theory in the 1970's, either completely abandoned the publication of these papers or very rarely published papers on these topics. We may refer to the following journals in this category: *Automatica, IEEE Transactions on Automatic Control, International Journal of Control, International Journal of Systems Sciences, SIAM Journal of Control, IEEE Transactions on Systems, Man and Cybernetics* and *Automation and Remote Control*. Moreover, a careful inspection of papers in specialized journals on economic applications of control theory, e.g. the *Journal of Economic Dynamics and Control*[1], reveals the fact that most of the published research work after the 1980's have been mainly theoretical in nature with no serious attempts in computations of optimal economic trajectories within an automatic control framework. This new development seriously weakened the early active collaborations of control engineers with economists on the application of advanced optimal control techniques aimed at the computation of optimal trajectories in econometric models.

The origin of the unfulfilled expectations in economic applications of optimal control theory lies in the fact that economic systems are not only more complex than physical systems but have different nature since they are inter-related to social, historical and political systems. Mathematical machinery is a system of logical reasoning based on the abstract concepts, hence the applications of optimal control theory to economic policy optimization necessarily requires two sets of abstract concepts which are derived from *i*) the economic realities to be controlled and *ii*) the policy-makers' priorities to be satisfied. The economic system dynamics and the objective function will then be constructed upon these abstract concepts in such a way as to become compatible with the standard control theoretic frameworks for mathematical tractability.

[1]. In 1978, the National Bureau of Economic Research (NBER) decided to discontinue publications of the *Annals of Economic and Social Measurement* which usually published selected papers from conferences and annual meetings on economic applications of optimal control theory. Shortly thereafter, North-Holland Publishing Company agreed to lunch the *Journal of Economic Dynamics and Control* to continue the publication of papers on economic applications of optimal control in a wider context. At the same time, the *Society for Economic Dynamics and Control* was established to promote and sponsor international conferences and research projects in this field. The first issue of this journal appeared in February 1979.

A proper method of abstraction should, therefore, reduce real-life economic complexities while preserving the underlying properties of relations between economic system and the related social, political, historical and cultural dimensions. It follows therefore that the application of optimal control theory to economic policy optimization are most promising only in those areas where the abstract economic concepts to be used in the mathematical formulation of economic control problem exhibit close approximations to economic realities. We may therefore conclude that the logic of abstraction in economic theorization plays the key role in the success or failure of economic applications of optimal control techniques. Further analysis of this point is beyond the scope of the present paper.[1]

Economic applications of optimal control have also faced another serious challenge in the late 1970's. In fact, the rational expectation hypothesis provided a strong criticism for the application of optimal control theory to economic policy optimization. Economic agents respond usually not to the signals which are mechanically generated by the controller in an engineering type environment but to their own expectations of economic state variables. Rational forward-looking expectations, in contrast to the case where expectations are functions of the past behavior, introduce serious difficulties in the standard formulation of policy optimization within control theoretic framework.

Standard optimal control approach does not accommodate the impact of expected future policies on the current values of state variables, hence Bellman's dynamic programming and Pontryagin's maximum principle do not produce optimal state and control trajectories for forward-looking models in which current state variables depend on the anticipated future policies. Economic systems are adaptive in nature in which actions and reactions constitute the mechanism of economic behavior. This problem, together with the consequences of the Lucas critique on the applications of optimal control theory to dynamic choice in economic models is examined in MasoudDerakhshan (2011).

8. Summary and Concluding Remarks

i) In the late 1930's and within the framework of Keynesian economics, the Government assumed the responsibility of regulating and stabilizing the economy on the ground that there is no inherent tendency in an economic system to generate stability and full employment. This motivated economists to study and apply classical control theory with particular emphasis on stabilization policies. However, the formal economic applications of engineering stability theory and self-regulating mechanisms did not take place until the 1950's when control engineers collaborated with economists.

ii) Classical control theory entered the new era of modern (or optimal) control theory in which controllers should posses an optimality character. This new

[1]. See MasoudDerakhshan (2014a and b) for an extended discussion on this point.

discipline emerged in the late 1950's and the early 1960's by the development of Pontryagin's maximum principle and Bellman's dynamic programming.

iii) Advances in mathematical modeling of economic growth in the 1950's and the compatibility of its mathematical structure with the optimal control formulation made the modeling of economic growth an appropriate field of applications for optimal control techniques. In this framework, the first order condition in Pontryagin's maximum principle proved to be the most powerful instrument in deriving the optimality conditions in models of economic growth. Hence, mathematical optimal control theory performed its most significant contribution to the theory of economic policy optimization in the 1960's.

iv) The 1970's was the decade of econometric applications of optimal control theory. Many control engineers collaborated closely with econometricians on the application of deterministic, stochastic and adaptive optimal control techniques to numerically compute the optimal state and control trajectories in medium to large scale linear or nonlinear economic models with quadratic and non-quadratic performance functions. However, these contributions were proved to be of academic interests only. In fact, policy-makers did not trust the outcome of mathematically sophisticated econometric control models, which generated the optimal economic policy trajectories within a black-box. On the contrary, policy-makers have always preferred conventional simulation models due to their much simpler mathematical structures and tractability, which allows better understanding the dynamics of economic policy formulations.

v) Automatic control has always been the basis of optimal control of physical systems. This is in sharp contrast to the impossibility of automation in economic policy optimization. The real-life economic systems are not only more complex than physical systems but have different structures since they are inter-related to social, historical, cultural and political dimensions. Optimal control theory is basically a mathematical structure, which is based upon pure mathematical relations abstracted from physical systems. This explains the success of the application of optimal control theory in engineering problems in general and space technology and remote control in particular. This also explains the failure of the application of optimal control to econometric models particularly for optimum economic policy design at macro-level.

The pure mathematical relations which are established by the instrumentality of mathematics through the observation of the behavior of economic variables cannot truly capture the underlying properties of economic relations at work in realities. In fact, the real economic relations and their properties, which exhibit the complex interactions of cultural, political, social and historical factors, are impossible to be identified by pure mathematical formulations. We may therefore conclude that the application of optimal control theory to economic policy optimization are most promising only in those areas where the abstract economic concepts and the mathematical economic relations to be used in optimal policy formulations constitute close approximations to economic realities. Many problems in micro-economics or finance where the impact of non-economic factors on the observed real economic performances are minimal fall in this category.

vi) The rational expectation hypothesis has raised a number of strong objections to the application of optimal control theory to economic policy optimization. Economic decision-makers do not usually respond to the signals generated by the control system in an engineering-type mechanism; they usually base their decisions on the expected values of economic state and control variables. Hence, the rational forward-looking expectations have put forward serious arguments against the applicability of optimal control theory to economic optimization problems and at the same time have opened new avenues of research work in economic optimization including the game-theoretic approach within conventional optimal control framework (Currei and Levine 1993).

vii) The promising collaborations of control engineers with econometricians on the computation of optimal state and control trajectories in econometric models, which started in the 1970's, soon fell flat due to the structural differences between economic and physical systems. Control theory applications to economic policy optimization has been proved to be more productive in the realm of mathematical economics with particular emphasis on the mathematical structure of economic system dynamics and the optimality conditions in models of economic growth. Economic applications of engineering optimal control aimed at the computation of optimal policies in econometric models have reasonably not received a warm welcome from the community of economists.

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