O

و هذه الإستبانة جعلها ثابت بن قرّه شكلاً من أصل الكتاب للإيضاح و لم تكن هي شكلاً منه في النسخ اليونانية والسريانية ولذلک لم يأت الحجاج به في نسخته والأليق بكتاب أقليدس و طريقته (= طريقيه في النسخة) في هذا الكتاب أن يكون من قبيل الأستبانة لا من أصل الكتاب إذ هو بالفروع أليق (و هذه صورته) و أنا أطنبت في بيّان الإستبانة للإيضاح.

P

و في نسخةثابت والأول عند عدد أخر هو الذي (لا) يعدهما عدد آخر.

لو

إن كلّ خطّين مستقيمين خارجا من نقطة خارجة من دائرة أحدهما قاطعاً إياها و الآخر منتهياً إليها غير قطع وكان سطح جميع القاطع فيما هو خارج منه عن‌الدائرة مساويا لمربع المنتهي فإنّ الخطّ المنتهي يماس الدائرة.

و الثابت بن قرة لتار أي أنّ أقليدس استعمله في الشكل المذكور ألحق بآخر هذه المقالة واللائق بالطريقة التي سلكها أقليدس في هذا الكتاب أنّ لاتفرد هذا الشكل بالذكر مع وجود هذه الإستبانات و لذلك الحجاج لم يذكره في نسخته لمّا لم يكن موجوداً في النسخ اليوناتية والسوريانية (في النسخة:والسورنانية) القديمة و نحن أشرنا إليه بالإستبانة ليعلم أنّه ليس من أصل‌الكتاب و ليس استعمل في الشكل العاشر من‌المقالة الرابعة ثم أنّ أذكر البرهان الذي ذكره الثابت.

J

و في ترتيب بعض هذه الأشكال خلاف و ما أوردناه على ترتيبة ثابت و أمّا الحجاج فقط أورد ما ذكرنا في شكلي يا، يب في شكل يا وحدة و ما أوردناه في شكل يج في شكل يب و أورد في شكل يج، يد الأحكام المذكورة في صدري شكلي يد، يه و في شكل يه الثانيات المذكورة فيهما ثم توفقا فيما بعد.

K

المقالة العاشرة ماية و خمسة أشكال و في نسخة ثابت (ماية) و تسعة أشكال أربعة منها كا، كب، كز، كح هي من زيادته و جعل شكل يز للحجاج شكلين هما كد، كه له و في الترتيب خلاف أيضاً.

L

و أورد ثابت بن قرّه برهاناً آخر لهذا الشكل في كتابه و حكى أنّه وجده في بعض النسخ اليونانية تركت ذكره لأنّ برهان الكتاب البسط و البراهين على أشكال الكتاب كثيرة إستنبطها المتقدمون والمتأخرون و الاليق بالإراد من البراهين في كتاب الأصول ليس إلا ما هو لابسط.

* * *

M

و قد أورد ثابت بن قرّه برهاناً آخر لهذا الشكل تركناه كمّا ذكرناه في آخر الشكل المتقدّم.

N

و في نسخة ثابت هي إضافة ما في القدر بين مقدارين متجانسين.

E

أقول: و لهذا الشكل إختلاف وقوع فإنالنقطة يمكن أن يقع مبانية غير مبانية إياه كمّا مرّ أو مسامته و يمكن أن يقع غير مبانية أمّا عليه أو على طرفه و هذه أربعة والوجه فيالجميع واحد.

أمّا الأوّل كمّامرّ.

ويمكن أنّ يقع فيه ا ب أمّا أقصر من ب ج فيقع المثلث داخل دائرة ج ح زكمّامرّ أو مساويا له فتمرّ الدائرة على ا، د أو أطول منه فيقع محيطها ضلعيا ب، ب د (و هما هكذا).

فأمّا الثانيفمثل الأوّل ويقع فيه (الصور هكذا).

وأمّا الثالث فلا يحتاج فيه إلى أن نصل بينالنقطة لأنّ ا ب يكون بعض ب ج فلا يقع فيه إلا صورة واحدة (و هي هكذا) و يمكن في جميع هذا الصور أن نرسمالمثلث (فيكلّتي) جنبتي خطّ ا ب ويجد أن بسببه أيضاً فى أوضاع الخطوط إختلاف.

وأمّا الرابع فلايحتاج فيه أيضاً إلى أن نصلالنقطة والطرف لإتحاد هما ولا إلى عملالمثلث لعدم البعد بينهما ولا إلى عمل الدائرتين لكونالمركزين و احداً بل يكفي فيه إجراج دائرة واحدة على طرفالخطّ ببعده ثمّ إخراج خطّ من مركزإلى المحيط كيف اتفق.

F

و هذا الشكل لم يذكره الحجاج في كتابه و قد وجد فينسخة ثابت والحقّ أنّه لايحتاج إليه بعدالشكل المتقدم و ذلک لأنّ طريقة أقليدس في كتابه هذا أنّه إذا كان شكل أو مقدّمة شكل يستبيّن منالأشكال المقدّمة لم يجعله شكلاً من أشكال كتابه و لاتخرجالمقدّمة من القوة إلى الفعل بل لم يذكر شياء منهما إعتماداً على أذهان من يحاول حلّ كتابه هذا لأنّه يتكلم على إستبانة منالشكل المتقدّم و إن كنت ذكرته بالفعل لأنّ طريقتي في هذا الكتاب تقتضي ذلک.

G

و أورد الحجاج هذه الإختلافات واقتصر ثابت على الآ خر.

H

و هذا الشكل ليس في نسخة الحجاج و هو ممّا زاده ثابت إذ وقع في عاشرالمقالة الرابعة إليه حاجة(في النسخة: خارجة).

I

و أقليدس لمّا يحفظ هذه المعاني لم يذكر الشكل الذي ألحقه ثابت بن قرة في آخر هذا المقالة و إنّ إستعمله في الشكل العاشر منالمقالة الرابعة إنّ عادته في هذا الكتاب أنّه يستعمل كثيراً منالمقدمات و لايفكر في الكتاب اءذ كانت منطوقة ممّا تقدّم من مسائله نفسها أو طريق الإستبانة و هو:

A

نريد أن نخرج من نقطة إلى خطّ غير محدود عموداً عليه.

B

كلّ نقطةمفروضة على سطح مفروض فيه خطّ مستقيم غير محدود في طرفيه ولاتكون النقطة علىالخطّ المفروض لنا أنّ نخرج من تلكالنقطة إلىالخطّ عموداً.

C

قال أبو علي: ولوكان مركز دائرة اب خارج من دائرة ا ج لصحّ الدعوى بهذا البرهان أيضاً لايتغير إلاّ بأن خطّ اه يخرج من دائرة ا ج و ه ج يقطع دائرة ا ج في موضوعين حيث يدخل الدائرة و حيث يخرج منه و أيضاً فإنّا نخطّ دائرة ا ج مماسة لدائرة ا ب من خارج على نقطة ا وليكن مركز دائرة ا ب نقطة ة، و مركز دائرة ا ج نقطة ز.

فأقول أنّ الخطّ الذي يجوز على نقطتيه، ز يمرّ بنقطة ا.

فإن لم يكن كذلك فليقع مثل خطّ ه ح ط ز و يخرج خطّي ا ز، ا ه مجموعين أطول من خطّ ز ه.

فأمّا خطّ ا ه فهو مثل خطّ ه ح و أمّا خطّ ا ز فهو مثل خطّ ط ز. فخطّا ه ح، ز ط مجموعين أطول من خطّ ه ز.

و هما أصغر منه. هذا خلف.

فالخطّ الذي يصل بين نقطتي ه، ز يمرّ بنقطة ا. و ذلک ما اردنا أن نبيّن.

D

كلّ دائرتين يتماسان فإن الخطّ الذي يمرّ بمركزيهما يمرّ بموضعالتماس.

و ليس فيه شکّ. و يمكن أن بيانه ببرهان آخر.

فليكن الدائرتان ا ب ج، ا د ج. وليكن ه مركز دائرة ا ب ج و ح مركز دائرة ا د ز. و ننفذه حتى يقطع دائرة ا ب ج وليلتها على نقطتيب، ج. فليكن نقطة ح على قطر دائرةا ب ج غيرالمركز و نصل ح ا. فيكون أعظم من ج ح كمّا يتبيّن من الشكل السابع و الثامن.

و ح ا مثل ح ز. ف ح ز أعظم من ج ح.

و هذا محال و هذا المحال عرض من فرضنا خطّ ه ح لايمرّ بموضع التماس. فالخطّ الذي يصل بينالمركزين يمرّ بموضع التماس. و ذلک ما أردنا أن نبيّن.

و هذا البرهان أحسن من برهان أقليدس لأنّه برهان على الوضعين جميعاً بلفظ واحد و علته هوالسابع والثامن.

to be incorporating ideas that were widely known and discussed, rather than making original intellectual contributions to the field. This does not indicate that he was a mathematician of lesser standing, since what we are considering here is one of a series of mathematical textbooks that constituted an integrated curriculum leading from elementary mathematics to the spherical geometry of Ptolemy's *Almagest.*

III, 34 & 36: (These have been discussed earlier.[91])

V, def. 3: And in the text of Thābit "it is some relation (*iḍāfa*) in respect to quantity (*qadr*) between two magnitudes (*miqdār*) of a species.[92]

VI, 11: This proposition (i.e., the equivalent of Euclid's VI, 12) is among those whic Thābit added.[93]

VII, def. 11: And in the text of Thābit "[That which is] prime to another number is such that another number [does not] measure the two of them."

VIII, Intro: Book VIII – Twenty five propositions. In the text of Thābit there are added to it two propositions, 24 and 24.

IX, 31: And it is noted, on the authority of Thābit, that this proposition and that which preceded it were not in the Greek manuscripts.

XIII, end: Thābit introduced at the end of this book a principle without an illustration.

CONCLUSION

The *Taḥrīr* of Naṣīr al-Dīn al-Ṭūsī is justly famous as one of the important mathematical treatises produced within the Arabic branch of the Euclidean tradition. As such, it is long overdue for a comprehensive study. This paper has attempted to outline some of the reasons for its importance to historians of mathematics. This *Taḥrīr*, although less well known to historians, illuminates part of the pedagogical aspect of the Euclidean tradition that has so far been ignored. It incorporates many of the results from earlier discussions, almost always without naming its sources. (Careful study of the major sources in this intellectual tradition will help to elucidate the complex interrelationships between scholars and their writings within the Arabic period.) In many cases, Ṭūsī seems

text, since the proposition is obviously present in the genuine Ṭūsī *Taḥrīr*. But he must have been looking only at the enunciation (or perhaps the illustration?), since the Pseudo-Ṭūsī *Taḥrīr* has the same result, but relegated to a porism at the end of the proposition.

91. London. BM, MS add 13,387, fol. 107b.(P)
The Pseudo-Ṭūsī has no comment on this definition.

92. London. BM, MS add 13,387, fol. 117b.
Cf. note 80. The Pseudo-Ṭūsī has no note here.

93. London. BM, MS add 13,387, fol. 135b.
The Pseudo-Ṭūsī *Taḥrīr* records no comment at this point.

X, Intro: Book X - 105 propositions. In the text of Thābit [there are 10]9 propositions. Four of these, 21, 22, 27, 28, are his additions. He made proposition 17 of al-Ḥajjāj into two, namely twenty four and twenty five in it (i.e., in his book), and in the order there is a difference as well.[86]

XI, Intro: There is no difference, in the case of the solids, between the texts of al-Ḥajjāj and Thābit.[87]

We also find ten brief statement about the contributions of Thābit to the Arabic text of the *Elements* – at least as known to Ṭūsī. I also catalogue these references.

III, Intro: Book III - 35 propositions. In the text of Thābit there is an additional proposition at the end of it.[88]

III, 9: Thābit said "In some manuscripts it has a different demonstration'

III, 10: In some manuscrips it has another demonstration which Thābit also introduced.[90]

Pseudo-Ṭūsī omits any mention of Thābit's addition.

86. London. BM, MS add 13,387, fol. 52b.

The comment in the Pseudo-Ṭūsī Taḥrīr explains why the editor has chosen not to include this material from Thābit)p. 71):

And Thābit ibn Qurra introduced another demonstration for this proposition in his book. He reports that he found it in some Greek manuscripts. I pass over his discussion because it is a proposition in the *Kitāb al-Basṭ* and the proofs which the ancients and moderns have contrived for this book are many. And the quotation of the proofs in the *Kitāb al-Uṣūl* is not appropriate except for those he does not explain.(L)

87. London. BM, MS add 13,387, fol. 53b.

The Pseudo-Ṭūsī adds a note parallel to that in the previous proposition (p. 72): Thābit ibn Qurra introduced another demonstration for this proposition. We pass over it just as we discussed at the end of the previous proposition.(M)

88. See notes 77 and 79.

89. London, BM, MS add 13,387, fol. 77b.(N)

Pseudo-Ṭūsī text has no note at this proposition.

90. London, BM, MS add 13,387, fol. 96b.

Pseudo-Ṭūsī *Taḥrīr*, following a porism to the tenth proposition of book VI (corresponding to Euclid's proposition VI, 11), notes (p. 144) that "Thābit ibn Qurra made this porism into a proposition of the essence of the treatise for the sake of clarification. But it is not a proposition in either the Greek or the Syriac manuscripts. Therefore, al-Ḥajjāj did not introduce it into his text. But the one familiar with the treatise of Euclid and [with] his technique in this treatise [recognizes] from the species of the porism that it is not of the essence of th treatise, since it is more proper for a subdivision. (This is its picture.) I have exaggerated in the demonstration of the porism for the sake of clarity."(O)

Heath claims [*Elements*, vol. 1, pp. 79-80] that Euclid's proposition VI, 12 has been omitted by Ṭūsī. Quite clearly, he has based this judgement on the Pseudo-Ṭūsī

I, 45: This proposition is not in the text of al-Ḥajjāj.[79]

III, 34: Al-Ḥajjāj mentioned these different [cases] and Thābit limited himself to the letter.[80], which does not even contain these two propositions.

III, 36: This proposition is not in the text of al-Ḥajjāj; it is among those added by Thābit since there occurs a need for it in the tenth [proposition] of book IV.[81]

VIII, 15: There exists a variation in the order of some of these propositions; what we have used is according to the order of Thābit. As for al-Ḥajjāj, he introduced what we discuss in proposition eleven and twelve as proposition eleven only, and what we have as proposition thirteen [he has] as proposition twelve. He has, in [his] propositions thirteen and fourteen, mentioned the first part of what we discuss in propositions fourteen and fifteen, and, in [his] proposition fifteen he has the second part of these two [propositions]. The two texts coincide in what follows.[82]

VIII, 25: These two propositions (i.e., 24 and 25) are not in the text of al-Ḥajjāj.[83]

IX, 12 In the text of al-Ḥajjāj this proposition precedes the one that lies before it.[84]

IX, 14: This proposition, in the text of al-Ḥajjāj, is twenty.[85]

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sition (i.e., proposition IV, 10) appended it to the end of this [third] book. [It is] appropriate to the technique Euclid follows in this treatise that this proposition not stand alone in the discussion together with the existence of these porisms. Therefore, al-Ḥajjāj does not mention it in his treatise, since it is not present in the ancient Greek or Syriac texts. We relegate it to the status of a porism in order to make clear that it is not of the essence of the book nor is it used in proposition ten of book four. Furthermore, I mention the demonstration which Thābit discussed.(I)

The demonstration that follows is the same as that found in Heath, *Elements*, vol. 2, p. 76.

79. London, BM, MS add 13, 387, fol. 123a.(J)
The Pseudo-Ṭūsī *Taḥrīr* has no comment with this proposition.

80. London, BM, MS add 13,387, fol. 125b.
This note does not appear in the Pseudo-*Taḥrīr*

81. London, BM, MS add 13,387, fol. 128b.
Pseudo-Ṭūsī has no note at this proposition.

82. London. BM, MS add 13,387, fol. 129b.
Pseudo-Ṭūsī has no note at this proposition.

83. London, BM, MS add 13,387, fol. 134a-b.(K)
Pseudo-Ṭūsī's heading merely states that there are 109 propositions.

84. London, BM, MS add 13,387, fol. 163b.
Pseudo-Ṭūsī has no note attached to this proposition.

85. London, BM, MS add 13,387, fol. 48a.

Taḥrīr tells us some information about the differences between the translation versions. But the genuine Ṭūsī version contains more information than its later rival. There are nine references to the translation of al-Ḥajjāj in the *Taḥrīr*. These are uniformly brief and refer only to instances in which there is a formal difference from the version of Isḥāq-Thābit. I briefly catalogue these references.

I, 45: This proposition is not in the text of al-Ḥajjāj.[76]

III, 34: Al-Ḥajjāj mentioned these different [cases] and Thābit limited himself to the latter.[77]

III, 36: This proposition is not in the text of al-Ḥajjāj; it is among those added by Thābit since there occurs a need for it in the tenth [proposition] of book IV.[78]

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76. London, BM, MS. add 13,387, fol. 27b.

Pseudo-Ṭūsī, at the end of the same proposition, says (p. 44): Al-Ḥajjāj does not discuss this proposition in his book, but it is found in the text of Thābit. The truth is that there is no need for it after the preceding proposition (I, 44). That is because the technique of Euclid, in this book of his, is that, if there be a proposition or lemma which is demonstrated already in a previous proposition, he does not make it a proposition of his book. And we do not draw out the enunciation from potentiality to actuality. Rather, he does not discuss anything about it, relying on the intellects of those who attempt the unravelling of this his treatise because he discusses it as a porism to the preceding proposition And I have discussed it in actuality because my technique in this treatise demands that. (F)

77. London, BM, MS add 13,387, fol. 64b.(G)

There is no equivalent statement found in the Pseudo-Ṭūsī Taḥrīr.

78. London, BM, MS add 13,387, fol. 66a.(H)

The Pseudo-Ṭūsī has (pp. 91-2), at this proposition, a more lengthy note. The microfilm of Teheran, Kutubhane Millī Malik 6264, on which I have relied for this paper, is unclear and some words are completely illegible. I have attempted to reconstruct these parts based on the portion of the text that I could read.

Euclid, since he did not retain these concepts (apparently those discussed in the immediately preceding porism), does not mention the proposition which Thābit ibn Qurra appended at the end of this book, even though he used it in proposition ten of book IV. It is his custom in this treatise that he uses many principles and he does not consider in the treatise whether they are expressed in terms of what preceded from its questions themselves or by way of the porism. It (the appended proposition) is:

36

Any two lines extending form a point external to a circle, one of which cuts it (the circle) and the other terminating at it without cutting, and the area of the entire cutting [line] by that [part] which is exterior to the circle being equal to the square on the terminating [line] the terminating [line] is tangent to the circle.

Thābit ibn Qurra, since he thought that Euclid used it in the mentioned propofour, thirty two and thirty four of book III and proposition five of book IV,[71] have been associated with the name of al-Ḥajjāj.[72] Curiously, although these cases are reported in the *Taḥrīr*, only in the note to proposition thirty four does Ṭūsī relate the cases to al-Ḥajjāj. Also noteworthy is Ṭūsī's omission of the cases that al-Ḥajjāj added to proposition thirty five of book III.[73] The table below summarizes the presence of these added cases in the Arabic Euclidean literature.

TEXT	III, 24	III, 32	III, 34	III, 35	IV, 5
al-Nayrīzī	Y	Y	Y	Y	Y
Ibn Sīnā	Y	Y	Y	Y	Y
Ibn al-Haytham	Y	Y	Y	Y	N(?)
al-Abharī	Y	Y	Y	Y	Y
al-Maghribī	Y	Y	Y	Y	Y
India Commentary	Y	?	?	?	Y
Pseudo-Ṭūsī	Y	Y	Y	Y	Y

The information in this table only serves to re-inforce our earlier observations. These propositions, whose added cases are explicitly referred to al-Ḥajjāj in quotations found in the Andalusian sub-family of manuscripts, were widely known and discussed by Arab geometers. Since these discussions routinely make no mention of sources, it can scarcely be surprising that Ṭūsī merely incorporates them into his own redaction without ascription (except to say, in proposition twenty four, that Thābit had truncated his discussion).

TUSI AS SOURCE FOR TEXTUAL HISTORY

Ṭūsī (or more correctly, the Pseudo-Ṭūsī, since most studies have been based on the latter's work) has been one of our most important sources of information about the translations of al-Ḥajjāj and Isḥāq-Thābit.[74] (He does not mention the putative translations of Abū'Uthmān al-Dimishqī or Thābit ibn Qurra.[75]) And it is true that each of these

71. These propositions correspond to Euclid's III, 25, 33, 35, and IV, 5.
72. For discussion of these al-Ḥajjāj cases, see G. De Young, "New Traces," pp. 650-7.
73. London, BM, MS. add. 13,387, fol. 64b.
74. Heath, *Elements*, vol. 1, pp. 78-82.
75. Brentjes, "Relevance," p. 202.

There are six discussions of added cases included by Ṭūsī in books III and IV. (One of these – the added case in proposition thirty of book III – we have already mentioned in the previous section as one of the additions ascribed to Thābit ibn Qurra.) The first example to occur in this section is proposition nineteen of book III.[63] Here Ṭūsī mentions the existence of the added cases but does not demonstrate them because "in each case it is clear from what occurred [in the given demonstration]."[64] Al-Nayrīzī's commentary contains a long discussion, attributed to Heron, concerning a method for generalizing Euclid's demonstration, but no discussion of added cases. The added cases are found, however, in ibn Sīnā,[65] ibn al-Haytham,[66] al-Abharī,[67] al-Maghribī.[68] These are each quite brief, but the discussion in the Indian commentary[69] and in the Pseudo-Ṭūsī *Taḥrir*[70] are more extensive and complete. Once again, it seems, Ṭūsī is importing a discussion that is wide-spread in the literature. Apparently, these cases were not of sufficient importance for his pedagogical purposes to warrant introducing their demonstrations. Further, Ṭūsī's brief note re-inforces the idea that he is uninterested in philosophical or logical debates, since he completely ignores the discussion of ibn al-Haytham – a discussion of which he could scarcely have been ignorant.

The remainder of the added cases in this section, propositions twenty

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63. "In a circle, the angle at the center is double of the angle at the circumference, when the two angles have the same circumference as a base." Heath, *Elements*, vol. 2, p. 46.

64. London, BM, MS. add 13,387, fol. 57b.

65. *Kitāb al-Shifā'*, pp. 107-9.

66. Istanbul, MS Fatih 3439, fol. 104b-105a. He continues by pointing out that Euclid, in his demonstration, assumes a relation which will only be demonstrated in book V. Cf. Heath, *Elements*, vol. 2, p. 49. In an attempt to rectify this apparent logical lapes, ibn al-Haytham introduces an alternative demonstration.

67. Dublin, Chester Beatty Lib., MS 3424, fol. 22a.

68. Oxford, Bod. Lib., MS Bodley Or. 448, fol. 23b-24a.

69. Hyderabad, SCL, MS riyāḍi 2, fol. 57a-b; Hyderabad, OU, MS Q'A 510 / Ash-R, fol. 113b-114b.

70. Pseudo-Ṭūsī, pp. 78-9. While it may be correct to claim (as does R. Lorch) that the Pseudo-Ṭūsī *Taḥrīr* is in some way based upon the genuine work of Ṭūsī, it is clearly an independent work in its own right. In this proposition, for example, Ṭūsī does not demonstrate the alternative positions, although such demonstrations are widely available in the literature. The Pseudo-Ṭūsī *Taḥrīr*, however, does include the full alternative demonstrations. It is, then, no merely slavish imitation of Ṭūsī. Cf. R. Lorch, "Somme Remarks on the Arabic - Latin Euclid," in C. Burnett (ed), *Adelard of Bath: An English Scientist and Arabist of the Early Twelfth Century* (London: Warburg Institute, 1987), p. 45.

cases were also discussed by al-Nayrīzī, who attributed them to Heron.[59] (Here the scalene case is further divided on the basis of which side is longer.) The case of the isosceles triangle (but not the scalene) is found in al-Abharī[60] and ibn al-Haytham.[61] This material is not present in ibn Sīnā, al-Maghribī, or the Pseudo-Ṭūsī. Ṭūsī himself chose not to include either case.

There are seven other propositions with added cases in Ṭūsī's book I: propositions seven, nine, twenty four, thirty five, thirty nine, forty two, and forty seven. Each of these added cases in attributed to Proclus in Heath's mathematical notes, except for the final two. (And perhaps Proclus also added cases to proposition forty two; we cannot know since there now exists a lacuna in the Greek text.[62]) Because of space limitations, I shall summarize the presence of these discussions in the Arabic Euclidean literature in a table.

TEXT	I, 7	I, 9	I, 24	I, 35	I, 39	I, 42	I, 47
al-Nayrīzī	N	Y	Y	Y	N	N	Y(?)
Ibn Sīnā	Y	N	Y	Y	N	N	N
Ibn al-Haytham	Y	Y	Y	Y	N	N	Y(?)
al-Abharī	Y	Y	Y	Y	N	Y	N
al-Maghribī	Y	Y	Y	N	N	N	N
India Commentary	Y	Y	Y	?	N	N	N
Pseudo-Ṭūsī	Y	Y	Y	Y	Y	Y	Y

It becomes clear that, although Ṭūsī cites no source for these added cases, in most instances they were widely known and discussed in the Arabic Euclidean tradition. Thus, they might legitimately be considered a form of common knowledge not needing an attribution. In these propositions, at least, Ṭūsī is merely reporting discussions already long familiar and widely repeated in the Greco-Roman mathematical tradition. What is equally interesting is Ṭūsī's (apparent) omission of other examples of added cases. Why did he choose to include some and exclude others? This is not easily answered because Ṭūsī remains silent on his selection criteria.

59. Codex Leidensis, fasc. 1, pp. 44-7. See also Tummers, pp. 39-41 and Curtze, pp. 42-6.

60. Dublin, Chester Beatty Lib., MS 3424, fol. 3b-4a.

61. Istanbul, MS. Fatih 3439, fol. 75a.

62. Heath, *Elements*, vol. 1, p. 341.

construct the equilateral triangle that Euclid's demonstration requires.[49]

It seems that most of what Ṭūsī includes in this note can be traced to Proclus. But does failure to cite Proclus explicitly indicate a lack of scholarly principle? I suggest that it does not. These Proclus cases were widely disseminated and often discussed in the Arabic Euclidean literature. The commentary of al-Nayrīzī, for example, discusses the two forms of the second case of Proclus, without mentioning a source.[50] The four basic cases of Proclus are noted by both Ibn Sina[51] and al-Abharī,[52] as well as in the *Taḥrīr* of al-Maghribī,[53] again without specifying a source. The anonymous Arabic commentary divides the demonstration into three parts, depending on whether the specified point is outside the given line, at its terminus, or within it.[54] Ibn al-Haytham makes the same distinctions in his demonstration.[55] And the discussion of Ṭūsī, including his terminology, is mirrored in the *Taḥrīr* of Pseudo-Ṭūsī.[56] Clearly, this material was widely available, so Ṭūsī may have included it without pursuing an historical inquiry into its original source.

Even from widely available material, Ṭūsī is selective in what he includes in his *Taḥrīr*. For example, Proclus had already added a kind of case study to proposition I, 1, noting that we can use the Euclidean technique to construct both an isosceles and a scalene triangle.[57] This discussion, too, was widely known. It appeared as two interpolated propositions in the Andalusian sub-family of Isḥāq-Thābit manuscripts.[58] These

49. London, BM, MS add 13,387, fol. 5b-6a. Ṭūsī's terminology (the use of *mabānī* and *musāma* especially) is not typical of the Arabic versions of the *Elements*. (E)

50. *Codex Leidensis*, fasc. 1, pp. 48 and 50. See also Tummers, pp. 41-2 and Curtze, pp. 47-8.

51. *Kitāb al-Shifā'*, pp. 22-4.

52. Dublin, Chester Beatty Lib., MS 3424, fol. 4a-b.

53. Oxford, Bod. Lib., MS. Bodley Or. 448, fol. 3b. The original text was either carelessly copied or copied from a corrupt exemplar. Most of the discussion has been added in the margin in what appears to be a hand other than that of the copyist.

54. Hyderabad, SCL, MS riyāḍi 2, fol. 21a; Hyderabad, OU, MS Q'A 510 / Ash-R, fol. 42a-43a. Again, there is no mention of Proclus, but, in discussing the case when the point in within the line itself, the commentator does refer to the construction of al-Nayrīzī. cf. Codex Leidensis, fasc. 1, p. 50.

55. Istanbul, MS Fatih 3439, fol. 75b-76a.

56. Pseudo-Ṭūsī, *Taḥrīr*, p. 10.

57. Heath, *Elements*, vol. 1, p. 243.

58. This subfamily includes Escurial, MS arabe 907; Rabat, al-Malik., MS 53 and Rabat, al-Malik., MS 1101. This group of manuscripts has been influenced at some point from the translation of al-Ḥajjāj. See the discussions by De Young and Brentjes mentioned in note 20.

strations found in the *Taḥrīr*. A full-scale study of the text is urgently needed. The few cases mentioned here do, at least, imply that not all of these alternative demonstrations are the work of Ṭūsī himself. Instead, he has borrowed from earlier sources, often without ascription. Why these particular alternative formulations have been chosen for inclusion is still not clear. It may reflect something of the ultimate purpose of the treatise-to be a first step toward mastery of the *Almagest* and spherical astronomy. The lack of formal attribution in many cases may also serve to underline anew the pedagogical nature of the treatise. (Even today, citations of earlier mathematical studies are not emphasized within introductory textbooks.)

ADDED CASES

We have already had occasion to mention one example (proposition I, 447) in which Ṭūsī expanded Euclid's text by working out the details of demonstrations not fully developed in the original. It is generally assumed that Euclid preferred to work out only the more difficult cases, leaving the remainder for his readers to solve. And many of his Greek and Arabic followers were, apparently, delighted to seize the opportunity to demonstrate their competences as geometers. Ṭūsī, in nearly twenty propositions, has a discussion of added cases. As in the case of alternative demonstrations, these added cases are not scattered at random throughout the text. Eight appear in book I, five in book III, and three in book XI, while books II and IV each have only one example.

Most of the added cases in book I seem to have been discussed as early as Proclus, althought his name is never mentioned by Ṭūsī. For example, proposition I, 2 has added cases in the commentary of Proclus.[48] Ṭūsī mentions the same cases: when the specified point falls (1) outside the given line, whether to one side (*mabāniyy*) or in line with it (*ghayr mabāniyy*) and if to one side, it is either as an azimuth (*musāma*) or not (*ghayr musāma*), or (2) if on the line, either within it or at its extremity. (Only the first case was demonstrated by Euclid.) And another set of cases arises when we consider whether the line to be produced be longer than, equal to, or shorter than the given line. Ṭūsī discusses all these cases, and adds still more, depending on which side of the given line we

48. "To place at a given point (as an extremity) a straight line equal to a given straight line." Heath, *Elements*, bol. 1, p. 244.

however. Instead, it turns out to be conceptually (although not verbally) very similar to the alternative demonstration included in Ṭūsī's *Taḥrīr* at this point! Once again, it Ṭūsī appears to borrow without ascription.

A final alternative demonstration attributed to Thābit appears after proposition thirty of book III (Euclid's proposition III, 31).[45] Thābit, relying once again on Greek manuscripts, offers an alternative only for the first case in the enunciation-the case of the semi-circle and the right angle. Again, in the Greek tradition there are two alternatives for this part of the demonstration. Isḥāq had accepted as his basic text the version which Heiberg relegated to the appendix. Thābit's alternative demonstration corresponds quite well with that found in Heiberg's text.[46] What we find in Ṭūsī's text as the alternative demonstration corresponds to that attributed to Thābit in the Isḥāq-Thābit manuscripts.[47] Once again, then, we see Ṭūsī apparently borrowing the work of one of his predecessors without providing an attribution. It would seem that this is his customary practice, and that the citation of Thābit in propositions III, 9 and 10 was quite unusual. Why he should have chosen to note his source in these two propositions but not in others remains unknown.

These are, of course, only a small sample of the alternative demon-

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We connect ***AH***. It is longer than ***HG***, as was shown in propositions seven and eight.

But ***HA*** is equal to ***HZ***. Thus, ***HZ*** is longer than ***HG***.

This is an impossibility, and this impossibility is the result of our specifying that line ***EH*** does not pass through the point of tangency.

Therefore, the line which connects the two centers passes through the point of tangency.

That is what we wanted to show.

This demonstration is better than Euclid's because it considers both cases together in a single expression, and its basis is propositions seven and eight (of book III). (D)

Cf. London, BM, MS. add 13,387, fol. 53b. (The text appears to be corrupt. I have followed the reading of Princeton University Library, MS Yehuda 4848.)

45. "In a circle, the angle in the semicircle is right, that in the greater segment is less than a right angle, and that in a less segment is greater than a right angle; and further the angle of the greater segment is greater than a right angle and the angle of the lesss segment is less than a right angle." Heath, *Elements*, vol. 2, p. 61.

46. Engroff, p. 24. The nature of the Greek source is more difficult to determine, however, In this case, al-Nayrīzī does not mention an alternative form of the demonstration, so there is no attribution, either. see *Codex Leidensis*, fasc. 3, pp. 104-9.

47. London, BM, MS add 13,387, fol. 61a-b.

is nearly a verbatim report of what we find attributed to "Abū'Alī" in the Isḥāq-Thābit manuscripts! This, in turn, is a close paraphrase of the demonstration of the case of external tangency attributed to Heron in al-Nayrīzī's commentary.[42] These varied ascriptions are a bit confusing, but none of this helps us to understand the source of the alternative demonstration reported in Ṭūsī's text, since it is quite different.

Who is the "Abū'Alī" mentioned in the Isḥāq-Thābit manuscripts? Abū'Alī ibn Sīnā? A quick check of his *Kitāb al-Shifā'*, however, indicates that his demonstration closely parallels that found in the commentary of al-Nayrīzī.[43] Or perhaps Abū'Alī ibn al-Ḥasan ibn al-Ḥasan ibn al-Haytham? His *Kitāb Ḥall Shukūk Kitāb Uqlīdis fī'l-Uṣūl* does contain an alternative demonstration for the eleventh proposition of book three.[44] This proposition is not identical to that ascribed to Thābit,

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the circle and where it exits from it; also, whether we draw circle ***AG*** tangent to circle ***AB*** from inside or outside, at point ***A***.

*Let the center of circle **AB** be **E** and the enter of circle **AG** be **Z**. I say: the line that passes through point **E** [and] **Z** passes through point **A**. If not, let it fall like line **EHTZ**.*

Let lines ***AZ*** [and] ***AE*** be extended. Then lines ***AE*** [and] ***AZ*** together are longer than line ***ZE***.

Now as for line ***AE***, it is equal to line ***EH***; line ***AZ*** is equal to line ***TZ***. Thus, lines ***EH, TZ*** are longer than line ***EZ***.

*But the two should be smaller than it. This is a contradiction. Therefore, the line which connects points **E** [and] **Z** passes through point **A**. That is what we wanted to show. (C)*

42. *Codex Leidensis*, fasc. 3, pp. 46-9. See also Tummers, pp. 98-9 and Curtze, pp. 121-2. Al-Nayrīzī, presumably basing his commentary on al-Ḥajjāj, presents a somewhat modified version of the Euclidean demonstration, relying on the application of proposition I, 20 for his argnment. Euclid, on the other hand, had utilized the (interpolated) common notion that equals subtracted from equals will be equal. His modification allowed al-Nayrīzī to unify the two parts of his proposition but did not eliminate the need for two demonstrations.

43. A. I. Sabra and A. H. Lotfy (eds), *Kitāb al-Shifā': al-Fann al-Awwal min Jumlat al-'Ilm al-Riyāḍī– Uṣūl al-Handasa* (Cairo: General Egyptian Book Organization, 1977), pp. 100-1.

44. Istanbul, MS Faith 3439, fol. 101b-102a:

In any two triangles that are tangent, the line passing through their centers passes through the point of tangency.

There is no doubt concerning this proposition, but it is possible to prove it with another demonstration:

Let there be two circles, ***ABG*** [and] ***ADZ***, and let ***E*** be the center of circle ***ABG***, ***H*** the center of circle ***ADZ***. We connect ***EH*** and extend it until it cuts circle ***ABG***, and let it meet it at points ***B*** [and] ***G***.

Now, let point ***H*** be on the diameter of circle ***ABG***, not the center.

manuscripts".

Both forms of the demonstration of Elements III, 9, occur in the Greek text but the version cited here by Thābit as an alternative has been judged by Heiberg to be not genuine and so it is relegated to an appendix in his edition.[35] The commentary ascribed to al-Nayrīzī and said to be based on the second version of the Arabic translation of al-Ḥajjāj, contains only Heiberg's accepted form of the demonstration, with no mention made of an alternative.[36] In the case of proposition III, 10, there are two demonstrations in the Greek tradition. As in the previous case, Heiberg relegates the alternative version, here associated with Thābit's name, to an appendix.[37] In this case, though, when we look into the commentary of al-Nayrīzī, we find the alternative demonstration present, but ascribed to Heron.[38]

The third Thābit alternative demonstration is added to proposition III, 11.[39] In this instance, however, none of the surviving Isḥāq-Thābit manuscripts contains a Thābit comment at this place. Our sole source is the anonymous Arabic commentary described earlier.[40] On the other hand three Isḥāq-Thābit manuscripts do contain a note attributed to "Abū'Alī" which discusses the second part of the proposition, the case when two circles touch externally (the equivalent of Heiberg's bracketed proposition twelve) in what appears to be a substitute for the bracketed proposition, which is universally missing from all the Arabic manuscripts and discussions.[41] The "Thābit" comment following proposition III, 11

35. Heath, *Elements*, vol. 2, p.22. Doubts about Heiberg's editing principles have been raised on several occasions. The most extensively argued position appeared in W. Knorr, "The Wrong Text of Euclid: On Heiberg's Text and its Alternatives," *Centaurus* (1996), 38: 208-76.

36. *Codex Leidensis*, fasc. 3, pp. 40 and 42.

37. Heath, *Elements*, vol. 2, pp. 23-4.

38. *Codex Leidensis*, fasc. 3, p. 44. See also Tummers, pp. 97-8; Curtze, pp. 120-1.

39. "If two circles touch one another internally, and their centres be taken, the straight line joining their centres, if it be produced, will fall on the point of contact of the circles." Heath, *Elements*, vol. 2, p. 24.

40. The anonymous Indian commentary (see note 30) mentions another addition by Thābit, this time to proposition eleven. This note does not appear elsewhere in the Arabic or the Latin versions. See Hyderabad, SCL, MS. riyāḍi 2, fol. 54a and Hyderabad, OU, MS. Q'A/Ash-R, fol. 106b.

41. In Dublin, Chester Beatty Lib., MS 3035; Teheran, Majlis, MS. 200; and Rampur, Rida Lib., MS 'Arshi 200 we find this appendix:

Abū'Alī said: And if the center of circle ***AB*** lie outside circle ***AG***, the enunciation of this proposition would still be correct without any changes except that line ***AE*** be extended from circle ***AG*** and line ***EG*** cuts circle ***AG*** at two points, where it enters demonstration is repeated in the margin of the *Taḥrīr* of the *Elements* by Muḥyi al-Dīn al-Maghribī. Here, again, it is without ascription, being introduced by the phrase "the master said" (*qāla al-sheikh*).[31] Ṭūsī, as is his custom, neglects to cite his source (assuming that he actually knew the source-the evidence seems to indicate that the demonstration was quite widely known, but its source was not). Most probably, Ṭūsī's alternative demonstration was borrowed from one of these Arabic sources, althought it is difficult to guess which one.

In surviving Arabic manuscripts of the Isḥāq-Thābit translation tradition, we find references to more than twenty instances in which Thābit added to or altered the text of Isḥāq.[32] In his *Taḥrīr*, Ṭūsī mentions two places (propositions III, 9 and 10) where Thābit added alternative demonstrations to Isḥāq's text,[33] but makes no explicit reference to two other places in book III (propositions eleven and thirty) in which Thābit also is credited with adding alternative demonstrations. Three of these Thābit passages are well-attested within the Arabic Euclidean tradition, but the addition to proposition eleven is cited only in the anonymous Indian commentary mentioned earlier.[34] Although in many of the Isḥāq-Thābit manuscripts Thābit's dependence on Greek manuscripts for these additions is stressed, Ṭūsī's statements mention only "other

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mentioned in Rahman, p. 421. The OU manuscript is copied directly from the SCL manuscript. Both, therefore, are incomplete at the beginning, and perhaps the end.

On ibn Hūd and his discussions of Euclidean geometry, see J. P. Hogendijk, "The Geometrical Parts of the *Istikmāl* of Yūsuf al-Mu'taman ibn Hūd (11th Century): An Analytical table of Contents," *Archives Internationales d'Histoire des Sciences* (1991), 41: 207-81.

31. Oxfort, Bod. Lib., MS. Bodley Or. 448, fol. 4a.

32. The notes attributed to Thābit have been cumulated and discussed in Engroff, pp. 20-39. Several mention that Thābit added propositions based on his collation of Isḥāq's Arabic text with Greek manuscripts. At least this is true until we reach book X, which contains nearly half the Thābit comments known to Engroff. Here, Thābit's comments are typically in the form of a gloss on the terminology of Euclid's text.

33. London, BM, MS. add 13,387, fol 52b and 53a.

34. Several manuscripts report that Thābit found these alternative demonstrations in his Greek manuscripts. Others (Rabat, al-Malak., MS 53; Teheran, Malik, MS 3586; Dublin, Chester Beatty Lib., MS 3035; Tehran, Majlis, MS 200; Rampur, Rida Lib., MS. 'Arshi 200) attribute these alternative demonstrations to Thābit but fail to mention Greek manuscripts. The translation attributed to Gerard of Cremona is the only Latin version to preserve these Thābit additions. In both cases, the text refers to finding them "in alia scriptura greca." See H. L. L. Busard, *The Latin Translation of the Arabic Version of Euclid's Elements Commonly Ascribed to Gerard of Cremona* (Leiden: Brill, 1984), col. 66 and 67.

ALTERNATE DEMONSTRATIONS

The largest group of commentary notes are Ṭūsī's alternative demonstrations of Euclid's propositions. There are ninety scattered throughout the text. Usually there is only one, but in a few cases we find two or three for a proposition. Each is introduced with the stereotypical phrase "and by another demonstration" (*wa-bi-wajḥ ākhar*). As we might expect, they are not randomly distributed throughout the treatise. More than two thirds of the ninety alternate demonstrations occur within the first six books (and only five of these occur in book V, Euclid's discussion of ratios, which differs markedly from the more geometrical content of the other books). The arithmetic books (VII-IX) contain only ten of these alternative formulations. Once again, it would appear that Ṭūsī's primary interest was in the geometry, rather than in the full range of mathematical topics developed by Euclid in the *Elements*.

The first case of an alternative demonstration occurs with proposition I, 5.[25] Ṭūsī remarks that it is possible to demonstrate the first part of Euclid's proposition without extending the two sides of the triangle.[26] What follows is the proof originally given by Proclus.[27] This proof of Proclus was also quoted as an alternative demonstration in the commentary on the *Elements* attributed to al-Nayrīzī.[28] It also appears as an alternative within the Latin form of this commentary, in the translation ascribed to Gerard of Cremona.[29] Neither of these texts associates this demonstration with Proclus. In an anonymous Arabic commentary on the *Elements* now extant in two copies in Hyderabad, India, this alternative demonstration is attributed to the "Master of the *Istikmāl*", ostensibly a reference to the mathematician Mu'taman ibn Hūd.[30] The

25. "In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another." Heath, *Elements*, vol. 1, p. 251.

26. London, BM, MS. add. 13,387, fol.7a.

27. See heath, *Elements*, vol. 1, p. 254.

28. *Codex Leidensis*, fasc. 1. pp. 60-3.

29. P. M. J. E. Tummers, *The Latin Translation of Anaritius' Commentary on Euclid's Elements of Geometry, Books I - IV* (Nijmegen: Ingenium, 1994), p. 43; M. Curtze, *Anaritii in decem libros priores Elementorum Euclidis Commentarii ex interpretatione Gherardi Cremonensis in codice Cracoviensi 569 servata* (Leipzig: Teubner, 1899), pp. 49-50.

30. Hyderabad, Oriental Manuscripts Library and Research Institute (formerly State Central Library), MS. riyāḍi 2, fol. 22b; Hyderabad, Osmania University Library, MS Q'A 510/Ash-R (acquisition no. 375), fol. 45b-46a. These two manuscripts are

numbers of the proposition using red for Thābit and black for al-Ḥajjāj whenever the two differ.[21] But these editorial comments are limited to variations within the order of propositions and sometimes to differences in terminology; they do not extend to noting (at least explicitly) internal differences in the content of demonstrations, even when these two traditions adopt completley different approaches to the same proposition, as occurs in book VIII, propositions twenty and twenty one.[22] Here, Isḥāq replaced the standard Euclidean demonstration with one based on book VIII, propositions fourteen and fifteen, while al-Ḥajjāj followed the Greek approach of treating these propositions as special cases of the preceding two.[23] Interestingly, the version of al-Ḥajjāj is given as an alternative demonstration by Ṭūsī. It is not introduced by the stereotypical "I say", but neither is it attributed to the work of al-Ḥajjāj.[24]

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della Scienza (1991), 28: 647-66; S. Brentjes, "Varianten einer Ḥaĝĝāūg-Version von Buch II der Elemente," in M. Folkerts and J. Hogendijk (eds), *Vestigia Mathematica: Studies in Medieval and Early Modern Mathematics in Honor of H. L. L. Busard* (Amsterdam: Rodopi, 1993), pp. 47-62; S. Brentjes, "Textzeugen und Hypothesen zum arabischen Euklid in der Überlieferung von al-Ḥaĝĝāĝ b. Yūsuf b. Maṭar (zwischen 786 und 833)," *Archive for History of Exact Sciences* (1994), 47: 53-92; Brentes, "The Relevance of Non-Primary Sources for the Recovery of the Primary Transmission of Euclid's *Elements* into Arabic," in F. J. Ragep (ed), *Tradition, Transmission, Transformation* (Leiden: Brill, 1996), pp. 201-25.

The Arabic text of a commentary ascribed to al-Nayrīzī and claiming to quote from the second version of the translation of al-Ḥajjāj has been published in R. O. Besthorn and J.. Heiberg (eds), *Codex Leidensis 399.1: Euclidis Elementa ex interpretatione al-Hadschdschadschii cum commentariis al-Nairizii* (Copenhagen: In Libraria Gyldendaliana, 1893-1932). The claim that this commentary contains verbatim quotation from al-Ḥajjāj has been doubted by J. Engroff, *The Arabic Tradition of Euclid's Elements: Book V* (Cambridge: Harvard University PhD dissertation, 1980, unpublished), pp. 5-19 and by H. L. L. Busard, "Einiges über die Handschrift Leiden 399,1 und die arabisch-lateinische Übersetzung von Gerard von Cremona," in J. W. Dauben (ed), *History of Mathematics: States of the Art* (San Diego: Academic Press, 1995), pp. 173-205, especially 176 ff.

On the question of the authorship of the second major Arabic translation, see G. De Young, "Isḥāq ibn Ḥunayn, Ḥunayn ibn Isḥāq, and the Third Arabic Translation of Euclid's Elements," *Historia Mathematica* (1992), 19: 188-99.

21. London, BM, MS. add. 13,387, fol. 2b.

22. These correspond to Euclid's propositions VIII, 22 and 23.

23. De Young, "New Traces," pp. 657-9. Cf. Heath, *Elements*, vol. 2, pp. 379-80.

24. London, BM, MS. add. 13,387, fol. 124b-125a.

attempts to prove the postulate.[17] These discussions are omitted from the *Taḥrīr*, and once again we sense an underlying pedagogical emphasis. Ṭūsī, apparently, does not wish to burden his students with unnecessary historical or philosophical discussions - his interest is in the geometric results and their demonstration.

The second lengthy note is appended to proposition forty seven, the famous "Pythagorean theorem", and illustrates that continuing fascination with additional cases that we so frequently encountered in both the Greek and Arabic Euclidean traditions.[18] In this note, Ṭūsī remarks that each of the squares constructed on the sides of the right triangle may lie on either side of the lines. Thus, there are eight defferent possibilities for the placement of these squares, each requiring a slightly different set of constructions of parallel lines in order to carry out the demonstration. (Euclid himself was content with the case when each of the three squares face outward from the triangle.[19]) Whether these represent the work of Ṭūsī himself or an adaptation from an earlier discussion is not evident. In the same discussion, Ṭūsī expe-riments with changing some of the parameters of the traditional demonstration, supposing that (1) line AL is not extended parallel to the side of the square, (2) we do not actually construct the squares on the two sides, (3) rather than construct the two squares, we construct a single square equal to the two of them, and (4) differences when one of the two lines bordering the right angle differs from the other in length.

Ṭūsī seems to imply, at the beginning of his *Taḥrīr*, that he has consulted both the version of al-Ḥajjāj, the first to render the *Elements* into Arabic, and of Thābit, who edited the later Arabic trauslation attributed to Isḥāq ibn Ḥunayn.[20] He will indicat, he informs us, the

17. For an example of Ṭūsī's criticism, directed toward Khayyām, see Jaouiche, pp. 88-98.

18. London, BM, MS, add. 13,387, fol. 28b-38a. A similar discussion, but limited to the eight cases arising from placement of the constructed squares, appears in the Pseudo-Ṭūsī *Taḥrīr*, pp. 46-8.

19. Heath, *Elements*, vol. 1, pp. 349-50.

20. The basic features of the Arabic translations of the *Elements*, drawn from the standard Arabic biobibliographical sources, are summarized in Heath, *Elements*, vol. 1, pp. 75-7; Murdoch, "Euclid: Transmission," pp. 438-9; Sezgin, *Mathematik*, pp. 83-104. Discussion of the translation of al-Ḥajjāj may be found in G. De Young, "The Arabic Textual Traditions of Euclid's *Elements*," *Historia Mathematica* (1984), 11: 147-60; G. De Young, "Al-Ḥajjāj in the Andalusian Euclidean Tradition," *Islamic Studies* (1991), 30: 171-8; G. De Young, "New Traces of the Lost al-Ḥajjāj Arabic Translations of Euclid's *Elements*," *Physis: Revista Internazionale di Storia*

notes-and of these, fully a dozen occur prior to proposition twenty six. Since Euclid begins his discussion of families of irrational lines immediately following proposition twenty six, this also seems to indicate that Ṭūsī did not find that topic particularly useful or interesting for his purposes.

Ṭūsī 's comments are very rarely attached to definitions and never to postulates or axioms. These exceptional comments appear

in conjunction with the first definition of book II, the third definition of book V, definitions two and eleven of book VII, and definitions sixteen and eighteen of book XI. All are brief, and typically deal with differences in terminology that occur between the translation traditions. None raises or discusses fundamental philosophical questions. The distinct impression given is that, at least within his *Taḥrīr ḥrīr*, Ṭūsī was little interested in the philosophical analysis of the logical underpinnings of Euclidean geometry.

The majority of Ṭūsī 's commentary notes are quite brief, but two (both in book I) stand out for being much more lengthy. The first follows proposition twenty eight.[15] (This, together with the preceding proposition, lays the groundwork for Euclid's consideration of parallel lines.) In this note, we find a collection of eight propositions which, together, purport to give a demonstration of Euclid's famous parallel line postulate. These propositions are virtually identical to the these found in Ṭūsī 's more widely studied *Risāla al-Shāfiyya ʻan al-shakk fī'l-Khuṭūṭ al-Mutawāziya.*[16] *The discussion in the Risāla al-Shāfiyya*, however, also includes mention of some of Ṭūsī's predecessors and criticisms of their

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15. London, BM, MS add 13,387, fol. 16b-21b.

16. The Arabic text has been published in *Rasā'il al-Ṭūsī* (Hyderabad: Dā'iratu'l-Ma'ārif, 1359 H.), vol. 2. This demonstration, along with that of the Pseudo-Ṭūsī, has been extensively discussed (in Arabic) in A. I. Sabra, "Burhān Naṣīr al-Dīn al-Ṭūsī 'alā muṣādarat Uqlīdis al-Khāmisa," *Majallat Kulliyyat al-Adab, Jāmi 'at al-Iskandariyya* (1959), 13: 133-70. Other discussions include G. D. Mamedbeili, *Mukhammad Nasruddin Tusi o Teorii parallelnykh liny: Teorii otnosheny* (Baku, 1959), reprinted in *Istoriko-mathematischeskie isledoveniya* (1960), 13: 457-532; Kh. Jaouiche, *La Théorie des Paralléles en Pays d'Islam* (Paris: Vrin, 1984), pp. 99-106 (a French translation of the first part of the Risāla, up to the discussion of his correspondence with Qayṣār ibn Abī'l-Qāsim, which is later translated by Jaouiche from original documents, is given on pp. 201-26). A. Daffā' and J. Stroyls, *Studies in the Exact Sciences in Medieval Islam* (New York: Wiley, 1984) give a brief English summary of Ṭūsī's argument on pp. 47-51. For a more detailed English discussion, see B. Rosenfeld, *A History of Non-Euclidean Geometry* (New York: Springer, 1988), pp. 74-80.

Ṭūsī with the surviving manuscripts of the Arabic Euclid, we can see at once why his treatise is called a *taḥrīr*, an editing or a redaction. While retaining both the structure of the Euclidean original and the basic logic of its propositions, Ṭūsī has re-phrased the text and has typically shortened the demonstrations by removing some of the intermediate steps. Thus, the over-all impression given is that he is interested in the geometric results rather than the detailed didactic argument and that his basic intention is to show that these results are valid, but he does not wish the student to become lost in a morass of logical minutiae. This is nowhere clearer than in book X, with its complex and often confusing discussions of families of irrational lines. There, demonstrations which, in the Greek, occupy pages of text are reduced by Ṭūsī to a matter of a few lines.[13]

Interspersed throughout the *Taḥrīr* are more than two hundred commentary notes, typically introduced by the Arabic phrase "I say" (*aqūlu*).[14] These are not distributed uniformly throughout the treatise. In book II, for example, there are comments attached to thirteen of the fourteen propositions. (Only proposition eleven has no comment.) In book X, whose 109 propositions make up nearly a quarter of the 468 propositions in the entire treatise, there are only fifteen commentary

13. This is, in many ways, not unlike the summary of the *Elements* prepared by ibn Sīnā for inclusion in his Kitāb al-Shifā'. Here, too, we find many intermediate logical steps have been omitted and this is especially true in book X. Perhaps some of these similarities are not so surprising if we consider that one of Ṭūsī's teachers at Nīshāpūr, was Farīd al-Dān al-Dāmad, "who through four intermediaries was linked to Ibn Sīnī and his school." [S. H. Nasr, "Al-Ṭūsī, Muḥammad ibn Muḥammad ibn al-Ḥasan," in C. C. Gillispie (ed), *Dictionary of Scientific Biography* (New York: Scribner, 1971/1980), vol. 13, p. 509.] In addition to omitting many intermediate logical steps, ibn Sīnā reduces the bulk of the Greek proposition by omitting the abstract statement of the problem, the enunciation in Proclus' division of the parts of the proposition. He begins immediately with the specific statement of the problem. [For a brief synopsis of the divisions of the demonstration as described by Procllus, see T. L. Heath, *The Thirteen Books of Euclid's Elements* (New York: Dovr, 1956), vol. 1, pp. 129-31; for a more extended discussion, see R. Netz, "Proclus' Division of the Mathematical Proposition into Parts: How and Why was it Formulated?" *Classical Quarterly* (1999) 49: 282-303.] Ṭūsī, however, retains the abstract enunciation statement.

14. A precise number is difficult to state. Not all notes begin with the stereotypical "I say". Moreover, not all manuscripts contain exactly the same set of notes. For example, Princeton University Lib., MS Yehuda 4848 contains two notes not found in London, BM, MS add 13,387. It also omits severl found in the BM copy. The omissions are somewhat easier to explain on the basis of scribal error or copyist editing, but the added notes require further study.

dition to this matter of the date, the diction and style of the two *Taḥrīr* are different.[10] Consider, for example, the enunciation of the problem in proposition twelve of book I:

Ṭūsī:

> We wish to extend from a point to a line not bounded, it (the point) not being on it (the line), a perpendicular.[11]

Pseudo-Ṭūsī:

> There being a specified point in a specified plane in which lies a straight line not bounded at either end and the point not being of that specified line, we are required to extend from that point to the line a perpendicular.[12]

From this typical example, it appears that the Pseudo-Ṭūsī takes considerably greater pains to emphasize logical precision and generality of statement in his formulation.

Differences appear in other aspects of the texts as well. The Pseudo-Ṭūsī text, for example, is replete with references to previous propositions, emphasizing the deductive logical structure of geometry. Ṭūsī rarely mentions earlier results. On the one hand, the *Taḥrīr* of Ṭūsī is filled with comments, typically introduced by the phrase "I say" which is often employed by Arabic commentators to indicate when they are reporting their own views. There are very few such comments in the Pseudo-Ṭūsī text. On the other hand, the Pseudo-Ṭūsī *Taḥrīr* contains more lemmas and corollaries than does the genuine Ṭūsī text.

GENERAL CHARACTERISTICS

The genuine Ṭūsī *Taḥrīr* is based on the thirteen books of Euclid's *Elements*, along with the two discussions added by Hypsicles and generally designated as books XIV and XV. If we compare the text of

10. B. A. Rosenfeld, A. K. Kubesov and G. S. Sobirov, "Kto by avtorom rimskogo izdnia 'Izlozhenia Evklida Nasir ad-Din at-Tusi'?" *Voprosy istorii estestvoznaniya tekniki* (1966), 20: 51-3.

11. London, British Museum, MS add 13, 387, fol. 10a. (A)

In general I shall rely on this manuscript, since it is one of the earliest copies known (dated from 656 H.) The ink has faded, but the microfilm is generally legible. At several points, the text appears to be corrupt. I have then revised the readings from other manuscript sources.

12. *Kitāb Taḥrīr Uṣūl li-Uqlīdis min ta'līfKhwajah Naṣīr al-Dīn al-Ṭūsī* (Rome, 1594), p. 17. (B)

teenth century (Istanbul, 1801; Calcutta, 1824; Lucknow/Delhi, 1873-4; Teheran, 1881; Fez, 1293 H.), indicating a continuing popularity and interest in the treatise across the centuries.[4] The *Taḥrīr* was also translated into Persian at least twice. The first was made by Quṭb al-Dīn al-Shīrāzī(634/1236-710/1311), a pupil of Ṭūsī; the second by Khayr Allah ibn Luṭf Allah Muhandis in 1144/1731-2. A Persian edition of books I-VI was also published by lithograph in Calcutta in 1824.[5] Ṭūsī's text was also translated into Sanskrit by Jagannātha Samrāt (fl. *ca.* 1720-1740), who also translated his *Taḥrīr* on the *Almagest*.[6] A substantial number of manuscript copies of the *Rekhāgaṇita* survive, and an edition has been published.[7]

When we consider the obvious importance of this *Taḥrīr* over the centuries, it seems incredible that there should still exist almost no scholarly discussion of its content or appreciation for its place in the history of mathematical thought. This apparent neglect does not arise from ignorance, however. Rather, it results from a most unfortunate historical accident-the publication in 1594 of an Arabic *Taḥrīr* of the *Elements* bearing on its title page the name Naṣīr al-Dīn al-Ṭūsī.[8] Despite the ascription, however, the text is not by Ṭūsī. (Who the author might be is still a mystery - I have opted to refer to him as the Pseudo-Ṭūsī.) A manuscript of this *Taḥrīr*, Laurentian Lib. MS. Or. 50, states the date of completion as 698/1298 - a quarter century after Ṭūsī's death![9] In ad-

4. F. Sezgin, *Geschichte des arabischen Schriftums, Band V: Mathematik* (Leiden: Brill, 1974), p. 113; J. E. Murdoch, "Euclid: Transmission of the Elements," in C. C. Gillispie (ed), *Dictionary of Scientific Biography* (New York: Scribner, 1971/1980), vol. 4, p. 435.

5. Sezgin, *Mathematik*, p. 114; C. A. Storey, *Persian Literature: A Biobibliographical Survey* (London: Luzac, 1957), vol. 2, pt. 1, pp. 1-2; A. Rahman, *et al.*, *Science and Technology in Medieval India – A Bibliography of Source Materials in Sanskrit, Arabic, and Persian* (New Delhi: Indian National Science Council, 1982), p. 386. The relation of this Persian translation to the Calcutta printed edition of Ṭūsī's *Taḥrīr* mentioned earlier is not yet clear. Perhaps they are identical? I have not been able to see either the Arabic or the Persian.

6. D. Pingree, *Census of the Exact Sciences in Sanskrit* (Philadelphia: American Philosophical Society, 1976), ser. A, vol. 3, pp. 56-8.

7. H. Dhruva and K. Trivedan (eds), *Rekhāgaṇita* (Bombay: Bombay SS, 1901-2). Two volumes.

8. R. Cassinet, "L'Aventure de l'Édition des Elements d'Euclide en arabe par la Société Typographique Médici vers 1594," *Revue Française d'Histoire du Livre* (1993), 88-89: 5-51.

9. See A. I. Sabra, "Simplicius' Proof of Euclid's Parallels Postulate," *Journal of the Warburg and Courtauld Institutes* (1969), 32: 18.

was about thirty years of ago and continued with them (perhaps without the freedom to leave) until the Mongol, Hūlāgu, broke their power in 1256. Hūlāgu's interests in judicial astrology combined with Ṭūsı's reputation as a scholar and astronomer came together in the founding of a major astronomical observatory and teaching institute at Marāgha. Begun in 1259, this institution, under the leadership of Ṭūṣī, became one of the greatest centers for research into the mathematical sciences, attracting scholars from across the Islamic world and beyond. The institution survived Ṭūsī's death, continuing at a somewhat lower intensity of research under the direction of his son.[3]

Ṭūsī wrote on many topics, from philosophy and ethics to pure mathematics and mathematical astronomy. In the field of mathematics, his primary interest was geometry. He composed a number of innovative treatises and carried on mathematical correspondence with fellow mathematicians, but his primary claim to historical fame is the redactions he produced to make the work of the Greek mathematical masters more readily accessible to Arabic-speaking/reading students. In addition to *Taḥrīr* devoted to Euclid's *Elements* and to Ptolemy's *Almagest*, he composed recensions of important works by such Greek mathematicians as Theodosius, Hypsicles, Autolycus, Aristarchus, Archimedes, and Menelaus. These latter *Taḥrīr*, because they formed a bridge connecting the fundamental treatises of Euclid and Ptolemy, were collectively called the "Intermediate Texts" (*almutawassiṭāt*). Together these treatises formed a complete introductory corpus to the study of mathematical sciences. And, probably because of their textual coherence and because of Ṭūsī's role as head of the most comprehensive institute for the study of mathematical sciences, they quickly came to dominate the pedagogical scene.

The popularity of Ṭūsī's *Taḥrīr* of the *Elements* is attested in part by the very large number of surviving manuscripts scattered in libraries from Morocco to India as well as in the major manuscript collections of Europe and North America, many begin heavily annotated by past users. The text was printed by lithograph several times during the nine-

3. For an introduction to the work of the observatory at Marāgha, see A. Sayili, *The Observatory in Islam and its Place in the General History of the Observatory,* 2nd ed., (Ankara: Turk Tarih Kurumu Basimevi, 1988); G. Saliba, "The Astronomical Tradition of Maragha: A Historical Survey and Prospects for Future Research," *Arabic Sciences and Philosophy* (1991), 1: 67-99; G. De Young, "Maragha" in H. Selin (ed), *Encyclopaedia of the History of Science, Technology and Medicine in the Non-Western World* (Dordrecht: Kluwer, 1997), pp. 599-601.

The Taḥrīr of Euclid's Elements by Naṣīr al-Dīn al-Ṭūsī: redressing the balance

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Muḥammad ibn Muḥammad ibn al-Ḥasan Naṣīr al-Dīn al-Ṭūsī (597/1201-672/1274) was one of the most outstanding polymaths of Islamic intellectual history.[1] Born into an intellectual family, he began his studies with the religious sciences, which he learned from his father, a legal scholar in the Twelve-Imam school of Shīʿism in his native Ṭūs. Encouraged by his father to explore the intellectual sciences as well, he probably also began his acquaintance with logic, natural philosophy, and metaphysics, as well as algebra and geometry, during this early period. He chose to complete his formal education in Nishāpūr, where he studied philosophy, medicine, and mathematics, the latter with Kamāl al-Dīn ibn Yūnus (551/1156-639/1242).[2]

Ṭūsī entered the service of the Ismāʿīlī rulers of Khurasān when he

1. B. H. Siddiqi, "Naṣīr al-Dīn al-Ṭūsī," in M. M. Sharif (ed), *A History of Muslim Philosophy* (Wiesbaden: Harrassowitz, 1963), vol. 1, pp. 564-80; Muḥammad Mudarris Riḍawī, *Aḥwāl wa-āthār ... Naṣīr al-Dīn* (Teheran: Farhang Iran, 1976); F. Jamil Ragep, "Naṣīr al-Dīn al-Ṭūsī," in H. Selin (ed), *Encyclopaedia of the History of Science, Technology and Medicine in non-Western Cultures* (Dordrecht: Kluwer, 1997), pp. 757-8.

2. On Kamāl al-Dīn, see H. Suter, *Die Mathematiker und Astronomen der Araber und ihre Werke* (Leipzig: Teubner, 1900), pp. 140-2.

Farhang, vol. 15-16, no. 44-45, pp. 117-143