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> my creations are far better than those of others and yet I have not achieved complete satisfaction. For here it is true that si paullum a summo discessit, vergit ad imun (moving away from the mountain peak will lead to the abyss). I turned back when I saw that no man can reach the bottom of this night. I turned back unconsoled, pitying myself and all mankind. I admit that I expect little from the deviation of your lines. It seems to me that I have been in these regions; that I have traveled past all reefs of this infernal Dead Sea and have always come back with broken mast and torn sail. The ruin of my disposition and my fall date back to this time. I thoughtlessly risked life and happiness – aut Caesar aut nihil. (either Caesar or nothing).

Finally, after 2000 years of vain effort some mathematicians began to show signs of skepticism and to doubt that a proof of the Parallel Postulate was at all possible. The thought was suggested that it might be possible to construct a new type of logically consistent geometry that rests on secure foundations and is constructed by impeccable deductions.

In the course of the time, three great mathematicians began to develop independently from each other this new type of geometry.

The German mathematician Carl Friedrich Gauss (1777-1855) was the first to grasp the possibility of a geometry different from Euclid's. However, he did not publish any of his thoughts or work on this matter. He did coin the term "non-Euclidean geometry".

Unware of Gauss's findings two other mathematicians, the Hungarian Janos Bolyai (1802-1860) and the Russian Nikolai Ivanovich Lobachevski (1793-1856), were also engaged in searching for a geometry different from that of Euclid. Fortunately, they published their descriptions of the same type of geometry.

Khayyām's research on the Parallel Postulate has thus formed part of the history of a problem whose investigation finally led to the discovery of non-Euclidean geometry in the nineteenth century.

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After introducing the terminology "hypothesis of the acute, obtuse, and right angles", Saccheri attempted to eliminate the first two hypotheses, (case a) and (case b), by demonstrating that they would lead to contradictions. But his "proofs" were not widely accepted. He did, however, succeed in showing that the sum of the angles in a triangle did not exceed two right angles (case c).

Like Khayyām, Saccheri, too, stumbled upon some non-Euclidean theorems, but was unable to understand the significance of his discovery. He failed because of his pious belief that Euclid's geometry was the only true one.

Conclusion

Ever since Euclid publised his *Elements*, generations of mathematicians puzzled over his Parallel Postulate. For centuries they believed that this postulate could be deduced from the others. The persistence displayed by them was due to the fact that they viewed the unproved Parallel Postulate as a shocking flaw in geometry. Although all their attempts to eliminate this postulate or to deduce it from the other axioms, that is, to find a simpler substitute, had failed, no one ever doubted its truth, because it had never led to a siugle inconsistency or contradiction.

In 1763 the German mathematician Georg Simon Klügel (1739-1812) brought together all significant attempts made by European mathematicians to prove the Parallel Postulate. He listed 28 "proofs" and rightly concluded that the alleged solutions were all unsound.

So Euclid's fifth axiom became *the* scandal of geometry by the eighteenth century. The following excerpt from a letter that the Hungarian mathematician Wolfgang von Bolyai (1775-1856) wrote to his son Janos, illustrates the state of miud of a desperate great mathematician who had engaged himself with the Parallel Postulate for many years. He was horrified when he learned that his son was also attracted to this problem.

> It is unbelievable that this stubborn darkness, this eternal eclipse, this flaw in geometry, this eternal cloud on virgin truth can be endured. You must not attempt this approach to parallels. I know this way to its very end. I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of parallels alone... . I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind. I accomplished monstrous, enormous labors;

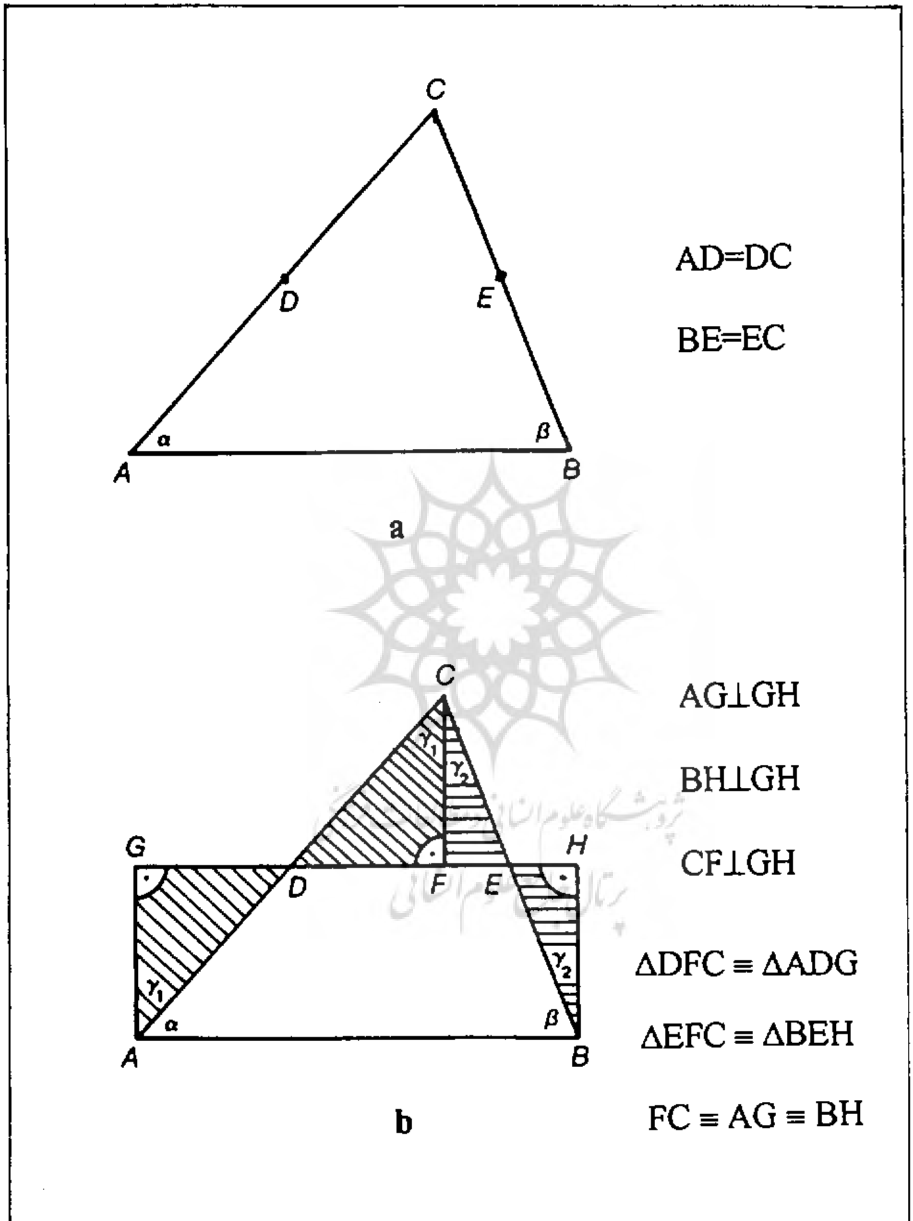


Fig. 8. Sacceri's proof of the Parallel Postulate

The Arabic version of Tusi's recension on Euclid's Elements was published in Rome in 1594. In 1613, it was then translated into Latin by Edward Pocock for the Oxford mathematician John Wallis (1616-1703). Through this publication the Italian Jesuit mathematician Gerolamo Saccheri (1667-1733) became acquainted with the proofs of the Parallel Postulate suggested by Muslim mathematicians. He discussed them in his important work entitled *Euclides ab omni naevo vindicatus* (Euclid Freed of All Blemish) which was published in 1733.

In his extensive investigations on the Parallel Postulate, Saccheri employed, like some Muslim mathematicians before him, the more sophisticated approach called reductio ad absurdum, that is, disproving a false proposition by logically deducing an absurd consequence. So he tried to establish the Parallel Postulate by denying it and seeking a consequent contradiction. Based on Fig. 8, his approach may be summarized as follows:

1. Let ABC be a triangle (Fig. 8a).

2. Let D and E be the midpoints of AC and BC.

3. Let F, G and H be the feet of the perpendiculars from C, A, B on the line DE prolonged in G and H (Fig. 8b).

4. It follows that AG, BH and FC are congruent, since the triangles $\triangle$DFC and $\triangle$ADG as well as $\triangle$ EFC and $\triangle$BEH are congruent.

5. The figure ABHG is a quadrilateral with right angles at G and H, that is, the angles $\angle$AGH and $\angle$ BHG are equal and each of them is equal to a right angle.

6. It follows that the angles $\angle$BAG and $\angle$ABH are congruent, that is, the angles $\angle(\alpha + \gamma_1)$ and $\angle(\beta + \gamma_2)$ are congruent.

7. Since $[\alpha + \beta + (\gamma_1 + \gamma_2)] = W$ is the sum of the angles in the triangle $\triangle$ABC, it follows that $\angle$BAG = $\angle$ABH = W/2.

8. Now the following three cases must be examined:

a) The sum of the angles in a triangle (W/2) is smaller than two right angles (W/2<2R, the hypothesis of the acute angle).

b) The sum of the angles in a triangle (W/2) is greater than two right angles (W/2>2R, the hypothesis of the obtuse angle).

c) The sum of the angles in a triangle (W/2) is equal to two right angles (W/2 = 2R, the hypothesis of the right angle).

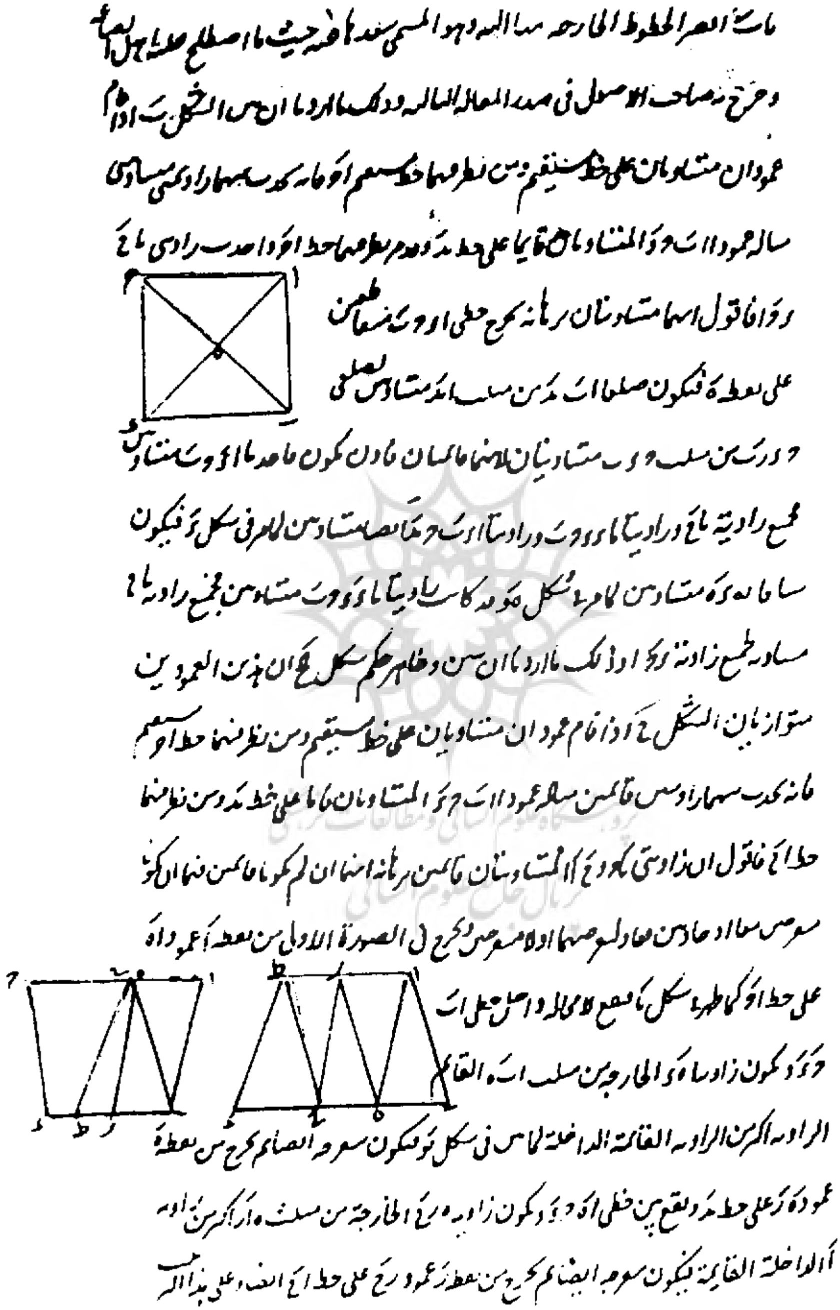


Fig. 7. One page of Tussi's treatise entitled *ar-Risala ash-Shafiya*

situations later known as the hypothesis of the acute angle (case a), that of the obtuse angle (case b) and that of right angle (case c). These three situations are now known to lead, respectively, to the non-Euclidean geometry of Bolyai-Lobacevskii, and to that of Rieman.[3]

Saccheri's Solution

Euclid was first made known in Europe through Latin translations of the Arabic versions of his mathematical work. The first complete Latin translation on *The Elements* was made about 1120 by the English scholastic philosopher Adelard of Bath (1090-1160), who obtained a copy of an Arabic version in Spain, where he traveled while disguised as a Muslim student. Soon after, Hermann von Kärnten (Hermanus Dalmata of Carinthia 1141-1193?) translated Books I-XII from the same Arabic version into Latin. Gerardo Cremona (1114-1187) prepared a complete Latin translation of the 15 books of Euclid based on the versions of Hunayn ibn Isḥāq and Thābīt ibn Qurra. The first Latin translation to be printed was made by Campanus da Novara (1210-1296) in 1255/59 and was published in Venice in 1482. The first direct translation from the Greek into Latin without the Arabic intermediary was prepared by Bartolomeo Zamberti and went into print in Vienna in 1505.

Although the problem of the Parallel Postulate was transmitted to Europe in the twelfth century through the Latin translations of Euclid's work, it did not provoke any keen interest among the European mathematicians until the seventeenth century, when they became acquainted with the research of Muslim mathematicians on this subject.

Among the most prominent mathematicians, who used to work intensively on the Parallel Postulate, was the Persian scholar and stateman Nasireddin Tuṣi. He had written his famous recension of *The Elements* into Arabic in the thirteenth century and carefully reviwed all the proofs that had been put forward by his fellow mathematicians in his famous treatise entitled *ar-Risala ash-Shafiya* (Fig. 7). He had quoted the method of approach suggested by Khayyām and had made explicit mention of the three cases a, b, and c discussed by him.

3. Struik, D.J.: "Omar Khayyam, mathematician", *The Mathematics Teachers*, 51 April 1958.

4. By prolonging EF its own length to G and drawing KH perpendicular to FG to interset AC and BD prolonged in K and H, the quadrilateral CDHK is constructed on ABCD.

5. By folding the figure about CD, it becomes obvious that both cases a and b must be rejected, since they contradict the previously stipulated principles. Hence, it must be concluded that only case c is true.

6. It follows that the angles ∠ ACD and ∠ BDC are equal and each of them is a right angle. Furthermore, it turns out that CD is equal to AB and, thus, the Parallel Postulate is established.

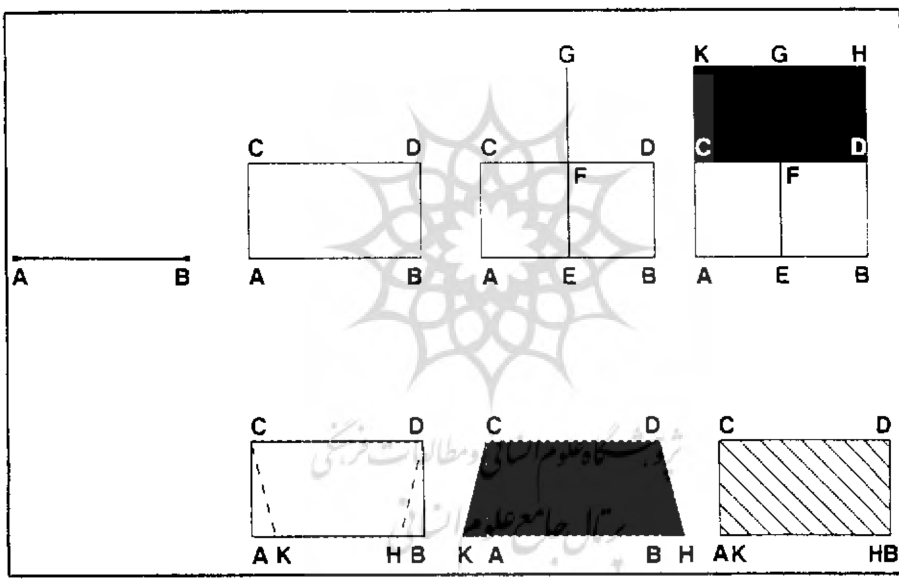
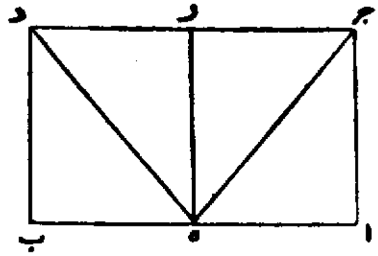


Fig. 6. Khayyām's proof of the Parallel Postulate

In the course of attempting to prove the Parallel Postulate, Khayyām was led to formulate and prove some non-Euclidean theorems. He did not, however, envisage the possibility of a non-Euclidean system of geometry. He failed to do so, because as a philosopher he believed that geometry was made up partly of intuition and partly of experience. He admired it for its beauty but regarded it as not altogether reliable.

The American mathematician D.J. Struik makes the following comment with regard to Khayyām's achievement:

We find here, apparently for the first time in history, the three



دیگر بار همان شکل (اب‌حـ د) را رسم می‌کنیم پس می‌گوییم که دو زاویهٔ (احـ د) و (ب‌د‌حـ) قائمه‌اند.
برهانش این است که خط (اب) را بر نقطهٔ (ه) به دو نیم تقسیم می‌کنیم و خط (ه‌ر) را عمود فرود می‌آوریم و او را راست امتداد می‌دهیم و خط (رک) را همانند (ره) قرار می‌دهیم و خط (حـ ک‌ط) را بر خط (ه‌ک) عمود می‌کنیم و خارج می‌کنیم دو خط (احـ) و...

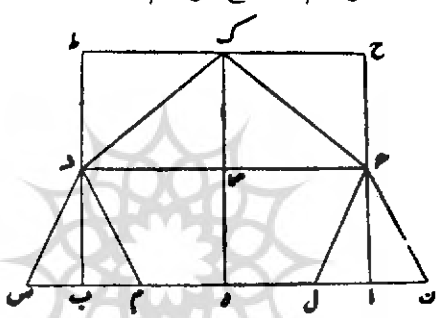


Fig. 5: Khayyām's proof of the Parallel Postulate
(→ Jalaleddin Homai 1967: pp. 236-238)

Based on his reasoning in this book, Khayyām's solution to the problem of Parallel Postulate be outlined as follows:

1. If two equal line segments AC and BD are drawn perpendicular to the line segment AB (Fig. 6) to construct the figure ABCD, then the angle ∠ ACD is equal to the angle ∠ BDC.

2. Now the following three cases must be examined:

a) The angles ∠ACD and ∠BDC are each less than a right angle ($< 90°$).

b) The angles ∠ACD and ∠BDC are each larger than a right angle ($> 90°$).

c) The angles ∠ACD and ∠BDC are each equal to right angle ($= 90°$).

3. To examine these cases let E be the midpoint of AB, and EF perpendicular to AB.

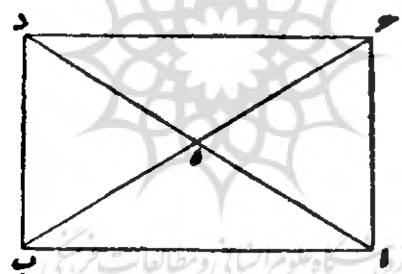
Now it is obvious that DF is equal to FC and EF is perpendicular to CD.

In order to prove the correctness of the Parallel Postulate, Khayyām, like many other Islamic mathematicians, adopted a definition of parallel lines as equidistant lines. Following this line of approach, he tried to connect the fifth axiom with the forth one to achieve his goal. To bridge the gap between the two axioms, he found that he had to introduce five principles and eight propositions into his line of reasoning.

Khayyām's approach towards proving the Parallel Postulate is elaborated in his book entitled *Commentaries on the Difficulties in the Postulates of Euclid's Elements.*[2] The original proof put forward by him in this book is outlined in Fig. 5.

شکل بیست و نهم از مقالهٔ اول اصول

خط (اب) فرض شده است و خط (اح) را بر (اب) عمود کنیم و خط (بد) را نیز بر خط (اب) عمود و مساوی خط (اح) اخراج کنیم؛ پس این دو خط چنانکه اقلیدس در شکل بیست و هفتم مقالهٔ اول بیان کرده است متوازیند؛ و خط (ح د) را وصل کنیم پس گوییم که زاویهٔ (اح د) مساوی زاویهٔ (بدح) است.



دیگر بار همان شکل (اب ح د) را رسم کنیم و خط (اب) را به دو نیم قسمت کنیم بر نقطهٔ (ه) و خط (ه ر) را بر (اب) عمود خارج کنیم پس گوییم که خط (ح ر) مثل خط (رد) است و خط (ه ر) عمود است بر خط (ح د).

2. The original Arabic text, from a Leiden manuscript, was published in Tehran in 1936 by Taghi Erani: *Discussion of Difficulties of Euclid by 'Omar Khayyām*, Edited with an Introduction by T. Erani. The Persian translation, from which Fig. 5 has been taken, was made by Jalaleddin Homai and published in Tehran in 1967.

There is an incomplete English translation of Khayyām's Discussions by Ali-Reza Amir Moéz:

Al-Khayyami Omar "Discussion of difficulties in Euclid" translated by Amir Moéz, *Scripta Mathematica*, 24 (1959), pp. 275-303.

In 1986 there appeared a French translation of the book devoted to the theory of parallels:

Jaouiche, K.: *La théorie des parallèles en pays d'Islam*, Paris 1986, pp. 185-199.

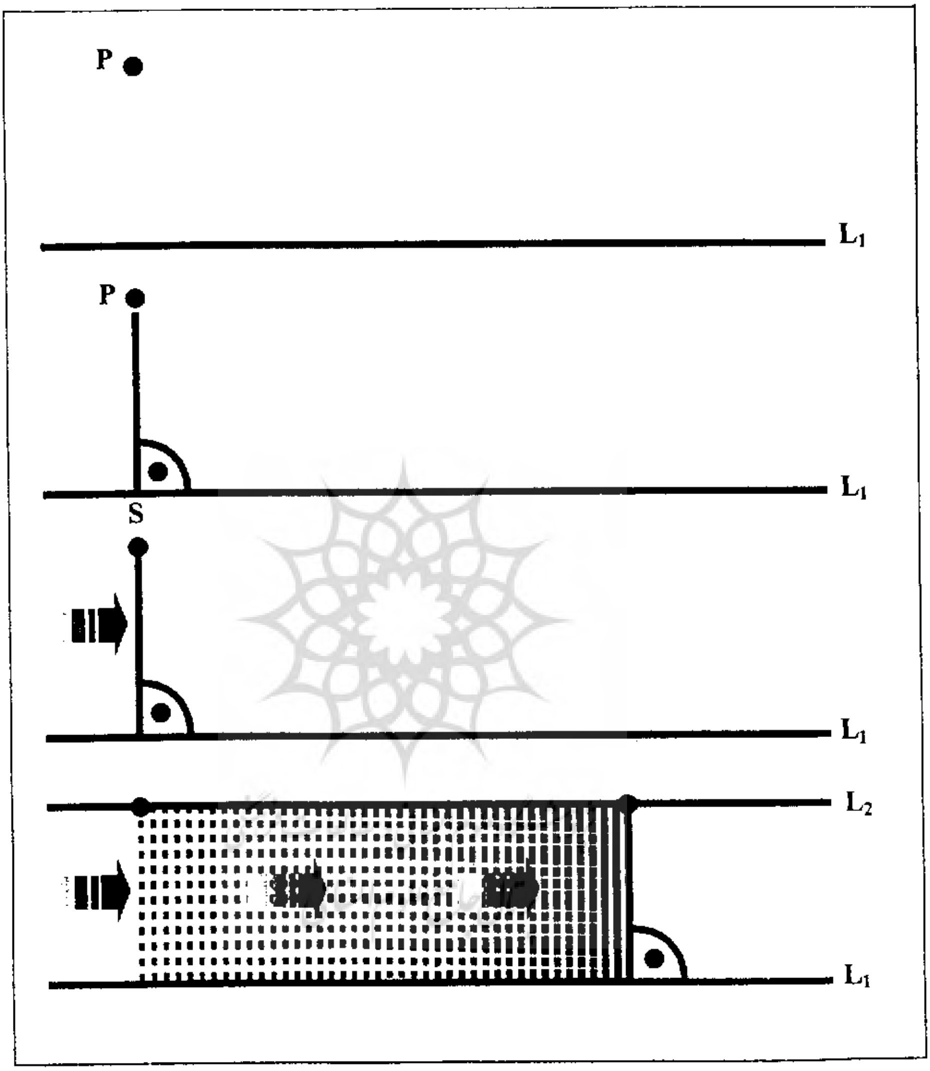


Fig. 4. Ibn al-Haytham's proof of the Parallel Postulate

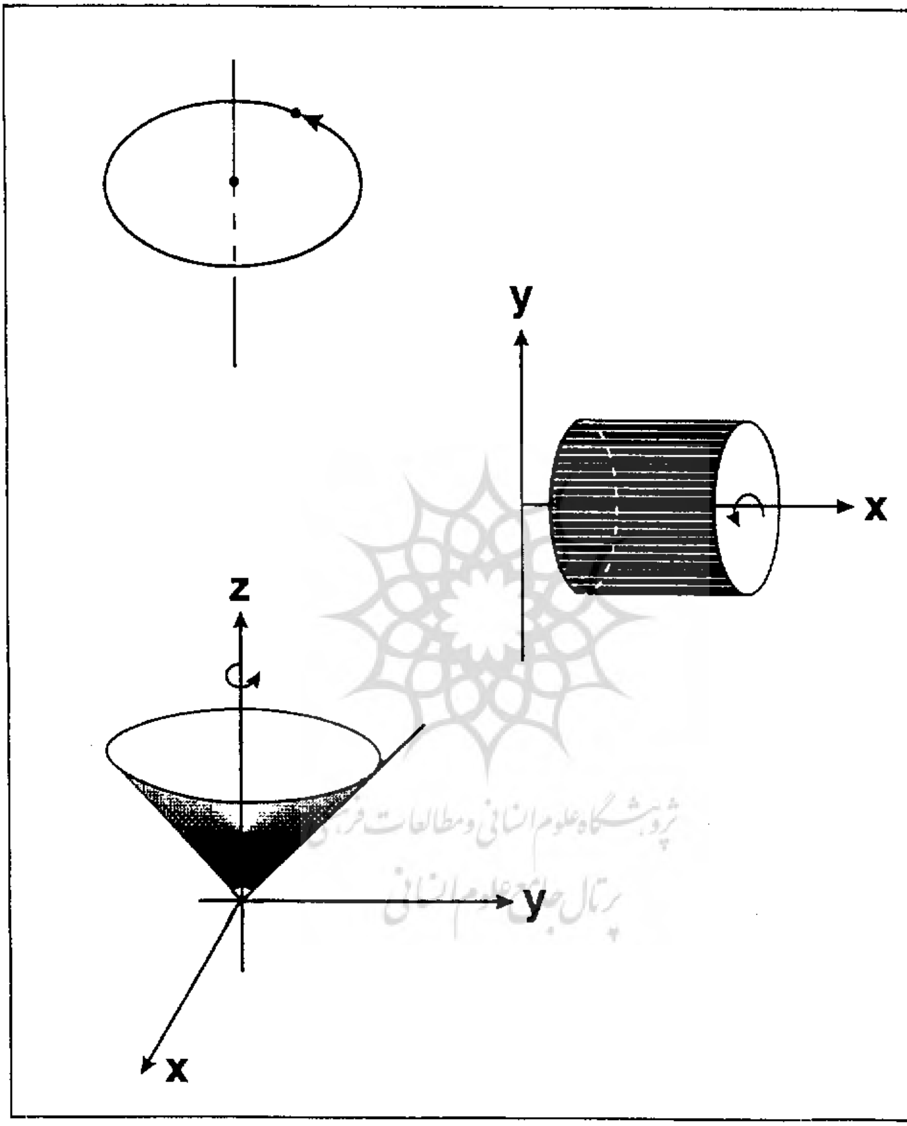


Fig. 3. Motion and Geometry

Haytham (965-1039), known generally as Alhazen, 'Omar Khayyām (1048-1131), Ibn Ṣalah al-Hamedāni (flourished about 1200), Athireddin Abhari (died 1265), Nasireddin Tuṣi (1201-1271), and Muhyaddin al-Maghribi (flourished about 1260).

All of them tried to prove the Parallel Postulate, that is, to derive it from other more obvious axioms, or to replace it by some principles which were less complicated. To this end, they employed different lines of approach. Some of them attempted to solve the problem employing the indirect method of proof. This mode of argument was expected to produce a contradiction. They argued that if the Parallel Postulate was false, then the sum of the angles in a triangle would turn out to be less than two right angles, an out-come that would come into conflict with the forth axiom. Others made use of the idea of motion, and in justification of their procedure, they argued that motion was in fact at the base of the whole of geometry. Fig. 3 shows the relationship between motion and geometry and demonstrates how different geometric figures and bodies can be constructed by rotating a point or a line around a center or an axis.

Ibn al-Haytham adopted a similar approach and succeeded in constructing one of the most interesting and striking solutions to the problem of Parallel Postulate. The quintessence of his proof is depicted in Fig. 4 and may be outlined as follows: Draw from the point P the line PS perpendicular to the line L_1. PS is obviously the shortest distance between P and L_1. Now start moving PS along the line L_1 in either direction, to the right or to the left. The point P will describe a line L_2 which is parallel to L_1 since the distance between P and L_1 remains unchanged throughout the movement of P.

Khayyām, however, objected to introducing the concept of motion into geometry. To his opinion, motion was an attribute of the matter and, therefore, should not be introduced into the immaterial world of mathematics. As a mathematician, he was intrigued with the purity and abstractness of geometry and strongly believed that fuzzy thinking, poor logic and unwarranted assumptions were errors that would have to be purged from mathematics.

clid's critics expressed their dissatisfaction strongly and explicity and were convinced that this axiom was unnecessary. Proclus (410-458), the last major Greek philosopher who was influential in disseminating of Neoplatonic ideas throughout the Byzantine, Islamic, and Roman worlds, made the following remark with regard to the Parallel Postulate:

> This postulate ought to be struck out of the postulates altogether; for it is a theorem involving many difficulties and it requires for the demonstration of it a number of definitions as well as theorems.[1]

Khayyām's Solution

Euclid's books survived in almost complete form because they were discovered by Muslim scholars of the Middle Ages and soon after translated into Arabic. Only centuries later they were translated from Arabic into Latin and then from these two languages into other European languages.

The translation of *The Elements* into Arabic was commissioned in Baghdad during the reign of the caliph al-Mamūn (813-833). The first Arabic translations were made by al-Hajjaj ibn Yusuf ibn Matar and Hunayn ibn Isḥaq. The work of the latter was later revised by Thabit ibn Qurrah.

Due to these translations, Muslim mathematicians became familiar with the Greek mathematics and geometry. And from now on, the systematic approach in mathematical thinking introduced by Euclid was to dominate the world of Islamic thinkers and philosophers. Nevertheless, Muslim mathematicians did make critical commentaries of *The Elements*, specifically, of the Parallel Postulate to which they devoted special attention. No one doubted its truth or the truth of any of the propositions derived by means of it. Muslim mathematicians, however, felt that this postulate could not be accepted without proof. Searching for a final solution to the problem became a great challenge to them. It is their credit that, having recognized the problem they inherited from their Greek predecessors, they persistently pusned its solution for more than five hundred years.

The mathematicians of medieval Islam who worked intensively on the Parallel Postulate were, to name only the most prominent ones, Mohammad ibn Isa al-Māhāni (flourished about 700), Abbas ibn Said al-Jawhari (flourished about 830), Thābīt ibn Qurra (836-901), al-Hassan ibn al-

1. Kline, M.: *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York, 1972, pp. 863-864.

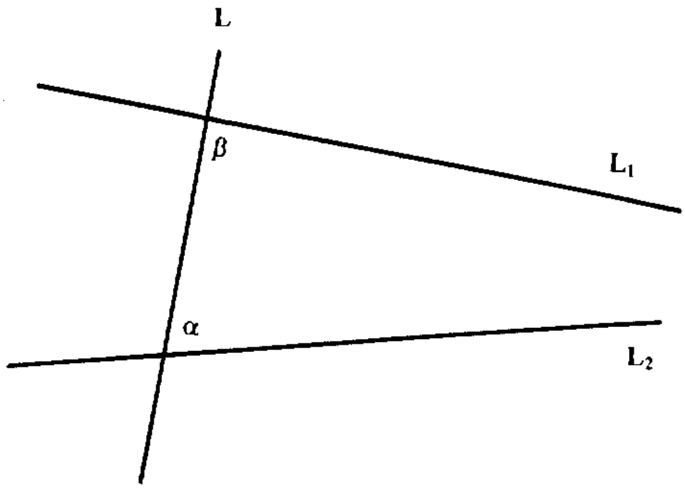


Fig. 1. Euclid's fifth axiom

Euclid's fifth axiom can be interpreted as if AB is any line and P is a point not on it, one line parallel to AB can be drawn through P.

The word parallel was defined by Euclid as follows:

> Parallel straight lines are lines, which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

The fifth axiom, in its original version, states in a rather complicated way that, as demonstrated in Fig. 2, **one and only one line** parallel to AB can be drawn through P. This is why this axiom is also often called Euclid's Parallel Postulate.

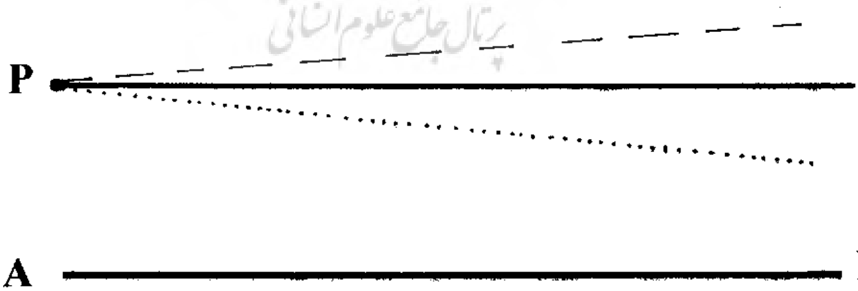


Fig. 2. Euclid's Parallel Postulate

From among all definitions and common assumptions put forward by Euclid, the Parallel Postulate provoked from the very beginning widespread interest among the mathematicians. It was widely attacked and criticized because all attempts to prove it resulted in failure. Eu-

In his *Elements* Euclid used two main approaches to present his mathematical thoughts: the synthetic approach in which he tried to proceed from the known to the unknown via logical steps, and the analytical approach in which he posited the unknown and treated it from the known, again via logical steps. Euclid's pioneering work was instrumental in developing a refined technique, called axiomatics, that has formalized the whole body of mathematics. Thus, he succeeded in designing a complete and consistent mathematical system.

The strength of Euclid's mathematical system lies not only in the small number of his definitions and assumptions, but also in the apparently self-evident truth of his fundamental propositions that have been admired through the history for their brevity and elegance. His propostitions are divided into *common notions* and *axioms.*

The common notions are five in number:

1. Things equal to the same thing are equal.
2. If equals are added to equals, the wholes are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. Things that coincide with one another are equal.
5. The whole is greater than a part.

As one can see, Euclid's common notions are assertions about magnitudes. The first three of them concern equals, the fourth is interpreted to mean that if two figures, such as intervals, angles, triangles, or circles, are such that one can be moved to coincide with the other, the figures are equal or congruent.

Euclid's axioms are also five in number:

1. Given two points there is an interval that joins them.
2. An interval can be prolonged indefinitely.
3. A circle can be constructed when its center, and a point on it, are given.
4. All right angles are equal.
5. If a straight line L falling on two straight lines L_1 and L_2 makes the interior angles α and β on the same side together less than two right angles ($\alpha + \beta < 180°$), the two straight lines, if produced indefinitely, meet on that side which the angles are together less than the two right angles.

The fifth axiom is demonstrated graphically in Fig. 1.

Introduction

During the third century BC, Alexandria of Egypt came to replace Athens as the focal point of classical mathematics. In Egypt of those days, as always in ancient Egypt, surveying for property lines was an important task. So the Greek mathematicians of Alexandria gave the name "geometry" to the Egyptians' empirical Methods of surveying. However, not being satisfied with the merely utilitarian aspects of the Egyptians' geometry, they tried to understand geometry for its own sake. The precise minds of the Greek thinkers sought rigorous deductive proofs of the general laws underlying the paractical applications of geometry.

Euclid and his Parallel Postulate

The Greek mathematician Euclid, who lived in Alexandria in the time of the first Ptolemy (323-285 BC), set out in 300 BC to accumulate all the knowledge in mathematics since the days of Tales of Milet (625-547 BC) and Pythagoras of Samos (570-480 BC) and their followers. He began to systematize the mathematical findings of his predecessors and succeeded in codifying the two and a half centuries of labor into one single work called *The Elements* (ta stoicheia). He drew together all the primary geometrical discoveries and drafted them into a systematic form. He was the first to attempt the construction of such a mathematical system that was to become an ever lasting edifice.

As a disciple of the Platonic school, Euclid was convinced that mathematics and geometry were purely mental and therefore perfectly rational and logical. Considering geometry as an abstract doctrine and a deductive theory, he realized that he first had to put forward certain definitions and a set of common assumptions without proof in order to systematize the existing geometrical knowledge. He knew that he could lay down any set of definitions and assumptions as long as they did not yield to contradictory results. His great accomplishment was to reduce the number of such fundamental assumptions to a minimum. At the same time, he managed to arrange them in such a manner that each mathematical theorem followed logically from its predecessors.

Euclid began his work with careful definitions of things such as point and line, describing a point as something with no part, a line as a length without breadth, and a line segment as one lying evenly between two points. He then laid down his common assumptions, before proceeding with the orderly arrangement of their consequences.

‘Omar Khayyām and the Parallel Postulate

Nasser Kanani
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The genesis of mathematical discovery is a problem which must inspire the psychologist with the keenest interest. For this is the process in which the human mind seems to borrow least from the exterior world, in which it acts, or appears to act, only by itself and on itself, so that by studying the process of geometric thought we may hope to arrive at what is most essential in the human mind.

Henri Poincaré
Science and Method

The present paper is solely involved with the description of the solution put forward by Omar Khayyām to the problem known as Euclid's Parallel Postulate. It does not concern itself with other scientific achievements of this great Iranian mathematician, philosopher, and poet that are highly remarkable in their own right. The paper consists of following parts: After a brief introduction describing the transfer of Greek mathematics to Egypt, mention is made of Euclid's life and work with special emphasis on his Parallel Postulate. Thereafter, the quest of Muslim mathematicians of the Middle Ages for a final solution to this problem will be outlined. Against this background Khayyām's proposals and views concerning the Parallel Postulate are discussed. Subsequently, the solution presented by the Italian scholar Saccheri will be mentioned to demonstrate, as a conclusion, that he in fact followed Khayyām's line of thought to proof the Parallel Postulate.

***Farhang*, vol. 12, no. 29-32, pp. 107-124**