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Analytical and Numerical Solutions for the Pricing of a Combination of Two Financial Derivatives in a Market Under Hull-White Model

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ABSTRACT

In this paper a combination of two financial derivatives in financial markets modelled of future interest rates is presented and evaluated. In fact, European option pricing is driven when zero-coupon bond is considered as underlying asset in a market under Hull-White model. For this purpose, the exact solutions of the valuation of this bond option are driven, using Lie group symmetries method. Then in the next part, the finite difference method is applied to find numerical solutions for assumed bond option pricing. Then the significance and usefulness of this approximated method is comparing with the exact solutions by some plotted graphs.

1 Introduction

The pricing of contingent claims is one of the most important topics in financial mathematics in order to their risk reduction affect in financial Markets. The option pricing methodology was developed by Black and Scholes in which the appreciation and volatility rate are supposed to be constant [2]. But in realistic, the asset price process distribution has a heavier tail comparing to lognormal distribution. So there are different distributions of call or put option prices which the differences are related to distribution tail. To obtain more actual model, many extensions of Black–Scholes model have been introduced in literatures. Some of them such as the stochastic volatility, jump diffusion models, regime-switching models are more famous and have been studied a lot by researchers [6]. As example in [2] Option pricing under two stochastic volatility models, double Heston model and double Heston with three jumps, is done. In [5] three types of power options have been priced under special stochastic markets which a risky underlying asset follows a model with two stochastic volatilities, two jumps, and a stochastic intensity measure.

In [3] the power option pricing is driven when the dynamic of underlying asset price follows a fractional double Heston model. A special stochastic market which the risky underlying asset follows a model with two stochastic volatilities, two jumps and stochastic intensity has been studied in [1]. In [6] the

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power option pricing was given in a market under fractional Heston model. In fact, power option pricing has been driven using fast Fourier transformation (FFT) and maximum likelihood method in Iran's gold market and the arbitrage existence possibility was investigated in it. Recently, the interest rate derivatives market has become the largest derivatives market in the world with the size up to trillions of dollars. In this financial market, almost all security prices relevant to the interest rate, a nonexchangeable asset and a crucial factor in each stochastic market [20]. Interest rate contingent claims such as caps, swaptions, bond options, captions, and mortgage backed securities have become increasingly popular. The valuation of these instruments is now a major concern of both practitioners and academics [14]. Bond options are a class of interest rate derivatives written on bonds at certain prices on or before the maturities of the options. They are the basic building blocks for other interest rate derivatives, such as call- or put-able bonds and convertible bonds. Consequently, from the angle of financial engineering, able pricing bond options represents a basis for pricing other related interest rate derivatives. Furthermore, caps and floors can be decomposed into a portfolio of options on zero-coupon bonds [19]. Some combinations of derivatives as bond options are dependent on interest rates and their value is often dependant on considered model and its assumptions.

The Hull-White model which is modelled of future interest rates, prices the financial derivative as a function of the entire yield curve, rather than at a single point. The Hull- White model as extension of Vasicek model belongs to a class of free arbitrage models that could be one of the most important advantages of this model. Lie theory of symmetry group is mainly used for the construction of similarity reductions, group invariant solutions and the conservations laws, so this theory plays an important and central role in geometric and financial mathematics [11]. In this paper, lie symmetries method has been used to find exact solutions of the system of coupled partial differential equations for the bond pricing, as a new method. In the sequel numerical solution have been driven using finite difference method. Finally, the usefulness of this approximated method is comparing with the exact solutions by some plotted graphs. This paper is organized as follows. The considered model is presented in section 2. Section 3 formulates the partial differential equation system for European zero-coupon bond option pricing via stochastic analysis. Section 4 derives analytical solutions for zero-coupon bond call option pricing using lie symmetries method. Numerical solutions applying finite difference method (FDM) are given in section 5. The section 6 presents the closed form of the considered bond option. The section 7 two treasury bills in Tehran stock market are priced. The paper is concluded in section 8.

Theorem 1 ***[11].*** Let X be a solution of the following stochastic differential equation.

$$dX(t) = \mu(t)dt + \sigma(t)dW(t),$$

where μ and σ are adapted processes, and f be $C^{1,2}$-function. Let $Z(t) = f(t,X(t))$ [1]. So Z is a solution of the following stochastic differential equation

$$df\big(t,X(t)\big) = \left\{\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right\}dt + \sigma\frac{\partial f}{\partial x}dW(t),$$

And

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX)^2.$$

Where the following formal multiplication table is used.

$$\begin{cases}(dt)^2 = 0,\\ dt.dW = 0,\\ (dW)^2 = 0.\end{cases}$$

2 The Hull-White Model

Assume (Ω,F,P) be a probability space where $\{F_t\}_t$ is the natural filtration generated by the Brownian motion at time t, $0 \le t \le T$ and Q is a risk neutral probability measure. The short-term interest rate r_t at time t is given by

$$dr_t = (b(t) - a(t)r_t)dt + \sigma(t)dW_t \,, \tag{1}$$

where the mean reversion $a(t) > 0$, the risk neutral drift $b(t)$ and the volatility $\sigma(t) > 0$ are deterministic continuous functions of t and W_t is a standard Brownian motion [15].

2.1 Bond option pricing under Hull-White Model

In this section, the pricing of zero-coupon bond (European) call option in a Hull-White model is presented. For the standard zero-coupon bond call option, the payoff is $\max\{p_c(T,S_i) - K\}$, where $p_c(T,S_i)$ is price of zero- coupon bond, K is strike price and T is maturity time. The price of European call option at time t with maturity time T is

$$C(r,t,T) = \Phi\big(\mathrm{r(T)}\big) = \max(p_c(T,S_i) - K, 0) = E^Q\left[e^{-\int_0^t r_s ds} C(T) \middle| r_t = r\right] \tag{2}$$
$$= E^Q\left[e^{-\int_0^t r_s ds}(p_c(T,S_i) - K)^+ \middle| r_t = r\right],$$

where $(p_c(T,S_i) - K)^+ = \max(p_c(T,S_i) - K, 0)$. Taking $C(r,t) = e^{-\int_0^t r_s ds} C(r,t,T)$ in the above formula,

$$C(r,t) = E^Q[e^{-\int_0^t r_s ds}(p_c(T,S_i) - K)^+ \mid r_t = r],$$
$$= E^Q\,[e^{-\int_0^t r_s ds}(p_c\,(T,S_i) - K)^+ \mid Ft]\,. \tag{3}$$

Using theorem 1 for $C(r,t)$,

$$dC(r_t,t) = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial r}dr + \frac{1}{2}\frac{\partial^2 C}{\partial r^2}d\langle r,r\rangle$$
$$= -r_t e^{-\int_0^t r_s ds} C(r_t,t)dt + e^{-\int_0^t r_s ds}\frac{\partial C}{\partial t}dt + e^{-\int_0^t r_s ds}\frac{\partial C}{\partial r}((b(t) - a(t)r_t)dt + \sigma(t)dW_t$$
$$+ \frac{1}{2}e^{-\int_0^t r_s ds}\frac{\partial^2 C}{\partial r^2}((\sigma(t))^2 dt.$$

Since $C(r,t)$ is Q martingale, the drift term value must be zero [13]. So

$$-r_t C + \frac{\partial C}{\partial t} + (b(t) - a(t)r_t)\frac{\partial C}{\partial r} + \frac{1}{2}\big(\sigma(t)\big)^2\frac{\partial^2 C}{\partial t^2} = 0. \tag{4}$$

In the next section, analytical and numerical solutions for Eq. (4) are presented.

2.2 Analytical Solutions via Lie Symmetries Method

Lie symmetry or sometimes called Lie group analysis is a powerful tool in order to find exact solutions of a given system of differential equations which is specially used to solve financial problem can be seen in a lot of literature [4, 7, 11, 19]. A symmetry group of a differential equation is a group that transforms solutions to other solutions [12]. Consider a system of DE (PDE or ODE) in the dependent variables $u^\alpha (1 \le \alpha \le m)$ and dependent variables $x^i (1 \le i \le n)$ of the form:

$$\Delta^s(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \ldots) = 0, \quad 1 \le s \le k, \tag{5}$$

where the subscripts denote partial derivatives (e.g. $u^\alpha_i = \partial u^\alpha / \partial x^i$). To determine continuous symmetries of (5), it is useful to consider infinitesimal Lie transformations of the following form

$$\tilde{x}^i = x^i + \varepsilon\xi^i + O(\varepsilon^2), \qquad \tilde{u}^i = u^\alpha + \varepsilon\eta^\alpha + O(\varepsilon^2), \tag{6}$$

that leave the equation system invariant to $O(\varepsilon^2)$. Lie point symmetries correspond to the case where the infinitesimal generators $\xi^i = \xi^i(x^i, u^\alpha)$ and $\eta^\alpha = \eta^\alpha(x^i, u^\alpha)$ depend only on the x^i and the u^α and not on the derivatives or integrals of the u^α. Generalized Lie symmetries are obtained in the case when the transformations (6) also depend on the derivatives or integrals of the u^α. The infinitesimal transformations for the first and second derivatives to $O(\varepsilon^2)$ are given by the prolongation formula:

$$\tilde{u}_i{}^\alpha = u_i^\alpha + \varepsilon\zeta_i^\alpha, \qquad \tilde{u}_{ij}{}^\alpha = u_{ij}^\alpha + \varepsilon\zeta_{ij}^\alpha, \tag{7}$$

where

$$\zeta_i^\alpha = D_i\hat{\eta}^\alpha + \xi^s u_{si}^\alpha, \qquad \zeta_{ij}^\alpha = D_i D_j\hat{\eta}^\alpha + \xi^s u_{sij}^\alpha. \tag{8}$$

In the other words

$$\hat{\eta}^\alpha = \eta^\alpha - \xi^s u_s^\alpha, \tag{9}$$

corresponds to the canonical Lie transformation for which $\tilde{x}^i = x^i$ and $\tilde{u}^\alpha = u^\alpha + \varepsilon\hat{\eta}^\alpha$. The symbol D_i in (8) denotes the total derivative operator with respect to x^i. Similar formula to (8) applies for the transformation of the higher order derivatives. The condition for invariance of the DE system (5) to $O(\varepsilon^2)$ under the Lie transformation (6) can be expressed in the following form.

$$L\boldsymbol{X}\Delta^s \equiv \tilde{\mathbf{X}}(\Delta^s) = 0, \qquad \Delta^s = 0, \qquad 1 \le s \le k, \tag{10}$$

where

$$\tilde{X} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} + \cdots, \tag{11}$$

is the prolongation of the vector field

$$\boldsymbol{X} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \tag{12}$$

associated with the infinitesimal transformation (6). The symbol $L\mathbf{X}\Delta^s$ in (10) denotes the Lie derivative of Δ^s with respect to the vector field $\mathbf{X}$ (i.e. $L\mathbf{X}\Delta^s = \frac{d\Delta^s}{d\varepsilon}|_{\varepsilon=0}$).[4]. The infinitesimal Lie transformation for (4) with independent variables (x,t) and dependent variable u ($u=C(S,t)$) are in the form of

$$r \to r + \epsilon\xi_1(r,t,u) + O(\epsilon^2),$$

$$t \to t + \epsilon\xi_2(r,t,u) + O(\epsilon^2)$$

$$u \to u + \epsilon\phi(r,t,u) + O(\epsilon^2),$$

with a small parameter $\epsilon \ll 1$. The vector field associated with the above group of transformation can be written as

$$X = \xi^1(r,t,u)\frac{\partial}{\partial r} + \xi^2(r,t,u)\frac{\partial}{\partial t} + \phi(r,t,u)\frac{\partial}{\partial u}. \tag{13}$$

Because of the order equation (4) we need to apply the second prolongation of operator (13) of the form

$$X^{(2)} = X + \phi^r\frac{\partial}{\partial u_r} + \phi^t\frac{\partial}{\partial u_t} + \phi^{rr}\frac{\partial}{\partial u_{rr}} + \phi^{rt}\frac{\partial}{\partial u_{rt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}, \tag{14}$$

where ϕ^r, ϕ^t, ϕ^{rr}, ϕ^{rt} and ϕ^{tt} are prolongation coefficients written by

$$\phi^r = D_r(\phi - \xi_1 u_r - \xi_2 u_t) + \xi_1 u_{rr} + \xi_2 u_{rt},$$
$$\phi^t = D_t(\phi - \xi_1 u_r - \xi_2 u_t) + \xi_1 u_{rt} + \xi_2 u_{tt},$$
$$\phi^{rr} = D_r(D_r(\phi - \xi_1 u_r - \xi_2 u_t)) + \xi_1 u_{rrr} + \xi_2 u_{rrt},$$
$$\phi^{rt} = D_r\big(D_t(\phi - \xi_1 u_r - \xi_2 u_t)\big) + \xi_1 u_{rrt} + \xi_2 u_{rtt},$$
$$\phi^{tt} = D_t(D_t(\phi - \xi_1 u_r - \xi_2 u_t)) + \xi_1 u_{rtt} + \xi_2 u_{ttt},$$

and D_r and D_t denote the total derivative with respect to r and t, respectively [11]. The Lie determining equations of (4) for the infinitesimal generators of the system (14) can be written as follows.

$$\xi^2_u = 0,\quad \xi^2_r = 0,\quad \xi^1_u = 0,\quad \xi^2_{ttt} = 4\xi^2_t a^2,\quad \xi^1_r = \frac{1}{2}\xi^2_t,\quad \xi^1_{tt} = \left(\frac{3}{2}a^2 r - \frac{3}{2}ab + \frac{3}{2}\sigma^2\right)\xi^2_t + \xi^1_r a^2,$$

$$\phi_{uu} = 0,\quad \phi_{ur} = \frac{1}{\sigma^2}\left(\left(\frac{1}{2}ar - \frac{1}{2}b\right)\xi^2_t + \xi^1_r a + \xi^1_t\right),$$

$$\phi_{tu} = \frac{1}{\sigma^2}\Big(-\frac{1}{4}\xi^2_{tt}\sigma^2 + \left(\frac{1}{2}a^2r^2 + \left(-br - \frac{1}{2}\sigma^2\right)a + \sigma^2 r + \frac{1}{2}b^2\right)\xi^2_t + (ar-b)\xi^1_t$$
$$+ (a^2 r - ab + \sigma^2)\xi^1_r\Big),$$

$$\phi_{rr} = \frac{1}{\sigma^2}\Big((2ar - 2b)\,\phi_r - 2\phi_u ru + 2\xi^2_t ru + 2\xi^1_r + 2\phi_r - 2\phi_t\Big).$$

Solving the above Lie determining equations leads to Lie symmetries and $X_1 = \frac{\partial}{\partial t}$ is one of them. For generator X_1, the classical similarity solution of (4) is obtained by integrating the group of trajectories,

$$\frac{dr}{d\varepsilon} = 0, \frac{dt}{d\varepsilon} = 1, \frac{du}{d\varepsilon} = 0,$$

where ε is a parameter along the trajectories. The above integrations yields $u = v(y)$ and $r = y$ as the invariant transformations. Substituting $u = v(y)$ into (4),

$$(b(t) - a(t)y)v' + \frac{1}{2}\big(\sigma(t)\big)^2 v'' - yv = 0. \tag{15}$$

So

$$v(y) = C_1 e^{-\frac{y}{a}} KummerM\left(\frac{2ab-\sigma^2}{4a^3}, \frac{1}{2}, \frac{(a^2y - ab + \sigma^2)^2}{a^3\sigma^2}\right)$$
$$+ C_2 e^{-\frac{y}{a}} KummerU\left(\frac{2ab-\sigma^2}{4a^3}, \frac{1}{2}, \frac{(a^2y - ab + \sigma^2)^2}{a^3\sigma^2}\right), \tag{16}$$

where $KummerM\,(\mu, \nu, z)$ and $KummerU\,(\mu, \nu, z)$ are Kummer functions that solve the following ODE

$$zy'' + (\nu - z)y' - \mu y = 0,$$

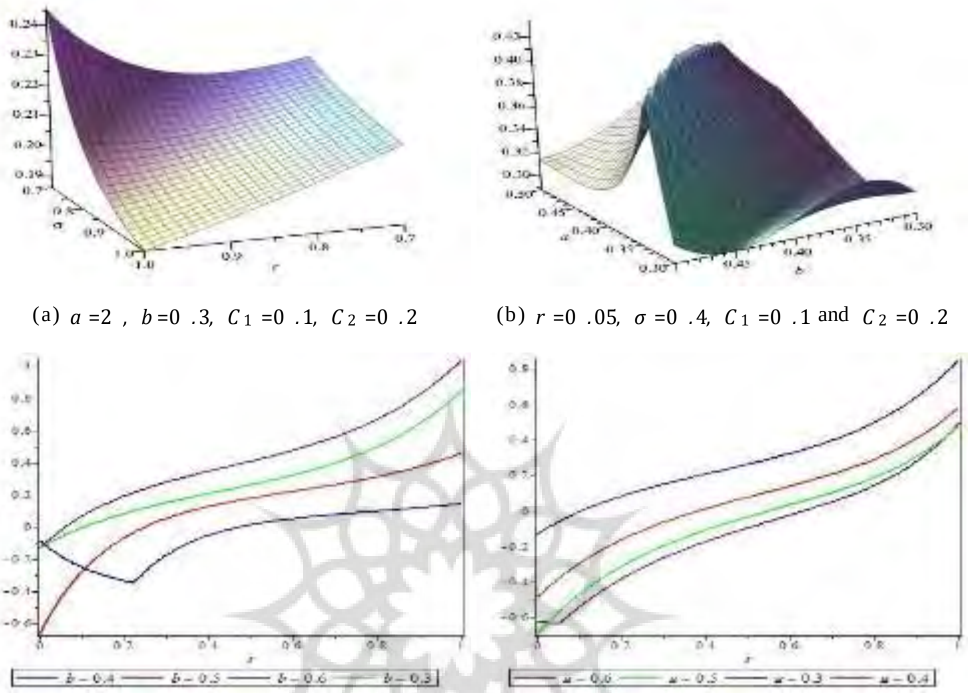
and define as

$$KummerM(\mu,\nu,z) = \sum_{k=0}^{\infty}\frac{P^*(\mu,k)}{P^*(\nu,k)}\frac{z^k}{k!}, \tag{17}$$

$$KemmerU(\mu,\nu,z) = \frac{1}{\Gamma(\mu)}\int_0^{\infty} e^{-zt}\, t^{\mu-1}(1+t)^{\nu-\mu-1}dt, \tag{18}$$

Where

$$P^*(z,a) = \frac{\Gamma(z+a)}{\Gamma(z)}. \tag{19}$$



(a) $a = 2$, $b = 0.3$, $C_1 = 0.1$, $C_2 = 0.2$ (b) $r = 0.05$, $\sigma = 0.4$, $C_1 = 0.1$ and $C_2 = 0.2$

(c) $a = 0.3$, $\sigma = 0.4$, $C_1 = 0.3$ and $C_2 = 0.7$ (d) $b = 0.3$, $\sigma = 0.4$, $C_1 = 0.3$ and $C_2 = 0.7$

Fig. 1: Some solutions of Eq. (16)

3 Numerical solutions using finite difference method

In this section, finite difference method is used to find a numerical simulation for PDE (4). By change of variable $\tau(t) = T - t$, , the backward equation transforms to a forward equation.

$$\frac{\partial U(r,t)}{\partial t} = \frac{\partial U(r,\tau(t))}{\partial \tau(t)} \tau'(t) = -\frac{\partial U(r,\tau)}{\partial \tau}. \tag{20}$$

Replacing $\tau(t)$ with t, (4) convert to;

$$\frac{\partial U}{\partial t} - (b(t) - a(t)r_t)\frac{\partial U}{\partial r} - \frac{1}{2}\big(\sigma(t)\big)^2 \frac{\partial^2 U}{\partial t^2} + r_t U = 0. \tag{21}$$

To solve 4, we divide the interval $[0,T]$ into M subintervals of length Δt. we also chose an upper bound ($r_{max} = 1$) for r. Now we divide [0,1] to N subintervals of length Δr. By this mesh, a grid point is denoted by $(n\Delta r, m\Delta t)$ where $n = 0,1,\ldots,N$, $m = 0,1,\ldots,M$. Using a explicit method for discretization of derivatives, a forward difference approximation for $\frac{\partial u}{\partial t}$ as Eq. (22), a central difference approximation for $\frac{\partial u}{\partial r}$ in Eq. (23) and as Eq. (24) a symmetric central difference approximation $\frac{\partial^2 u}{\partial r^2}$[8].

$$\frac{\partial U}{\partial t}(n\Delta r, m\Delta t) = \frac{u_n^{m+1} - u_n^m}{\Delta t} + O(\Delta t), \tag{22}$$

$$\frac{\partial U}{\partial r}(n\Delta r, m\Delta t) = \frac{u_{n+1}^{m} - u_{n-1}^{m}}{2\Delta r} + O((\Delta r)^2), \tag{23}$$

$$\frac{\partial^2 U}{\partial r^2}(n\Delta r, m\Delta t) = \frac{u_{n+1}^{m} - 2u_{n}^{m} + u_{n-1}^{m}}{(\Delta r)^2} + O((\Delta r)^2). \tag{24}$$

Now we have

$$\frac{u_n^{m+1} - u_n^m}{\Delta t} - (b(t) - a(t)n\Delta r)\frac{u_{n+1}^{m} - u_{n-1}^{m}}{2\Delta r} - \frac{1}{2}\big(\sigma(t)\big)^2 \frac{u_{n+1}^{m} - 2u_{n}^{m} + u_{n-1}^{m}}{(\Delta r)^2} + n\Delta r u_n^m = 0, \tag{25}$$

for $n = 1,...,N - 1$, $m = 1,...,M - 1$. Equation (25) yields

$$u_n^{m+1} = u_n^m + \frac{(b(t) - a(t)n\Delta r)\Delta t}{2\Delta r}(u_{n+1}^{m} - u_{n-1}^{m}) + \frac{\Delta t\big(\sigma(t)\big)^2}{2(\Delta r)^2}(u_{n+1}^{m} - 2u_{n}^{m} + u_{n-1}^{m})$$
$$-n\Delta t \Delta r u_n^m = 0. \tag{26}$$

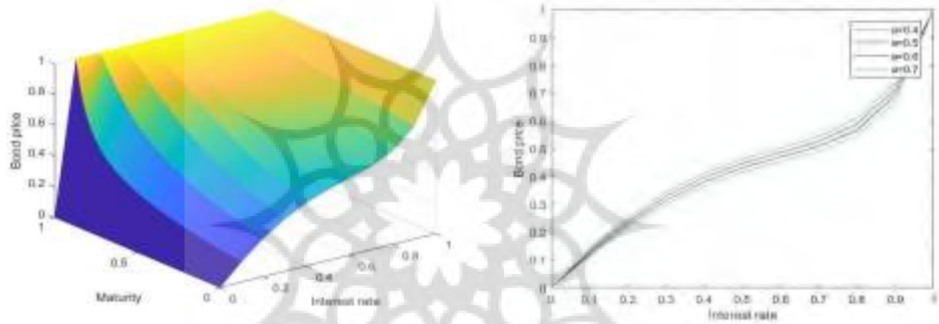


Fig. 2: The Plots of numerical solutions with b =0.067 and σ =0.4

In each time step the term u_n^{m+1} evaluated from one time step back. Values u_n^0, u_0^m, u_N^m for $n = 1,...,N$ and $m = 1,...,M$, are known from initial and boundary conditions.

4 Closed Form

Using Feynman-Kac formula [15] the solution of Eq. (4) is

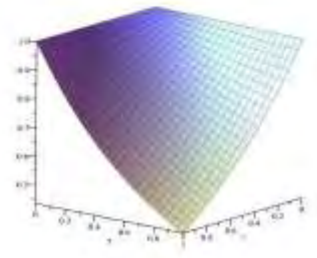
$$C(r,t) = A(\tau)e^{-B(\tau)r} \tag{27}$$

where $\tau = T - t$,

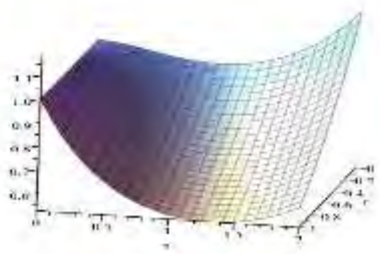
$$A(\tau) = \exp\Big(\frac{\sigma^2}{2a^2}\Big(\tau + \frac{2}{a}(e^{-a\tau} - 1) - \frac{1}{2a}(e^{-2a\tau} - 1)\Big) - \int_0^{\tau} b(T - \tau^*)B(\tau^*)d\tau^*\Big),$$

And

$$B(\tau) = \frac{1 - e^{-a\tau}}{a}$$



(a) $a = 0.4$, $b = 0.067$ and $\sigma = 0.4$



(b) $a = 1.4$, $b = 0.67$ and $\sigma = 0.4$

Fig. 3: Solutions of Eq. (27)

Table 1: Some numerical examples

	τ= 0.1	τ= 0.3	τ= 0.5	τ= 0.7	τ= 0.9
r = 0.02	0.9977	0.9920	0.9858	0.9800	0.9755
r = 0.05	0.9948	0.9836	0.9725	0.9622	0.9536
r = 0.1	0.9899	0.9698	0.9507	0.9333	0.9183
r = 0.2	0.9802	0.9428	0.9086	0.8780	0.8513
r = 0.4	0.9612	0.8910	0.8298	0.7771	0.7320

5 Pricing of Treasury Bills

Treasury bills are classified as securities that Indicate the government's commitment to repay their face amount in the future. Treasury bills are financial instruments with a debt nature that are issued without interest coupons and their main purpose is to provide the governments's budget deficit. Treasury bills are the main tool of the money market to implement monetary policy. These bonds are issued with the aim of settling government debts to non-government creditors, controlling market liquidity, implementing monetary policy, financing the budget deficit and managing the market. It is also the main tool of the money market for the implementation of monetary policy by the Central Bank of the Islamic Republic of Iran. Islamic treasury bills are securities with the name that the government entrusts to non-governmental creditors in order to settle its debts for capital asset acquisition plans with a par value and a certain maturity. Treasury bills in the world are sold to buyers by the government at a lower price than the par value, and government debts are paid from the financial resources obtained from the sale, but due to religious problem in this way, the government of the Islamic Republic of Iran issues these bonds and assignees directly to non-governmental creditors. If the holder of the bonds needs cash, he/she sells these bonds in the new OTC financial instruments market of Iran.

The biggest concern of capital market activists regarding Islamic treasury bills is the non-fulfillment of government obligations at maturity. To address this concern, the government prioritizes this debt at the same level as the salaries and benefits of its employees and is considered a privileged debt of the government. The treasury of the country is obliged to pay the face amount of Islamic treasury bills at the maturity of the bonds. What are the benefits of publishing Islamic treasury bills?

- Discovering the risk-free interest rate expected by the market in the country's economy
- Provide the necessary platform for the development of market financial instruments
- Assist monetary and fiscal policies to implement optimal policies to regulate and control market expectations
- Help maintain the country's independence by using government domestic financing and reducing international borrowing

- •Extract market expectations from risk-free interest rates with different maturities
- •Tax-free income and transactions of these securities

In general, by issuing Islamic treasury bills simultaneously, the government benefits from debt management, the central bank through liquidity control, and the public through risk-free extraction in a competitive market. This article deals with the price of two kinds of Iranian treasury bonds whose current prices are available. The real and calculated price of the considered two kinds bond have been compared. The maturity of the bonds is supposed one year and the calculated price from relations (26) and (27) is compared with the real price. For this purpose, firstly it is necessary to estimate the parameters of the Hall White model. The following theorem is used to estimate the parameters.

Theorem 2. If R_{t_0} is the first given interest rate in the time series and R_{t_n} is the current interest rate, the parameters a, b and σ^2 in the hull-white models are given by the ML-estimators [10]

$$\hat{a} = -\frac{1}{dt}\log\left(\frac{\sum_{i=1}^{n} R_{t_i}R_{t_{i-1}} - \sum_{i=1}^{n} R_{t_i}\sum_{i=1}^{n} R_{t_{i-1}}}{n\sum_{i=1}^{n} R_{t_{i-1}} - (\sum_{i=1}^{n} R_{t_{i-1}})^2}\right),$$

$$\hat{b} = \frac{\hat{a}}{n(1-e^{-\hat{a}dt})}\left(\sum_{i=1}^{n} R_{t_i} - e^{\hat{a}dt}\sum_{i=1}^{n} R_{t_{i-1}}\right),$$

$$\hat{\sigma}^2 = \frac{2\hat{a}}{n(1-e^{-2\hat{a}dt}}\sum_{i=1}^{n}(R_{t_i} - R_{t_{i-1}}e^{-2\hat{a}dt} - \hat{r}\left(1-e^{-\hat{a}dt}\right))^2,$$

To use the above theorem, the monthly bond return rate has been considered. The results of estimating the parameters are shown in the table below. As can be seen in Table 2, treasuryBill210111 and treasuryBill231028 have 1000000 Rials Par Value of each Security. TreasuryBill210111 has been published at 2018-06-12 and payed at 2021-01-11 and treasuryBill210111 has been published at 2020-01-26 and payed at 2023-10-28.

Table 2: Parameters Estimation

	a	b	σ
TreasuryBill210111	1.7659	0.0134	0.0204
TreasuryBill231028	1.3683	0.0302	0.0178

Table 3 shows the results of pricing with two methods; finite difference (FDM) and closed form. Real price of bonds from published date to one year later have been shown in Table 3, as well.

Table 3: Pricing Results

	FDM	Close form	Real price
TreasuryBill210111	994164	979935	995000
TreasuryBill231028	604245	601110	609000

As can be seen in the table, the bond pricing using the finite difference method is closer to the real market price. The reason for the difference in the closed form method can be due to the turbulence in the market and the random conditions that are derived from political and economic events and etc.

6 Conclusion

Financial derivatives have a key role in hedging strategy and increase efficiency. So their pricing is one of the most important concepts in financial mathematics. The pricing of options and bonds, as two types of financial derivatives, is related to a PDE solution. In this paper, a bond and an option are combined together. In fact, a zero coupon bond is considered as an underlying asset which is equipped with a European call option to reduce the investment risk in a market under Hull-White model. In the sequel the exact and numerical solutions for this bond option pricing are driven. For these Lie symmetries and finite difference method are applied to find analytical and numerical solutions, respectively. Then the usefulness of the approximated method is comparing with the exact solutions by some plotted graphs. Also the closed form for option prising is presented as well. Finally, a case study of treasury bills as an example of zero coupon bonds was discussed. Treasury bills, on the one hand, are one of the most appropriate methods of financing the government budget deficit, and on the other hand, are a good opportunity for market participants to invest with less risk and higher profits. Therefore, their pricing is very sensitive. This is because the price of bonds creates excitement in the market that should not be so high as to intensify market volatility and should not be so low that it does not attract the attention of investors to buy. In this article, during a case study of two samples of treasury securities in Tehran Stock Exchange, numerical and exact pricing methods were compared and it was found that pricing with finite difference method was closer to the real price of treasury bills. In fact, the efficiency of the difference method in this case study can help the investor to have the best investment in bonds by predicting the price.

References

[1] Bjork, T., *Arbitrage Theory in Continuous Time*, 2nd edn, Oxford University Press, 2004.

[2] Dastranj, E., Latifi, R., *A comparison of option pricing models*, International Journal of Financial Engineering, 2017, 4.02n03, 1750024. Doi: 10.1142/S2424786317500244

[3] Sahebi Fard, H., Dastranj, E., Hejazi, S.R., *Power Option Pricing Under Fractional Double Heston model*, Annual Iranian Mathematics Conference, 2018, P.3911–3916.

[4] Dastranj, E., Hejazi, S.R., *Exact solutions for Fokker-Plank equation of geometric Brownian motion with Lie point symmetries*, Computational Methods for Differential Equations, 2018, **6**.3, P.372-379.

[5] Dastranj, E., Sahebi Fard, H., Abdolbaghi, A., Latifi, R., *A comparison between Capped and power option pricing with prevention the arbitrage opportunity: Evidence from a stochastic market with double stochastic volatility, double jump and stochastic intensity*. Asset Management and Financing, 2020, Doi: 10.22108/amf.2020.119750.1479.

[6] Dastranj, E., Sahebi Fard, H., Abdolbaghi, A., Hejazi, S.R.,*Power option pricing under the unstable conditions (Evidence of power option pricing under fractional Heston model in the Iran gold market)*, Physica A: Statistical Mechanics and its Applications, 2020, **537**, 122690. Doi: 10.1016/j.physa.2019.122690.

[7] Dastranj, E., Sahebi Fard, H., *Exact solutions and numerical simulation for Bakstein-Howison model*, Computational Methods for Differential Equations, 2021 Apr 24.

[8] Dura, G., and Moşneagu, A. M., *Numerical approximation of Black-Scholes equation*, Annals of the AlexandruIoanCuza University-Mathematics, 2010, **56**(1), P.39-64. Doi: 10.2478/v10157-010-0004-x

[9] Elliott, R. J.,van der Hoek,J., *An application of hidden Markov models to asset allocation problems,* Finance and Stochastics, 1997, **3**, P.229-–238. Doi: 10.1007/s007800050022

[10] Gerebrink, A., Lundgren, J., Malmström, F., Thorén, O., *Maximum Likelihood calibration of the Vasicek model to the Swedish interest rate market*,2018, BS thesis.

[11] Habibi, N., Lashkarian, E., Dastranj, E.,Hejazi, S.R., *Lie symmetry analysis, conservation laws and numerical approximations of time-fractional Fokker–Planck equations for special stochastic process in foreign exchange markets*, Physica A, 2019, **513**, P.750–766. Doi: 10.1016/j.physa.2018.08.155

[12] Hejazi, S. R., *Lie group analysis, Hamiltonian equations and conservation laws of Born-Infeld equation*, Asian-European Journal of Mathematics, 2014, **7**(3), 1450040. Doi: 10.1142/S1793557114500405

[13] Heynen, R. C., Kat, H. M., *Pricing and hedging power options*, Financial Engineering and the Japanese Markets, 1996, **3**(3), P. 253–261. Doi: 10.1007/BF02425804

[14] Hull, J.,White, A., *Pricing interest-rate-derivative securities*, The review of financial studies, 1990, **3**(4), P.573-592. Doi: 10.1093/rfs/3.4.573

[15] Øksendal, B., *Stochastic Differential Equations*, Springer, New York, NY, USA, 2003. Doi: 10.1007/BF02925504

[16] Pan, J., Zhou, X., *Pricing for options in a mixed fractional Hull-White interest rate model*, international Journal of Financial Engineering, 2017, **4**(01), 1750011. Doi: 10.1142/S2424786317500116

[17] Tong, Z., *Option pricing with long memory stochastic volatility models*, Diss. Universitéd'Ottawa/University of Ottawa, 2012.

[18] Mpanda, M.M., *Pricing European and American Bond Options under the Hull-White extended Vasicek Model*, Diss. 2013.

[19] Sahebi Fard, H., Easapour, H., Dastranj, E., *Stochastic analysis to find exact solutions of rainbow option under 2-D Black-Scholes model*, Journal of Interdisciplinary Mathematics, accepted.

[20] Yang, S., Siu, T., *Pricing bond options under a Markovian regime-switching Hull–White model*, Economic Modelling, 2013, **30**, 933940. Doi: 10.1016/j.econmod.2012.09.041