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Estimation of Count Data using Bivariate Negative Binomial Regression Models

Pouya Faroughi[1], Mohammad Sharif Karimi[2], Noriszura Ismail[3] and Asrin Karimi[4]

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Abstract:
Negative binomial regression model (NBR) is a popular approach for modelling over-dispersed count data with covariates. Several parameterizations have been performed for NBR, and the two well-known models, negative binomial-1 regression model (NBR-1) and negative binomial-2 regression model (NBR-2), have been applied. Another parameterization of NBR is negative binomial-P regression model (NBR-P), which has an additional parameter and the ability to nest both NBR-1 and NBR-2. This paper introduces several forms of bivariate negative binomial regression model (BNBR) which can be fitted to bivariate count data with covariates. The main advantages of having several forms of BNBR are that they are nested and allow likelihood ratio test to be performed for choosing the best model, they have flexible forms of mean-variance relationship, they can be fitted to bivariate count data with positive, zero or negative correlations, and they allow over-dispersion of the two dependent variables. Applications of several forms of BNBR have been illustrated on two sets of count data; Australian health care and Malaysian motor insurance.

JEL classification**:** C35, C54, C60

Keywords: *Bivariate Poisson Regression Model, Bivariate Negative Binomial Regression Model, Correlation, Over Dispersion*

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[1] Assistant Professor, (Corresponding Author), Department of Statistics, Sanandaj Branch, Islamic Azad University, Sanandaj, Iran. Email: (pouya.faroughi@gmail.com)
[2] Assistant Professor, Department of Economics, Faculty of Social Sciences, Razi University, Kermanshah, Iran.
[3] Professor, School of Mathematical Sciences, Faculty of Science and Tchnology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor,Malaysia.
[4] Faculty of Economics and Management, University Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia.

1. Introduction

In statistics, count data is a type of data in which observations can take only the non-negative integer values, and these integers arise from counting, such as the number of occurrences of an event within a fixed period. In this situation, statistical modelling of count data has been used in many areas such as economics, insurance, biology, finance, environment, engineering and social sciences. Poisson distribution can be considered as a standard distribution for modelling count data. Poisson regression model is a standard model for fitting count data with covariates. However, to handle over-dispersed count data, a situation where the variance exceeds the mean, negative binomial regression model (NBR) has been used as an alternative. Several parameterizations have been performed for NBR, and the two well-known models are negative binomial-1 regression model (NBR-1) and negative binomial-2 regression model (NBR-2) (Cameron and Trivedi 1986; Cameron and Trivedi 2013; Hilbe 2011; Lawless 1987; Winkelmann 2008). Negative binomial-P regression model (NBR-P), which has an additional parameter and the ability to nest both NBR-1 and NBR-2, is another parameterization of NBR (Greene 2008; Zulkifli et al. 2013). Besides NBR, the generalized Poisson regression model (GPR) has also been suggested for handling under- or over-dispersed count data (Zamani and Ismail 2012; Karimi et al. 2015).

Bivariate count data can be fitted using bivariate models such as bivariate Poisson distribution (BPD) (Campbell 1934). BPD described in Johnson et al. (1997) is based on a trivariate reduction method, a method which only allows positive correlation. Several applications of bivariate Poisson regression model (BPR) based on the trivariate reduction method can be found in Gourieroux et al. (1984), King (1989), Jung and Winkelmann (1993) and Kocherlakota and Kocherlakota (2001). The limitations of bivariate models from the trivariate reduction method can be found in several studies in which a few models were suggested, such as modelling dependence through correlated random effects (Berkhout and Plug 2004), fitting bivariate data using bivariate generalized negative binomial regression model (Gurmu and Elder 2000), modelling bivariate data with bivariate negative binomial distribution (BNBD) that allows a restricted range of negative correlations (Mitchell and Paulson 1981), and modelling dependence through copula functions (Cameron et al. 2004; Lee 1999; Ophem 1999).

BPD which handles negative, zero or positive correlations was introduced by Lakshminarayana et al. (1999) who defined the distribution from the product of Poisson marginals with a multiplicative factor parameter. Using the same method, bivariate negative binomial regression model (BNBR) was introduced in Famoye (2010). The BNB distribution has an unrestricted correlation structure. Furthermore, the BNB distribution is superior to the BP distribution since the BNB distribution can be applied to describe bivariate count data that exhibits over-dispersion. Famoye (2010) also defined a BNB regression model based on the new BNB distribution. Famoye (2012) compared some bivariate regression models including the BNB regression and bivariate Poisson log-normal (BPL) regression models.

Based on literatures, this paper introduces several forms of BNBR models which can be fitted to bivariate and correlated count data with covariates. The main advantages of having several forms of BNB regression are that they are nested and allow likelihood ratio test to be performed for choosing the best model, they have flexible forms of mean-variance relationship, they can be fitted to bivariate count data with positive, zero or negative correlations, and they allow over-dispersion of the two dependent variables.

The rest of this paper is organized as follows. Section 2 discusses the problem statement and research objectives. Section 3 proposes the joint p.m.f. of BPR. Several forms of BNBR are proposed in section 4, while section 5 discusses several tests for testing over-dispersion, for testing independence, and for choosing the best model. Numerical illustrations are provided in section 6 where several forms of BNBR are fitted to two sets of data, each with a negative and a positive correlation. The two datasets are the Australian health survey data (Cameron et al. 1988) and the Malaysian motor insurance claims data.

2. Problem Statement

Poisson distribution is characterized by a parameter that its mean is equal to its variance. As the mean and variance of Poisson distribution are equal, we say that the distribution satisfies equi-dispersion property. This property is often violated in real-life count data. We have over-dispersion (under-dispersion) when the variance is greater (less) than the mean. When the principle of complete randomness fails (that is, when the data is either over or under-dispersed), it is important to use a probability model that can handle the situation.

Mixed distribution can be considered as one of the important approaches to obtain a new distribution for count data in statistics and probability studies. In particular, mixed Poisson and mixed negative binomial distributions provide a more flexible alternative for modelling over-dispersed count data compared to Poisson distribution. Examples of mixed Poisson and mixed negative binomial distributions are negative binomial which is a mixture of Poisson and gamma, negative binomial-Pareto, Poisson-inverse Gaussian, Poisson weighted exponential and Poisson-Lindley.

For handling over-dispersed count data, a situation in which the variance exceeds the mean, negative binomial (NB) regression has been used. Several parameterizations have been performed for NB regression, and the two well-known models are NB-1 and NB-2. NB-P regression, which has an additional parameter and has the advantage of nesting both NB-1 and NB-2 regressions, is another parameterization of NB regression. GP distribution is obtained from the limiting form of a generalized NB distribution. Based on literatures, different forms of GP regressions have been proposed using different parameterization of GP regression. The classical GP regression which is also known as GP-1 regression, The GP-2 regression which is another parameterization of GP regression, The GP-P regression which has an additional parameter and has the advantage of nesting both

GP-1 and GP-2 regressions, are other parameterizations of GP regression.

If we have bivariate or multivariate count data, the bivariate or multivariate models can be fitted. There are many forms of bivariate and multivariate discrete distribution, such as bivariate Poisson (BP) distribution and multivariate Poisson (MP) distribution. This study extends the Poisson-weighted exponential (P-WE) distribution from the univariate to the bivariate case. There are several forms of bivariate discrete distributions based on the method of trivariate reduction. The disadvantage of using this method is that it only admits positive correlation. Based on literatures, the BP distribution, BGP distribution based on GP-1 and BNB based on NB-2 have been proposed which BP, BGP and BNB distributions allow negative, zero or positive correlations can be derived from the product of Poisson, GP-1 and NB-2 marginals with a multiplicative factor parameter.

The main objectives of this research are to define several forms of bivariate negative binomial (BNB) regression namely BNB-1, BNB-2 and BNB-P, which can be fitted to bivariate count data with covariates. The main advantage of having several forms of BNB regression is that they are nested and allow the likelihood ratio test to be performed to choose the best model. The other advantage is that BNB regressions discussed in this study can be fitted to bivariate and over-dispersed count data with flexible correlations.

3. Bivariate Poisson Regression Model (BPR)

Lakshminarayana et al. (1999) defined a BPD that allows the correlation structure to be positive, negative or zero. The joint p.m.f. was derived from the product of Poisson marginals with a multiplicative factor parameter

$$\Pr(y_1, y_2) = e^{-\mu_1-\mu_2} \frac{\mu_1^{y_1}\mu_2^{y_2}}{y_1! y_2!} \{1+\phi[(g_1(y_1)-\bar{g}_1)(g_2(y_2)-\bar{g}_2)]\} \quad (1)$$

$$y_1, y_2 = 0,1,2,..., \quad \mu_1, \mu_2 > 0$$

Where ϕ is a multiplicative factor (or correlation) parameter, and $g_1(y_1)$ and $g_2(y_2)$ are bounded functions in y_1 and y_2 respectively. Non-negativity of $\{.\}$ in p.m.f. (1) is ensured by defining

$$g_t(y_t) = e^{-y_t} \text{ and } \bar{g}_t = E[g_t(Y_t)] = E(e^{-Y_t}), \quad t = 1,2. \qquad (2)$$

Suppose Y_{i1} and Y_{i2} $(i = 1, 2, \dots n)$ are count response variables. Following (1)-(2), the joint p.m.f. of BPR is

$$\Pr(y_{i1}, y_{i2}) = \prod_{t=1}^{2} e^{-\mu_{it}} \frac{\mu_{it}^{y_{it}}}{y_{it}!}[1 + \phi \prod_{t=1}^{2}(e^{-y_{it}} - e^{-d\mu_{it}})],$$
$$i = 1,2,\dots,n, \quad \mu_{i1}, \mu_{i2} > 0$$

Where $d = 1 - e^{-1}$ and ϕ is the correlation parameter. The marginal means, marginal variances and covariance are $E(Y_{it}) = Var(Y_{it}) = \mu_{it}$, $t = 1,2$ and $Cov(Y_{i1}, Y_{i2}) = \phi \mu_{i1} \mu_{i2} d^2 e^{-d(\mu_{i1} + \mu_{i2})}$. When $\phi = 0$, random variables Y_{i1} and Y_{i2} are independent, each is distributed as a marginal Poisson regression model. When $\phi > 0$ and $\phi < 0$, we have positive and negative correlations respectively. The covariates can be incorporated using log link functions

$$E(Y_{i1}) = \mu_{i1} = \exp(\mathbf{x}_i^T \boldsymbol{\beta}) \text{ and } E(Y_{i2}) = \mu_{i2} = \exp(\mathbf{x}_i^T \boldsymbol{\gamma}) \qquad (3)$$

Where $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are the regression parameters for y_{i1} and y_{i2} respectively, and $\mathbf{x}_i$ is the vector of covariates.

4. Bivariate Negative Binomial Regression Model (BNBR)

The p.m.f. of univariate NBR is

$$\Pr(y_i) = \frac{\Gamma(y_i + v_i)}{y_i!\Gamma(v_i)} \left(\frac{v_i}{v_i + \mu_i} \right)^{v_i} \left(\frac{\mu_i}{v_i + \mu_i} \right)^{y_i}, \quad y_i = 0,1,2,...,n \qquad (4)$$

Where $v_i^{-1} = a$ is the dispersion parameter. The mean and variance of NBR are $E(Y_i) = \mu_i$ and $Var(Y_i) = \mu_i(1 + v_i^{-1}\mu_i) = \mu_i(1 + a\mu_i)$. NBR in p.m.f. (4) is also referred to as NBR-2. NBR-2 reduces to Poisson regression model in the limit as $a \to 0$, and displays over-dispersion when $a > 0$.

If we replace $v_i = a^{-1}\mu_i^{2-P}$ in p.m.f. (4), univariate NBR-P is obtained. The p.m.f. of NBR-P is (Cameron and Trivedi 2013; Ridout et al. 2001; Greene 2008; Zamani and Ismail 2013)

$$\Pr(y_i) = \frac{\Gamma(y_i + a^{-1}\mu_i^{2-P})}{y_i!\Gamma(a^{-1}\mu_i^{2-P})} \left(\frac{a^{-1}\mu_i^{2-P}}{a^{-1}\mu_i^{2-P} + \mu_i} \right)^{a^{-1}\mu_i^{2-P}} \left(\frac{\mu_i}{a^{-1}\mu_i^{2-P} + \mu_i} \right)^{y_i}, \qquad (5)$$

$$y_i = 0,1,2,...,n$$

Where a is the dispersion parameter and P is the functional parameter. NBR-P is a flexible model that nests NBR-1 and NBR-2 by including an additional parameter, P (functional parameter). The mean and variance of NBR-P are $E(Y_i) = \mu_i$ and $V(Y_i) = \mu_i(1 + a\mu_i^{P-1})$. NBR-P reduces to NBR-1 and NBR-2 when $P = 1$ and $P = 2$ respectively, reduces to Poisson regression model in the limit when $a \to 0$, and allows over-dispersion when $a > 0$.

Using the same approach suggested by Lakshminarayana et al. (1999), bivariate negative binomial-P regression model (BNBR-P) can be derived from the product of two NBR-P marginals and a mulitiplicative factor parameter. The p.m.f. of BNBR-P is

$$\Pr(y_{i1},y_{i2})=\left[\begin{array}{l}\prod_{t=1}^{2}\frac{\Gamma(y_{it}+a_t^{-1}\mu_{it}^{2-P})}{y_i!\Gamma(a_t^{-1}\mu_{it}^{2-P})}\left(\frac{\mu_{it}}{a_t^{-1}\mu_{it}^{2-P}+\mu_{it}}\right)^{y_{it}}\\ \left(\frac{a_t^{-1}\mu_{it}^{2-P}}{a_t^{-1}\mu_{it}^{2-P}+\mu_{it}}\right)^{a_t^{-1}\mu_{it}^{2-P}}\end{array}\right] \times\left[1+\phi(e^{-y_{i1}}-c_{i1})(e^{-y_{i2}}-c_{i2})\right] \quad (6)$$

where a_t, $t=1,2$, are the dispersion parameters, P_t, $t=1,2$, are the functional parameters, ϕ is the multiplicative factor (correlation) parameter, and $c_{it}=E(e^{-Y_{it}})=\left(\frac{a_t^{-1}\mu_{it}^{2-P_t}}{a_t^{-1}\mu_{it}^{2-P_t}-e^{-1}+1}\right)^{a_t^{-1}\mu_{it}^{2-P_t}}$, $t=1,2$. The joint p.m.f. of bivariate negative binomial-2 regression model (BNBR-2), where $P_1=P_2=2$, is already defined in Faroughi and Ismail (2014).

The marginal means and variances for the new BNBR-P are $E(Y_{it})=\mu_{it}$, $t=1,2$, and $Var(Y_{it})=\mu_{it}(1+a_t\mu_{it}^{P_t-1})$, $t=1,2$. The covariance is $Cov(Y_{i1},Y_{i2})=\phi c_{i1}c_{i2}D_{i1}D_{i2}$, where $D_{it}=\frac{a_t^{-1}\mu_{it}^{2-P_t}e^{-1}}{a_t^{-1}\mu_{it}^{1-P_t}-e^{-1}+1}-\mu_{it}$, $t=1,2$. From the covariance, it can be seen that Y_{i1} and Y_{i2} are independent if $\phi=0$. When $\phi<0$, the correlation between response variables is negative and when $\phi>0$, the correlation is positive. BNBR-P reduces to bivariate negative binomial-1 regression model (BNBR-1) and BNBR-2 when $P_1=P_2=1$ and $P_1=P_2=2$ respectively, reduces to BPR where $a_t\to 0$, $t=1,2$, and allows over-dispersion when $a_t>0$, $t=1,2$.

It is interesting to see that BNBR-P allows the two dependent variables to have flexible forms of mean-variance relationship. The mean-variance relationship for BPR is equal where $Var(Y_{it})=\mu_{it}$, $t=1,2$, for BNBR-1 it is linear where $Var(Y_{it})=\mu_{it}(1+a_t)$, $t=1,2$, for BNBR-2 it is quadratic where $Var(Y_{it}) = \mu_{it}(1+a_t\mu_{it})$, $t=1,2$, and for BNBR-P it is to the *P*-th power where $Var(Y_{it}) = \mu_{it}(1+a_t\mu_{it}^{P_t-1})$, $t=1,2$.

Table 1 provides the joint p.m.f., marginal means, marginal variances and covariance for BPR, BNBR-1, BNBR-2 and BNBR-P.

5. Tests

Test for Independence

Response variables Y_{i1} and Y_{i2} are independent when the multiplicative factor parameter, ϕ, is zero. Therefore, we can use likelihood ratio test (LRT) for testing independence where the hypothesis are $H_0:\phi=0$ against $H_1:\phi\neq 0$. The LRT is

$$T = -2(\ln L_0 - \ln L_1) \tag{7}$$

Where L_0 and L_1 are the likelihood functions when H_0 and H_1 are true respectively. The equation (7) is asymptotically distributed as chi-square with one degree of freedom.

Test for Over dispersion

LRT in (7) can also be performed to test over dispersion in BPR against BNBR-1 (or BNBR-2) where the null hypothesis is $H_0: a_1 = a_2 = 0$. Since BNBR-1 and BNBR-2 reduce to BPR in the limit when $a \to 0$, the null hypothesis is on the boundary of parameter space. We can use the results of Chernoff (1954), Self and Liang (1987) and Famoye (2010), where the statistic is asymptotically distributed as 0.25 of probability mass at zero, 0.5 of chi-square with one degree of freedom and 0.25 of chi-square with two degrees of freedom.

Test for BNBR-1 (or BNBR-2) against BNBR-P

We can also use the LRT in (7) for testing BNBR-1 (or BNBR-2) against BNBR-P where $H_0: P_1 = P_2 = 1$ (or $H_0: P_1 = P_2 = 2$). The equation (7) is asymptotically distributed as a chi-square with two degrees of freedom.

Table 1: Joint p.m.f., marginal means, marginal variances and covariance for BPR, BNBR-1, BNBR-2 and BNBR-P.

Model	Joint p.m.f	Marginal means, marginal variances and covariance
BPR	$d=1-e^{-1}$ $\Pr(y_{i1},y_{i2})=\left[\prod_{t=1}^{2} e^{-\mu_{it}}\frac{\mu_{it}^{y_{it}}}{y_{it}!}\right]\left[1+\phi\prod_{t=1}^{2}(e^{-y_{it}}-e^{-d\mu_{it}})\right]$	$E(Y_{it})=Var(Y_{it})=\mu_{it}$ $Cov(Y_{i1},Y_{i2})=\phi\mu_{i1}\mu_{i2}d^{2}e^{-d(\mu_{i1}+\mu_{i2})}$,
BNBR-1	$\Pr(y_{i1},y_{i2})=\left[\prod_{t=1}^{2}\frac{\Gamma(y_{it}+a_t^{-1}\mu_{it})}{y_i!\,\Gamma(a_t^{-1}\mu_{it})}\left(\frac{\mu_{it}}{a_t^{-1}\mu_{it}+\mu_{it}}\right)^{y_{it}}\left(\frac{a_t^{-1}\mu_{it}}{a_t^{-1}\mu_{it}+\mu_{it}}\right)^{a_t^{-1}\mu_{it}}\right]\left[1+\phi\prod_{t=1}^{2}(e^{-y_{it}}-c_{it})\right]$ $c_{it}=\left(\frac{a_t^{-1}\mu_{it}}{a_t^{-1}\mu_{it}-e^{-1}+1}\right)^{a_t^{-1}\mu_{it}}$	$E(Y_{it})=\mu_{it}$ $Var(Y_{it})=\mu_{it}(1+a_t)$ $Cov(Y_{i1},Y_{i2})=\phi c_{i1}c_{i2}D_{i1}D_{i2}$ $D_{it}=\frac{a_t^{-1}\mu_{it}e^{-1}}{a_t^{-1}-e^{-1}+1}-\mu_{it}$
BNBR-2	$\Pr(y_{i1},y_{i2})=\left[\prod_{t=1}^{2}\frac{\Gamma(y_{it}+a_t^{-1})}{y_i!\,\Gamma(a_t^{-1})}\left(\frac{\mu_{it}}{a_t^{-1}+\mu_{it}}\right)^{y_{it}}\left(\frac{a_t^{-1}}{a_t^{-1}+\mu_{it}}\right)^{a_t^{-1}}\right]\left[1+\phi\prod_{t=1}^{2}(e^{-y_{it}}-c_{it})\right]$ $c_{it}=\left(\frac{a_t^{-1}}{a_t^{-1}-e^{-1}+1}\right)^{a_t^{-1}}$	$E(Y_{it})=\mu_{it}$ $Var(Y_{it})=\mu_{it}(1+a_t\mu_{it})$ $Cov(Y_{i1},Y_{i2})=\phi c_{i1}c_{i2}D_{i1}D_{i2}$ $D_{it}=\frac{a_t^{-1}e^{-1}}{a_t^{-1}\mu_{it}^{-1}-e^{-1}+1}-\mu_{it}$

BNBR-P

$$\Pr(y_{i1}, y_{i2}) = \left[\prod_{t=1}^{2} \frac{\Gamma(y_{it} + a_t^{-1}\mu_{it}^{2-P_t})}{y_i!\,\Gamma(a_t^{-1}\mu_{it}^{2-P_t})} \left(\frac{\mu_{it}}{a_t^{-1}\mu_{it}^{2-P_t} + \mu_{it}} \right)^{y_{it}} \left(\frac{a_t^{-1}\mu_{it}^{2-P_t}}{a_t^{-1}\mu_{it}^{2-P_t} + \mu_{it}} \right)^{a_t^{-1}\mu_{it}^{2-P_t}} \right] \left[1 + \phi \prod_{t=1}^{2} (e^{-y_{it}} - c_{it}) \right]$$

$$c_{it} = \left(\frac{a_t^{-1}\mu_{it}^{2-P_t}}{a_t^{-1}\mu_{it}^{2-P_t} - e^{-1} + 1} \right)^{a_t^{-1}\mu_{it}^{2-P_t}}$$

$$E(Y_{it}) = \mu_{it}$$

$$Var(Y_{it}) = \mu_{it}(1 + a_t \mu_{it}^{P_t - 1})$$

$$Cov(Y_{i1}, Y_{i2}) = \phi c_{i1} c_{i2} D_{i1} D_{i2}$$

$$D_{it} = \frac{a_t^{-1}\mu_{it}^{2-P_t} e^{-1}}{a_t^{-1}\mu_{it}^{1-P_t} - e^{-1} + 1} - \mu_{it}$$

Wald test

The test of over-dispersion in BPR against BNBR alternatives (BNBR-1 or BNBR-2), $H_0: a_t = 0,\ t = 1,2$, can also be performed using Wald test, $\frac{\hat{a}_t^{\,2}}{Var(\hat{a}_t)}$, $t = 1,2$, where $\hat{a}_t$, $t = 1,2$, is the estimated dispersion parameter. Since BNBR-1 (or BNBR-2) reduces to BPR in the limit when $a \to 0$, the null hypothesis is on the boundary of parameter space. The Wald statistics is asymptotically distributed as a mixture of 0.5 of probability mass at zero and 0.5 of chi-square with one degree of freedom (Lawless 1987).

The independence of response variables Y_{i1} and Y_{i2} can also be tested using Wald test, $\frac{\hat{\phi}^2}{Var(\hat{\phi})}$, where $\hat{\phi}$ is the estimated multiplicative factor (or correlation) parameter. The statistics is asymptotically distributed as a chi-square with one degree of freedom.

For testing the adequacy of BNBR-1 against BNBR-P, $H_0: P_t = 1,\ t = 1,2$, the Wald test, $\frac{(\hat{P}_t - 1)^2}{Var(\hat{P}_t)}$, $t = 1,2$, where $\hat{P}_t$ is the estimated functional parameter, can be used. For testing the adequacy of BNBR-2 against BNBR-P, $H_0 : P_t = 2,\ t = 1,2$, the Wald test is $\frac{(\hat{P}_t - 2)^2}{Var(\hat{P}_t)}$. Both statistics asymptotically follow a chi-square distribution with one degree of freedom.

In terms of preference between LRT and Wald test, the LRT may be better used for the bivariate data. The hypothesis for testing over-dispersion under LRT is $H_0 : a_1 = a_2 = 0$ compared to $H_0 : a_t = 0,\ t = 1,2$, under Wald test. For testing adequacy of BNBR-1 (or BNBR-2) against BNBR-P, the hypothesis is $H_0 : P_1 = P_2 = 1$ (or $H_0 : P_1 = P_2 = 2$) under LRT, compared to $H_0 : P_t = 1$ (or $H_0 : P_t = 2$), $t = 1,2$, under Wald test.

AIC

Akaike Information Criteria (AIC) is defined as $AIC = 2k - 2\ln(L)$, where k is the number of parameters and $\ln(L)$ is the log likelihood for the estimated model. The model with the smallest AIC is the best model. The log-likelihood for BNBR-P is

$$\begin{aligned}\log L = \sum_{i=1}^{n}\Bigg\{\sum_{t=1}^{2}\Big[& y_{it}\log\mu_{it} + a_t^{-1}\mu_{it}^{2-P_t}\log(a_t^{-1}\mu_{it}^{2-P_t}) \\ & -(y_{it} + a_t^{-1}\mu_{it}^{2-P_t})\log(\mu_{it} + a_t^{-1}\mu_{it}^{2-P_t}) \\ & -\log(y_{it}!) + \sum_{j=0}^{y_{it}-1}\log(a_t^{-1}\mu_{it}^{2-P_t} + j)\Big] \\ & + \log\Big[1 + \phi(e^{-y_{i1}} - c_{i1})(e^{-y_{i2}} - c_{i2})\Big]\Bigg\}\end{aligned} \quad (8)$$

Where $c_{it} = \left(\frac{a_t^{-1}\mu_{it}^{2-P_t}}{a_t^{-1}\mu_{it}^{2-P_t} - e^{-1} + 1}\right)^{a_t^{-1}\mu_{it}^{2-P_t}}$, $t = 1,2$. Log likelihood in (8) can also be used for BNBR-1 and BNBR-2, by replacing $P_1 = P_2 = 1$ for BNBR-1 and $P_1 = P_2 = 2$ for BNBR-2.

6. Applications

Applications of several forms of BNBR are illustrated on two sets of count data with negative and positive correlations; the Australian health care data and the Malaysian motor insurance data. Australian health care and Malaysian motor insurance data are the most useful data sets which are used in this kind of studies. Another point is that the correlation coefficient between two response variables is negative in the Australian health care data and Malaysian motor insurance data has a positive correlation coefficient between two response variables. These examples imply that the new models have flexible structures. Several tests are also applied to choose the best model. Finally, the test of independence is performed to indicate whether the data should be fitted jointly under bivariate regression model or independently under univariate regression model.

Australian Health Data (1977-1978)

We consider the Australian health survey data (Cameron et al. 1988) for fitting BPR and several forms of BNBR. The same data was also used by Cameron and Johansson (1997) for fitting several univariate models, by Gurmu and Elder (2000) who fitted bivariate generalized negative binomial regression model and by Famoye (2010) who fitted BNBR-2.

We consider two possibly dependent and negatively correlated response variables namely Y_1, which is the total number of prescribed medications used in the past two days (PRESCRIBED), and Y_2, which is the total number of non-prescribed medications used in the past two days (NON-PRESCRIBED). The mean and standard deviation for prescribed medications are 0.863 and 1.415respectively, the mean and standard deviation for non-prescribed medications are 0.356 and 0.712 respectively, and the correlation between y_{i1} and y_{i2} is -0.043. The negative correlation indicates possible negative dependency between the two response variables. The regressors are:

(1) Socio-economic variables: an indicator variable for whether female (SEX), age in years (AGE), age-squared (AGESQ), annual income in hundreds of dollars (INCOME).

(2) Health insurance status indicator variables: private insurance cover (LEVYPLUS), free government insurance cover due to low income (FREEPOOR) and free government cover due to old age, disability or veteran status (FREEREPA). The omitted category is the default government Medibank insurance cover paid for by an income levy (LEVY).

(3) Recent health-status measures: The number of illnesses in the past two weeks (ILLNESS) and the number of days of reduced activity the in past two weeks due to illnesses or injuries (ACTDAYS). (4) Long-term health status measures: general health questionnaire score using Goldberg's method with high score indicating bad health (HSCORE), indicator variable for chronic condition not limiting activity (CHCOND1), and indicator variable for chronic condition limiting activity (CHCOND2). The most notable feature of the data is over-dispersion.

Table 2 provides the estimates and standard errors for BPR, BNBR-1, BNBR-2 and BNBR-P. We use *R programming* with *nlm* function to maximize the log likelihood of BNBR.

Table 2: BPR, BNBR-1, BNBR-2 and BNBR-P (Australian health data)

Parameter	BPR		BNBR-1		BNBR-2		BNBR-P	
	est.	**s.e.**	**est.**	**s.e.**	**est.**	**s.e.**	**est.**	**s.e.**
Y_1, PRESCRIBED								
Intercept	-2.70	0.13	-2.66	0.15	-2.75	0.15	-2.70	0.15
Sex	0.48	0.04	0.55	0.04	0.55	0.04	0.55	0.04
Age	2.41	0.62	2.27	0.71	2.37	0.73	2.34	0.72
Agesq	-0.64	0.64	-0.56	0.74	-0.59	0.78	-0.62	0.76
Income	0.00	0.06	0.00	0.06	0.02	0.06	0.00	0.07
Levyplus	0.29	0.05	0.27	0.06	0.26	0.06	0.27	0.06
Freepoor	-0.05	0.12	-0.09	0.14	-0.05	0.13	-0.09	0.14
Freerepa	0.30	0.06	0.27	0.07	0.29	0.07	0.28	0.07
Illness	0.20	0.01	0.20	0.01	0.21	0.01	0.20	0.01
Actdays	0.03	0.01	0.03	0.00	0.03	0.01	0.03	0.00
Hscore	0.02	0.01	0.02	0.01	0.02	0.01	0.02	0.01
Chcond1	0.77	0.05	0.75	0.05	0.77	0.05	0.77	0.05
Chcond2	1.01	0.05	0.99	0.06	1.02	0.06	1.01	0.06

Y_2, NON-PRESCRIBED								
Intercept	-2.03	0.17	-2.02	0.19	-2.04	0.19	-1.98	0.19
Sex	0.27	0.05	0.27	0.06	0.27	0.06	0.27	0.06
Age	2.86	0.95	3.08	1.05	2.86	1.08	2.86	1.06
Agesq	-3.90	1.07	-4.19	1.19	-3.86	1.22	-3.93	1.20
Income	0.17	0.08	0.13	0.09	0.16	0.09	0.13	0.09
Levyplus	-0.03	0.06	-0.03	0.06	-0.04	0.07	-0.04	0.06
Freepoor	0.00	0.12	-0.04	0.14	-0.02	0.14	-0.06	0.14
Freerepa	-0.29	0.09	-0.26	0.10	-0.29	0.10	-0.29	0.10
Illness	0.20	0.02	0.20	0.02	0.21	0.02	0.20	0.02
Actdays	0.01	0.01	-0.00	0.01	0.01	0.01	-0.00	0.01
Hscore	0.03	0.01	0.03	0.01	0.03	0.01	0.03	0.01
Chcond1	0.15	0.06	0.13	0.06	0.15	0.06	0.14	0.06
Chcond2	0.02	0.08	0.03	0.09	0.02	0.09	0.03	0.09
a_1, dispersion	-	-	0.39	0.04	0.30	0.03	0.40	0.04
a_2, dispersion	-	-	0.29	0.03	0.74	0.08	0.37	0.14
P_1, functional	-	-	1.00	-	2.00	-	1.24	0.12
P_2, functional	-	-	1.00	-	2.00	-	1.22	0.35
ϕ, correlation	-0.89	0.13	-0.91	0.13	-0.93	0.13	-0.91	0.13
Log likelihood	-9522.59		-9332.744		-9351.481		-9330.392	
AIC	19099.18		18723.49		18760.96		18722.78	

The LRT for testing BPR against BNBR-1 and BPR against BNBR-2 are 379.74 and 342.22 respectively, indicating over-dispersion in both data. Therefore, BNBR-1 and BNBR-2 are better than BPR.

The LRT for testing BNBR-1 against BNBR-P is 4.70, which is significant under 0.10 level since the *p*-value is 0.095. The LRT for testing BNBR-2 against BNBR-P is 42.18, which is significant. Based on LRT and AIC, the best model is BNBR-P, followed by BNBR-1, BNBR-2 and BPR.

The estimates of correlation parameter under all models are negative, indicating negative dependence between the two data. The absolute values of *t*-ratio for the correlation parameter, $\left|\frac{\hat{\phi}}{s.e.(\hat{\phi})}\right|$, under BPR, BNBR-1, BNBR-2 and BNBR-P respectively are 6.66, 6.90, 7.08 and 6.96, indicating that the two response data are significantly dependent. Therefore, the response data is suggested to be fitted jointly under BNBR-P, which is the best model compared to BNBR-1, BNBR-2 and BPR.

Table 3: Univariate Poisson regression model, NBR-1, NBR-2 and NBR-P
(Australian health care data)

Parameter	Poisson est.	Poisson s.e.	NBR-1 est.	NBR-1 s.e.	NBR-2 est.	NBR-2 s.e.	NBR-P est.	NBR-P s.e.
Y_1, PRESC								
Intercept	-2.74	0.13	-2.63	0.15	-2.73	0.15	-2.67	0.15
Sex	0.48	0.04	0.55	0.04	0.55	0.04	0.55	0.04
Age	2.65	0.61	2.11	0.71	2.16	0.73	2.07	0.72
Agesq	-0.89	0.64	-0.41	0.75	-0.35	0.78	-0.33	0.76
Income	0.00	0.06	0.00	0.06	0.03	0.06	0.01	0.07
Levyplus	0.28	0.05	0.27	0.06	0.27	0.06	0.28	0.06
Freepoor	-0.05	0.12	-0.10	0.14	-0.05	0.14	-0.09	0.14
Freerepa	0.30	0.06	0.28	0.07	0.29	0.07	0.29	0.07
Illness	0.20	0.01	0.20	0.01	0.21	0.01	0.20	0.01
Actdays	0.03	0.00	0.03	0.00	0.03	0.00	0.03	0.00
Hscore	0.02	0.01	0.02	0.01	0.02	0.01	0.02	0.01
Chcond1	0.78	0.05	0.76	0.05	0.77	0.05	0.78	0.05
Chcond2	1.01	0.05	1.00	0.06	1.02	0.06	1.02	0.06
a, dispersion	-	-	0.39	0.04	0.30	0.03	0.40	0.04
P, functional	-	-	-	-	-	-	1.24	0.11
Log likelihood	-5530.767		-5424.565		-5441.482		-5422.376	
AIC	11087.53		10877.13		10910.96		10874.75	
Y_2, NONPRESC								
Intercept	-2.31	0.17	-2.25	0.19	-2.32	0.19	-2.28	0.15
Sex	0.24	0.05	0.24	0.06	0.25	0.06	0.24	0.04
Age	4.69	0.94	4.82	1.04	4.77	1.08	4.84	0.72
Agesq	-5.93	1.07	-6.12	1.18	-6.00	1.22	-6.15	0.76
Income	0.12	0.08	0.06	0.08	0.11	0.09	0.07	0.07
Levyplus	-0.03	0.06	-0.05	0.06	-0.04	0.07	-0.04	0.06
Freepoor	-0.02	0.12	-0.08	0.14	-0.02	0.14	-0.06	0.14
Freerepa	-0.28	0.09	-0.28	0.10	-0.29	0.10	-0.29	0.07
Illness	0.20	0.02	0.20	0.02	0.21	0.02	0.21	0.01
Actdays	0.00	0.01	0.00	0.01	0.00	0.01	0.00	0.00
Hscore	0.03	0.01	0.03	0.01	0.03	0.01	0.03	0.01
Chcond1	0.16	0.06	0.14	0.06	0.16	0.06	0.14	0.05

Chcond2	0.01	0.08	0.01	0.09	0.00	0.09	0.01	0.06
a , dispersion	-	-	0.29	0.03	0.74	0.08	0.38	0.04
P , functional	-	-	-	-	-	-	1.25	0.11
Log likelihood	-4011.105		-3929.086		-3931.781		-3928.78	
AIC	8048.209		7886.172		7891.562		7887.56	

As to compare, the univariate Poisson regression model, NBR-1, NBR-2 and NBR-P were fitted separately to the two response variables so that LRT could be performed for testing independence. The estimates and standard errors for the fitted models have been shown in Table 3.

Using the results from Tables 2-3, the LRT for testing independence, where $H_0 : \phi = 0$ against $H_0 : \phi \neq 0$, can be implemented for univariate Poisson against BPR, univariate NBR-1 against BNBR-1, univariate NBR-2 against BNBR-2 and univariate NBR-P against BNBR-P. The LRT are 38.58, 41.81, 43.56 and 41.53 respectively, indicating that the two response data are dependent under all models (BPR, BNBR-1, BNBR-2 and BNBR-P).

Comparing the estimates of covariates between bivariate and univariate regression models shows that most covariates have similar estimates. However, there are several covariates that indicate otherwise. If we focus on BNBR-P (which is the best model) and univariate NBR-P, the estimates of Age and Agesq in Y_1 and the estimates of Intercept, Age, Agesq and Income in Y_2 are different for both models.

Comparing the significance of estimates of regression parameters between BNBR-P and univariate NBR-P shows that both models provide the same insignificant estimates at 0.05 level, namely Agesq, Income and Freepoor in Y_1, and Income, Levyplus, Freepoor, Actdays and Chcond2 in Y_2.

Malaysian Motor Insurance Data (2001-2003)

The data for Own Damage (OD) and Third Party Property Damage (TPPD) claims obtained and compiled from ten insurance companies in Malaysia is also considered in this section. The data is based on 1.01 million private car policies with comprehensive coverage (2002) and is supplied by Insurance Services Malaysia (ISM). The exposure is expressed in a car-year unit and the incurred claims consist of claims already paid as well as outstanding. Table 4 shows the rating factors and rating classes for the exposures and incurred claims.

Table 4: Rating factors and rating classes (Malaysian motor insurance data)

Rating factors	Rating classes
Vehicle year	0-1
	2-3
	4-5
	6-7
	8+
Vehicle c.c.	0-1000
	1001-1300
	1301-1500
	1501-1800
	1801+
Vehicle make	Local type 1
	Local type 2
	Foreign type 1
	Foreign type 2
	Foreign type 3
Location	North
	East
	Central
	South
	East Malaysia

The mean and standard deviation for OD (Y_1) claim counts are 114.60 and 235.55, and the mean and standard deviation for TPPD (Y_2) claim counts 41.43 and 99.89. The correlation between y_{i1} and y_{i2} is 0.949. The positive correlation indicates possible positive dependence between the two response variables.

Table 5 provides the estimates and standard errors for BPR and the new BNBR-1, BNBR-2 and BNBR-P for the fitted data.

Table 5: BPR, BNBR-1, BNBR-2 and BNBR-P (Malaysian insurance data)

Parameter	BPR		BNBR-1		BNBR-2		BNBR-P	
	est.	s.e.	est.	s.e.	est.	s.e.	est.	s.e.
(OD)								
Intercept	-3.31	0.03	-3.24	0.07	-3.74	0.14	-3.30	0.08
2-3 years	0.54	0.01	0.54	0.03	0.58	0.11	0.54	0.04
4-5 years	0.51	0.01	0.51	0.04	0.40	0.11	0.51	0.04
6-7 years	0.44	0.01	0.45	0.04	0.32	0.11	0.45	0.04
8+ years	0.24	0.01	0.24	0.03	0.24	0.11	0.25	0.04
1001-1300 c.c.	-0.03	0.02	-0.10	0.06	0.19	0.12	-0.04	0.07
1301-1500 c.c.	0.16	0.02	0.08	0.06	1.28	0.12	0.15	0.07
1501-1800 c.c.	0.41	0.02	0.31	0.06	0.74	0.11	0.37	0.07
1801+ c.c.	0.47	0.03	0.37	0.06	0.92	0.11	0.45	0.07
Local type 2	-0.19	0.02	-0.27	0.06	0.62	0.11	-0.21	0.06
Foreign type 1	-0.23	0.01	-0.21	0.03	-0.36	0.09	-0.22	0.04
Foreign type 2	0.15	0.02	0.18	0.05	0.26	0.09	0.20	0.05
Foreign type 3	-0.08	0.02	-0.04	0.05	-0.27	0.13	-0.07	0.07
East	0.32	0.02	0.38	0.05	0.36	0.11	0.37	0.05
Central	0.29	0.01	0.29	0.03	0.64	0.10	0.30	0.04
South	0.24	0.01	0.24	0.04	0.48	0.10	0.27	0.04
East Malaysia	0.09	0.01	0.10	0.04	0.13	0.10	0.10	0.05
(TPPD)								
Intercept	-4.70	0.04	-4.65	0.08	-4.79	0.08	-4.74	0.08
2-3 years	0.94	0.02	0.95	0.04	1.09	0.06	0.98	0.05
4-5 years	0.90	0.03	0.92	0.05	1.00	0.06	0.95	0.05
6-7 years	1.02	0.02	1.03	0.04	1.13	0.06	1.06	0.05
8+ years	0.99	0.02	1.01	0.04	1.13	0.06	1.04	0.05
1001-1300 c.c.	0.04	0.04	-0.03	0.06	0.08	0.06	0.04	0.06
1301-1500 c.c.	-0.00	0.04	-0.08	0.06	0.04	0.07	-0.01	0.07
1501-1800 c.c.	0.01	0.04	-0.06	0.07	0.05	0.07	0.01	0.07
1801+ c.c.	0.11	0.04	0.05	0.07	0.17	0.07	0.12	0.07
Local type 2	-0.24	0.04	-0.32	0.06	-0.26	0.06	-0.26	0.06
Foreign type 1	-0.13	0.02	-0.16	0.03	-0.25	0.04	-0.17	0.04
Foreign type 2	-0.10	0.03	-0.11	0.05	-0.19	0.05	-0.12	0.06
Foreign type 3	-0.47	0.04	-0.44	0.07	-0.45	0.08	-0.44	0.07
East	0.06	0.03	0.11	0.06	0.07	0.06	0.09	0.06
Central	0.45	0.02	0.46	0.03	0.49	0.05	0.46	0.04
South	0.24	0.02	0.25	0.04	0.30	0.05	0.26	0.04
East Malaysia	0.20	0.02	0.21	0.04	0.19	0.05	0.21	0.05
a_1, dispersion	-	-	6.62	0.42	0.42	0.04	3.41	0.37
a_2, dispersion	-	-	2.27	0.22	0.06	0.01	1.31	0.18

P_1 , functional	-	-	1.00	-	2.00	-	1.21	0.03
P_2 , functional	-	-	1.00	-	2.00	-	1.18	0.04
ϕ , correlation	0.28	0.45	4.16	0.57	3.79	0.46	3.97	0.40
Log likelihood		-5908.89		-3778.88		-4061.16		-3734.20
AIC		11887.77		7631.75		8196.32		7546.39

The LRT for testing over-dispersion in BPR against BNBR-1 and BPR against BNBR-2 are 4260.02 and 3695.46, indicating that the data are over-dispersed. Therefore, BNBR-1 and BNBR-2 are better models than BPR.

The LRT for testing BNBR-1 against BNBR-P and BNBR-2 against BNBR-P are 89.36 and 653.92, indicating that BNBR-P is better than both BNBR-1 and BNBR-2. Based on LRT and AIC, the best model for the Malaysian motor insurance data is BNBR-P, followed by BNBR-1, BNBR-2 and BPR.

The absolute values of *t*-ratio for the correlation parameter, $\left|\frac{\hat{\phi}}{s.e.(\hat{\phi})}\right|$, under BNBR-1, BNBR-2 and BNBR-P are 7.30, 8.24 and 9.93 respectively, indicating that the two response variables are significantly dependent under these models. Therefore, it is suggested that the response variables are fitted jointly under BNBR-P, which is the best model compared to BNBR-2 and BNBR-1.

The univariate NBR-1, NBR-2 and NBR-P are fitted to both response variables although the complete results have not been shown here. The log likelihood of univariate NBR-1, NBR-2 and NBR-P can be used to perform LRT for testing independence, against $H_1 : \phi \neq 0$. The LRT for testing univariate NBR-1 against BNBR-1, univariate NBR-2 against BNBR-2 and univariate NBR-P against BNBR-P respectively are 55.52, 39.06 and 49.14, indicating that the two response variables are dependent under BNBR-1, BNBR-2 and BNBR-P.

7. Conclusions

This study has defined several new forms of BNBR which are nested and allow LRT to be applied for choosing the best model. The new forms of BNBR have flexible mean-variance relationship, can be fitted to bivariate count data with positive, zero or negative correlations, and allow over-dispersion of the two response variables.

We have fitted BP and several forms of BNBR to two sets of data, each with a negative and a positive correlation; the Australian health survey (Cameron et al. 1988) and the Malaysian motor insurance claim. Based on LRT, the best model for the Australian data is BNBR-P regression, followed by BNBR-1, BNBR-2 and BPR. The estimates of correlation parameter under all models are significantly negative, suggesting both responses to be fitted jointly under BNBR-P, which is the best model compared to BNBR-1, BNBR-2 and BPR. Comparison between BNBR-P and univariate NBR-P shows that several estimates of covariates in Y_1 and Y_2 are different under both models. Comparing the significance of estimates of covariates shows that both models provide the same insignificant estimates at 0.05level.

As for the Malaysian data, the results from LRT implied that BNBR-P is the best model, followed by BNBR-1 and BNBR-2. The estimates of correlation parameter under BNBR are significantly positive, suggesting both response variables to be fitted jointly using BNBR-P, which is the best model compared to BNBR-1 and BNBR-2.

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