Journal of Production and Operations Management
University of Isfahan E-ISSN: 2423-6950
Vol. 11, Issue 4, No. 23, Winter 2021, p 115-147
Submit Date: 2022-03-13 / **Accept Date:** 2022-04-04
http://dx.doi.org/10.22108/jpom.2022.133033.1435

(Research Paper)

Optimizing the inventory control decisions under multiple constraints for deteriorating products: An application of meta-heuristic algorithms

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Abstract

Purpose: This study aims to investigate the influence of known price increases on the inventory model regarding both uniform and an exponential distribution of replenishment intervals with the partial backorder. It examines the optimization of inventory control decisions for deteriorating products considering a known price increase, probabilistic replenishment interval, warehouse capacity constraint, and partial back-ordering.

Design/methodology/approach: To obtain the specific inventory order quantity, the problem has been modeled in such a way that the total cost savings function is obtained from the differences in the optimal order policy for both special and regular orders. The two situations discussed in this study are: i) unconstrained problem modeling, and ii) constrained problem. Some computational experiments have been performed to examine the effects of various parameters on cost savings performance. For the constrained problem, Genetic Algorithm (GA) and Particle Swarm Optimization (PSO) have been

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used and their results have been compared in terms of the cost savings values and computation time.

Findings: Findings indicated that for the constrained problem, GA has a better performance than PSO. Accordingly, for an unconstrained problem, by using the derivative of the profit function and performing sensitivity analysis, the influence of parameters such as demand, price, holding cost after the price increase, λ in exponential distribution, length of periods in uniform distribution, and deterioration rate on the decision variables including order quantity and the profit were obtained,

Practical implications: The model’s generated policy is more effective and profitable for retailers when demand and deterioration rate are higher and replenishment periods are decreased.

Originality/value: This study completes the previous inventory control models that were under the policy of known price increase and is closer to the real environment by utilizing deteriorating items, capacity constraints, and meta-heuristic approaches.

Keywords: Inventory control, Partial back-ordering, Probabilistic replenishment intervals, Deteriorating items, Genetic Algorithm (GA), Particle Swarm Optimization (PSO)

1. Introduction

Inflation and rising prices for some raw materials, oil, on the one hand, and the introduction of various incentive policies by some suppliers, on the other hand, have a great influence on stock decisions. Therefore, taking into account the increase in commodity prices in the future is inevitable for the supplier. When a supplier announces a price increase in the future and allows the retailer to buy a surplus of goods at the current price. Deciding whether to buy or not to buy and the amount of purchase is necessary for the retailer. Therefore, retailers must decide on their inventory based on the increased rate of product prices in the coming months, random time to the next savings, amount of deterioration items, warehouse capacity and use the optimal use of the special order provided by the supplier (Zhang, X.L & Shi., 2018; Janssen, 2018a; Tashakkor, Mirmohammadi, & Iranpoor, 2018; Li and Teng, 2018; Bounkhel et al., 2019; Soni and Suthar, 2020; Babangida and Baraya, 2020).

This study aims to determine replenishment level values in response to a price increase by maximizing the total cost saving between special and regular orders. We provide several numerical examples for both constrained and unconstrained inventory models as the simplest type of optimization is unconstrained and the unconstrained-optimization technique is so efficient, it has been used as the point of departure for constructing a more realistic constrained inventory model (Bradley et al., 1977; Malik & Sarkar, 2018). Furthermore, a sensitivity analysis of the optimal solution is conducted to show the effects of some parameters on replenishment levels and total saving. In this paper, we will use the decision variables and expected total saving structure as is shown in Taleizadeh, Zarei, & Sarker. (2016).

According to previous research, including Taleizadeh et al. (2013a), Yang et al. (2015), Karimi-Nasab & Wee (2015), and Taleizadeh, Zarei, & Sarker (2016), it can be seen that they did not include real-world constraints and more attention has been paid to the increase in price over time due to a random delivery period and a motivational policy. In several studies, there are no limitations and the problem is modeled and solved with integer decision variables and linear programming (Zeballos, Seifert & Protopappa-Sieke, 2013; Sarkar & Moon, 2014; Giri & Sharma, 2016; Braglia, Castellano & Frosolini, 2016; Braglia, Castellano, & Song, 2017). Therefore, in studies that focus exclusively on known price increases, the gap is quite evident when a study is aimed at getting a better understanding of the real-world conditions (Cimen & Kirkbride, 2017). Therefore, this research seeks to consider goods that do not have a stable lifespan, as well as storage space limitations and problem-solving by using meta-heuristics algorithms.

The rest of the paper is organized as follows: In Section 2, a brief literature review is presented after which the problem along with assumptions is defined in section 3. In Section 4, a proposed model of the problem is devised. To do this, first, the parameters and the variables of the problem are introduced. Next, an unconstrained model with both uniform and exponential distributions is presented and the solution method is elaborated. Then, a constrained problem and the algorithms used to solve it are described. In Section 5, through the numerical examples, both constrained and unconstrained models are implemented and the results are presented in the relevant tables. A sensitivity analysis is performed and its results are shown in section 6, in section 7, the conclusions and also recommendations for future research are presented. Finally, a discussion is presented in section 8 in which the findings of this study are compared with previous research.

2. Literature review

For many researchers and management, decisions about inventory control of *deteriorating items* have always been challenging due to their specific characteristics. Goyal & Giri (2001) discussed developments of deteriorating inventory from 1990 to 2001. They indicated that most of the models had been classified on the base of demand, constraint, and condition. Yang & Wee (2003) developed a mathematical multi-lot-size production model for a deteriorating item in which the perspective of buyers and sellers has been considered. Moon, Giri, & Ko (2005) studied the EOQ model for two kinds of products (deteriorating/ameliorating) under situations such as finite planning horizon, time-dependent demand, inflation, and time value of money. Prekopa (2006) used the model which so-called Hungarian inventory control to obtain that optimal safety stock level. In his model, production was continuous without disruption. Caloieroa, Strozzia & Comenges (2008) investigated the bullwhip effect on demand in the supply chain; they focused on a single product in a serial supply chain. Another work that expanded the EOQ model is Ouyang et al. (2008) which linked permissible delay in payment to deteriorating EOQ. In recent studies, such as Amorim, Costa, & Almada-Lobo (2013), Yu et al. (2012), Abad (2008), Maihami & Karimi (2014), Chen et al. (2016), Neeraj & Kumar (2017), Jaggi, Tiwari & Goel (2017), Zhang, X.L & Shi (2018), Janssen (2018b), Tashakkor, Mirmohammadi, & Iranpoor (2018), Li & Teng (2018), Asif and Biswajit (2018), Bounkhel et al. (2019), Soni & Suthar (2020), Babangida & Baraya (2020), demand (deterministic or stochastic) has been found as a very significant factor in diversifying inventory control models for deteriorating items. To get closer to the real world, Tiwari, et al (2017) developed the model for deteriorating seasonal products with ramp-type demand. They formulated their model with some considerations such: as stock-dependent consumption rate and partial backordering. The main model variable was the preservation technology cost.

In the literature, many studies have focused on the *announcement of a price increase* problem. Naddor (1966) was one of the first researchers who considered the price increase in the future. He modeled an EOQ (economic order quantity) model that highlighted the rise in prices and offered a chance to buy to the buyer. Ghosh (2003) and Huang, Kulkarni & Swaminathan (2003) considered the effect of the infinite horizon on the increase of known price problems. In their studies, buyers could have spatial order before the price increase. In inventory management literature, few studies consider the constant price change. Yang (2006) developed a two-warehouse inventory model for deterioration items with different rates and linear demands under inflationary conditions. Sarker & Kindi (2006) developed economic order quantity (EOQ) models with a discounted price. In their work, they attempt to obtain the order value in five different cases: a) coincidence of sale period with replenishment time, b) non-coincidence of sale period with replenishment time, c) sale period longer than a cycle, d) discounted price as a function of the special ordering quantity, and (e) incremental discount. Sharma (2009) proposed a composite model for the environment with fractional back-ordering. Hsu & Yu (2011) developed an EOQ model for imperfect quality items under an announced price increase where a 100% screening process was performed; then defectives were screened out, and at the end of the inspection process, the defectives were sold as a single batch. They obtained optimal ordering policies under this situation and by some examples, illustrated their proposed model. Taleizadeh, Akhavan Niaki & Makui (2012) described an economic order quantity model in which there were costs in advance and divided the prepayment into multiple equal-size parts during a fixed lead time. Taleizadeh, et al (2013a) formulated and modeled the multiple partial prepayments of the EOQ problem with partial back-ordering. They considered the level of inventory at the time of special order and provided scenarios to explain it. Then, Taleizadeh (2014), Wang et al. (2015), Tsao & Linh (2016), Diabat, Taleizadeh, & Lashgari (2017), Lashgari, Taleizadeh & Sadjadi (2018), Tiwari et al. (2018), and Taleizadeh et al. (2020), developed another EOQ models in which they consider partial back-ordering and prepayment policy. For inflation and the time value of money or deteriorating items, Singh, Kumar & Kumari (2011) developed a two-warehouse model. Ouyang (2016), Palanivel, Uthayakumar & Finite (2015), Herbon (2017), Banerjee & Agrawal (2017), Herbon & Khmelnitsky (2017), Jaggi, Tiwari & Goel (2017), and Kaya & Ghahroodi (2018), considered various situations to obtain optimal order quantity for deteriorating items under changing prices.

In many studies, the consideration of *probabilistic replenishment* intervals is very common (Rabbani, Pourmohammad, & Rafiei, 2016; Chen et al., 2016; Pal, Bardhan & Giri, 2018; Palak, Sioglu, & Geunes, 2018; Janssen, 2018b). For example, Sazavar et al. (2016), investigated multi-period with single item model with restricted order size. The model included a multi-period/multi-product optimal ordering problem considering the expiry date. Pan (2017), investigated a medical resource inventory model for emergency preparation with uncertain demand and stochastic occurrence time, considering different risk preferences.

Meta-hubristic algorithms such as Genetic Algorithm (GA) and Particle Swarm Optimization (PSO) can be used in inventory control to obtain optimal reorder points (Dye, 2012; Mousavi et al., 2014; Buhnia, Shaikh & Gupta, 2015; Bhunia & Shaikh, 2015; Vandani, Niaki, & Aslanzade, 2017; Akbari Kaasgari, Imani, & Mahmoodjanloo, 2017; Azadeh, 2017; Hiassat, Diabat & Rahwan, 2017; Tiwari, 2017). In general, scholars suggest that hybrid meta-heuristic algorithms have gained considerable attention for their capability to solve difficult problems in different fields of science especially to solve the inventory problems and due to the non-linearity of the proposed model of this study, particle swarm optimization (PSO) and genetic algorithm (GA), are implemented as optimizing solvers instead of analytical methods (Talezadeh et al., 2013b; Alejo-Reyes, et al., 2020).

Yao & Chu (2008) developed an improved inventory control model with the GA approach to obtain the optimal value of replenishment cycles. They minimized maximum warehouse space and allowed the warehouse to replenish at any time. Hong & Kim (2009) used a Genetic algorithm to optimize a joint replenishment model. Chiang (2013) completed previous work and considered partial back-ordering and fix–cost-ordering for replenishment. Taleizadeh et al. (2013b) assumed that in several products, the time between two replenishments is the same and follows random variables. They also considered shortages including back-order and emergency orders. Muthuraman, Seshadri, & Wu (2014) modeled an inventory system with stochastic demand and stochastic delivery lags as Quasi-Variation Inequality (QVI). Their model had an infinite-dimensional state-space and was intractable. Bischak et al. (2014) developed an analytical model to obtain optimum inventory levels under lead time constraints. They summed that crossover orders occur. Shu, Huang, & Fu (2015) developed a production–delivery lot-sizing model in which, the time between two delivery was stochastic. Karimi-Nasab & Wee (2015) formulated an inventory model with stochastic replenishment intervals and deterministic sale offers in which replenishment intervals had an exponential distribution and shortage was partially backordered.

3. Problem definition and assumptions

Consider a periodic inventory control model in which a supplier announces a price increase for all items in the future at or before the next scheduled ordering time of the buyer.

This paper developed and formulated an inventory control problem in which:

(i) The time between two consecutive visits is stochastic and follows exponential and uniform distributions

(ii) There are several products

(iii) The level of inventory at the beginning of the spatial period is zero

(iv) The supplier offers special sales

(v) The price increase is not temporary

(vi) The shortage is in the partial back-ordering form

(vii) Goods are deteriorating items and the deteriorating rate is constant

(viii) The demand rate is constant

(ix) There is a warehouse space restriction

(x) The holding cost per unit will increase after the price is increased

(xi) There are holding, shortage, and purchase costs

(xii) Only one provider exists

4. Mathematical modeling

The following notation is used to model the problem:

For i=1, 2,..., n, the parameters and the variables of the model are defined as

Parameters:

D_i	Demand market rate **of the** i^{th} **product** (units/year)
A_i	The fixed ordering cost of i^{th} product (in dollars)
P_i	The selling price **of the** i^{th} **product** (unit /year)
C_i	**Regular purchasing price per unit of the** i^{th} **product** (in dollars)

C_{ki}	**Purchasing price in future per unit of the *i*th product** (in dollars)
α_i	The fraction of shortages that will be back-ordered **of the *i*th product** (Percent)
π_i	**Unit backorder cost of the ith product ($/unit/year)**
π'_i	Lost sale cost in normal price **per unit of the ith product** ($/unit)
π'_{ki}	Lost sale cost in increased **price per unit of the ith product ($/unit**)
h_i	**Inventory holding cost per unit of the ith product** ($/unit/year)
h_{ki}	**Inventory holding cost when pricing increases per unit of the ith product** ($/unit/year)
θ_i	**Deteriorating rate of the *i*th product**
T_i	The time between two consecutive replenishments **of the *i*th product**
t_{maxi}	The maximum amount of time between two consecutive replenishments in uniform distribution **of the *i*th product**
t_{mini}	The minimum amount of time between two consecutive replenishments in uniform distribution **of the *i*th product**
λ_i	Mean number of replenishments per year for the exponential probability distribution function **of the *i*th product**
$f(t_i)$	Probability distribution function (pdf) of *L* **of the *i*th product**
$F(t_i)$	The cumulative distribution function of *L* **of the *i*th product**
m_i	The required warehouse space per unit of the ith product
M	**Total available warehouse space**

Variables:

R_i	The replenish-up-to level in the regular order **of the *i*th product** (unit)
R_{ki}	The replenish-up-to level when the price increase **of the *i*th product** (unit)
R_{si}	The replenish-up-to level in the special sale **of the *i*th product** (unit)
Q_{1Si}	Expected number of units replenished per cycle in the special sale before price increases **of the *i*th product (**Unit/order)
Q_{2Si}	Expected number of units sold per cycle in the special sale before price increases **of the *i*th product** (unit)
$\overline{B}_{si}$	Expected back-ordered quantity per cycle in the special sale **of the *i*th product**, (unit)
$\overline{L}_{si}$	Expected lost sale quantity per cycle in the special sale **of the *i*th product**, (unit)
$\overline{I}_{si}$	Expected inventory per cycle in the special sale **of the *i*th product** (unit)
Q_{1ki}	Expected number of units replenished per cycle when the price increased of the ith product (Unit/order)
Q_{2ki}	Expected number of units sold per cycle in special sale when the price increased of the ith product (Unit)
$\overline{B}_{ki}$	Expected back-ordered quantity per cycle of the ith product when the price increased (unit)
$\overline{L}_{ki}$	Expected lost sale quantity per cycle of the ith product when the price increased (unit)
$\overline{I}_{ki}$	Expected inventory per cycle of the ith product when the price increased (unit)
Q_{1i}	Expected number of units replenished per cycle in the special sale before price increases of the ith product (Unit/order)
Q_{2i}	Expected number of units sold per cycle in the special sale before price increases of the ith product (Unit)
$\overline{B}_i$	Expected back-ordered quantity per cycle in the special sale of the ith product (unit)

$\overline{L}_i$	Expected lost sale quantity per cycle in the special sale of the i^{th} product (unit)
$\overline{I}_i$	Expected inventory per cycle in the special sale of the i^{th} product (unit)
ECP_N	Expected cyclic profit without ordering a special sale ($)
ECP_S	Expected cyclic profit when special order is placed ($)
TS	**Expected total saving function**

Unconstrained Modeling

If the retailer or buyer decided to place a special order, the total cost of the order is computed as follows. Taleizadeh, Zarei, & Sarker (2016) proposed the following total cost function when an order is placed:

$$ECP_S = \sum_{i=1}^{n} P_i Q_{2si} - [A_i + Q_{1s} C_i + h_i \overline{I_{Si}} + \overline{L_{Si}} \pi_i' + \overline{B_{Si}} \pi_i + C_{ki}(\overline{B}_{Si} - \overline{B}_{ki})] \quad (1)$$

If the special order is not placed:

$$ECP_N = \sum_{i=1}^{n} - \overline{B}_i \pi_i - \overline{L}_i \pi'_i + (P_i - C_{Ki})\overline{B}_i + \quad (2)$$

$$[P_i Q_{2ki} - (A_i + Q_{1ki} C_{ki} + h_i \overline{I_{ki}} + \overline{L_{ki}} \pi'_i + \overline{B_{ki}} \pi_i)] (\frac{Q_{1si}}{Q_{1ki}} - \frac{\overline{L_i} + \overline{B_i}}{Q_{1ki}})$$

To calculate the optimal size of the replenishment level, the difference in total costs must be maximized:

$$TS = \sum_{i=1}^{n} ECP_{Si} - ECP_{Ni} \quad (3)$$

$$TS = \sum_{i=1}^{n} \{P_i Q_{2si} - [A_i + Q_{1s} C_i + h_i \overline{I_{Si}} + \overline{L_{Si}} \pi_i' + \overline{B_{Si}} \pi_i + C_{ki}(\overline{B}_{Si} - \overline{B}_{ki})]\} \text{-} \quad (4)$$

$$\{-\overline{B_i} \pi_i - \overline{L_i} \pi'_i + (P_i - C_{Ki})\overline{B}_i +$$

$$[P_i Q_{2ki} - (A_i + Q_{1ki} C_{ki} + h_{ki} \overline{I_{ki}} + \overline{L_{ki}} \pi_{ki}' + \overline{B_{ki}} \pi_i \ +)] (\frac{Q_{1Si}}{Q_{1Ki}} - \frac{\overline{L_i} + \overline{B_i}}{Q_{1ki}})\}$$

By simplifying the equation (4):

$$TS = \sum_{i=1}^{n} [Q_{2si}\ \varphi_{1i} - Q_{1si}\ \varphi_{2i} - \overline{I_{Si}}\ \varphi_{3i} - \overline{B_{Si}}\ \varphi_{4i} - \overline{L_{Si}}\ \varphi_{5i} - A_i + \varphi_{6i}] \quad (5)$$

Where

$$\varphi_{1i} = P_i \quad (6)$$

$$\varphi_{2i} = \left(C_i + \frac{1}{Q_{1ki}}\left(P_i Q_{2ki} - [A_i + Q_{1ki}C_{ki} + h_{ki}\overline{I_{ki}} + \overline{L_{ki}}\pi_{ki}' + \overline{B_{ki}}\pi_i\,]\right)\right) \tag{7}$$

$$\varphi_{3i} = h_i \tag{8}$$

$$\varphi_{4i} = (\pi_i + C_{ki}) \tag{9}$$

$$\varphi_{5i} = \pi_i' \tag{10}$$

$$\varphi_{6i} = C_{ki}\,\overline{B}_{ki} + \overline{B_i}\pi_i + \overline{L_i}\pi_i' - (P_i - C_{Ki})\overline{B_i} + \left(P_i Q_{2ki} - [A_i + Q_{1ki}C_{ki} + h_{ki}\overline{I_{ki}} + \overline{L_{ki}}\pi_{ki}' + \overline{B_{ki}}\pi_i\,]\right)\left(\frac{\overline{L_i} + \overline{B_i}}{Q_{1ki}}\right) \tag{11}$$

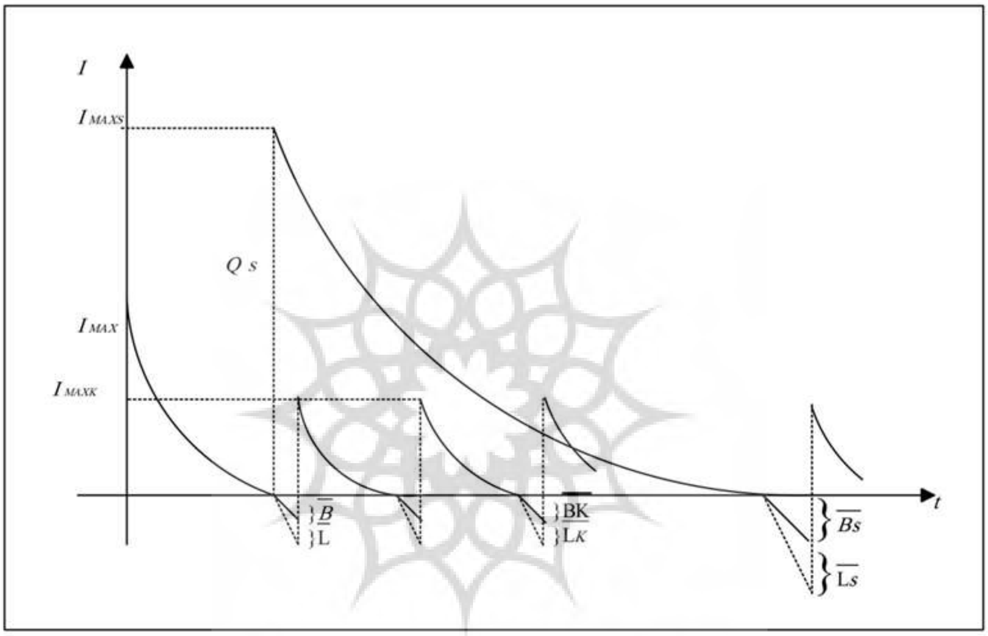


Fig 1- The inventory system scheme for one product

Considering that both normal price period and increased price period are probabilistic, $Q_{1ki}, Q_{2ki}, \overline{B}_{ki}, \overline{L}_{ki}, \overline{I}_{ki}, Q_{1si}, Q_{2si}, \overline{B}_{si}, \overline{L}_{si}, \overline{I}_{si}, Q_{1i}, Q_{2i}, \overline{B}_i, \overline{L}_i, \overline{I}_i$ are computed as below.

Uniform distribution

When the time follows a uniform distribution, Q_{1si}, Q_{2si}, $\overline{B}_{si}$, $\overline{L}_{si}$, $\overline{\ }$ Isi, and total cost saving would be:

$$Q_{1Si} = \frac{1}{t_{maxi} - t_{mini}}\{(1-\alpha_i)R_{si}t_{maxi} + \frac{D_i}{2}(\alpha_i t_{maxi}^2 - t_{mini}^2) + \frac{R_{si}^2}{2D_i}\;\theta_i(t_{maxi} - t_{mini}) - \frac{R_{si}^2}{2D_i}(1-\alpha_i)\,\} \tag{12}$$

$$Q_{2Si}=\frac{1}{t_{maxi}-t_{mini}}\{(1-\alpha_i)R_{si}t_{maxi}+\frac{D_i}{2}(\alpha_i t_{maxi}^2-t_{mini}^2)-\frac{R_{si}^2}{2D_i}(1-\alpha_i)\} \quad (13)$$

$$\overline{B}_{si}=\frac{\alpha_i}{t_{maxi}-t_{mini}}\{\frac{D_i t_{maxi}^2}{2}+\frac{R_{si}^2}{2D_i}-R_{si}t_{maxi}\} \quad (14)$$

$$\overline{L}_{si}=\frac{1-\alpha_i}{t_{maxi}-t_{mini}}\{\frac{D_i t_{maxi}^2}{2}+\frac{R_{si}^2}{2D_i}-R_{si}t_{maxi}\} \quad (15)$$

$$\overline{I}_{Si}=\frac{1}{t_{maxi}-t_{mini}}\{\frac{R_{si}^2}{2D_i}(t_{maxi}-t_{mini})\} \quad (16)$$

$$TS=\sum_{i=1}^{n}[(\frac{\varphi_{1i}}{t_{maxi}-t_{mini}}\left\{(1-\alpha_i)R_{si}t_{maxi}+\frac{D_i}{2}(\alpha_i t_{maxi}^2-t_{mini}^2)-\frac{R_{si}^2}{2D_i}(1-\alpha_i)\right\})$$
$$-(\frac{\varphi_{2i}}{t_{maxi}-t_{mini}}\left\{(1-\alpha_i)R_{si}t_{maxi}+\frac{D_i}{2}(\alpha_i t_{maxi}^2-t_{mini}^2)+\frac{R_{si}^2}{2D_i}\theta_i(t_{maxi}-t_{mini})-\frac{R_{si}^2}{2D_i}(1-\alpha_i)\right\})$$
$$-(\frac{\varphi_{3i}}{t_{maxi}-t_{mini}}\left\{\frac{R_{si}^2}{2D_i}(t_{maxi}-t_{mini})\right\})$$
$$-(\frac{\varphi_{4i}\,\alpha_i}{t_{maxi}-t_{mini}}\left\{\frac{D_i t_{maxi}^2}{2}+\frac{R_{si}^2}{2D_i}-R_{si}t_{maxi}\right\})$$
$$-(\frac{\varphi_{5i}\,(1-\alpha_i)}{t_{maxi}-t_{mini}}\left\{\frac{D_i t_{maxi}^2}{2}+\frac{R_{si}^2}{2D_i}-R_{si}t_{maxi}\right\})-A_i+\varphi_{6i}] \quad (17)$$

The value of $Q_{1ki}, Q_{2ki}, \overline{B}_{ki}, \overline{L}_{ki}, \overline{I}_{ki}$ are

$$Q_{1ki}=\frac{1}{t_{maxi}-t_{mini}}\{(1-\alpha_i)R_{ki}t_{maxi}+\frac{D_i}{2}(\alpha_i t_{maxi}^2-t_{mini}^2)+\frac{R_{ki}^2}{2D_i}\theta_i(t_{maxi}-t_{mini})-\frac{R_{ki}^2}{2D_i}(1-\alpha_i)\} \quad (18)$$

$$Q_{2ki}=\frac{1}{t_{maxi}-t_{mini}}\{(1-\alpha_i)R_{ki}t_{maxi}+\frac{D_i}{2}(\alpha_i t_{maxi}^2-t_{mini}^2)-\frac{R_{ki}^2}{2D_i}(1-\alpha_i)\} \quad (19)$$

$$\overline{B}_{ki}=\frac{\alpha_i}{t_{maxi}-t_{mini}}\{\frac{D_i t_{maxi}^2}{2}+\frac{R_{ki}^2}{2D_i}-R_{ki}t_{maxi}\} \quad (20)$$

$$\overline{L}_{ki}=\frac{1-\alpha_i}{t_{maxi}-t_{mini}}\{\frac{D_i t_{maxi}^2}{2}+\frac{R_{ki}^2}{2D_i}-R_{ki}t_{maxi}\} \quad (21)$$

$$\overline{I}_{ki} = \frac{1}{t_{maxi} - t_{mini}}\{\frac{R_{ki}^2}{2D_i}(t_{maxi} - t_{mini})\} \tag{22}$$

Then $Q_{1i}, Q_{2i}, \overline{B}_i, \overline{L}_i, \overline{I}_i$:

$$Q_{1i} = \frac{1}{t_{maxi} - t_{mini}}\{(1-\alpha_i)R_i t_{maxi} + \frac{D_i}{2}(\alpha_i t_{maxi}^2 - t_{mini}^2) + \frac{R_i^2}{2D_i}\theta_i(t_{maxi} - t_{mini}) - \frac{R_i^2}{2D_i}(1-\alpha_i)\} \tag{23}$$

$$Q_{2i} = \frac{1}{t_{maxi} - t_{mini}}\{(1-\alpha_i)R_i t + \frac{D_i}{2}(\alpha_i t_{maxi}^2 - t_{mini}^2) - \frac{R_i^2}{2D_i}(1-\alpha_i)\} \tag{24}$$

$$\overline{B_i} = \frac{\alpha_i}{t_{maxi} - t_{mini}}\{\frac{D_i t_{maxi}^2}{2} + \frac{R_i^2}{2D_i} - R_i t_{maxi}\} \tag{25}$$

$$\overline{L_i} = \frac{1-\alpha_i}{t_{maxi} - t_{mini}}\{\frac{D_i t_{maxi}^2}{2} + \frac{R_i^2}{2D_i} - R_i t_{maxi}\} \tag{26}$$

$$\overline{I}_i = \frac{1}{t_{maxi} - t_{mini}}\{\frac{R_i^2}{2D_i}(t_{maxi} - t_{mini})\} \tag{27}$$

In the total saving equation described above, the $\varphi_{1,}\varphi_{2,}\varphi_{3,}\varphi_{4,}\varphi_{5,}\varphi_{6,}$ are considered constant and do not have any effect on the concavity of the functions. TS is a quadratic equation and its second derivative is negative (see Appendix). Therefore, the concavity of profit function is proven. To obtain the optimal value of replenishment level value, the first-order derivative of TS must be equal to zero, therefore:

$$R_{si}^* = \frac{-(1-\alpha_i)t_{maxi}(\varphi_{1i} - \varphi_{2i}) - t_{maxi}(1-\alpha_i)\varphi_{5i} - t_{maxi}\alpha_i\varphi_{4i}}{\frac{1}{D_i}\{(1-\alpha_i)t_{maxi}(\varphi_{1i} - \varphi_{2i}) - \theta_i(t_{maxi} - t_{mini})(\varphi_{2i}) - (1-\alpha_i)\varphi_{5i} - \alpha_i\varphi_{4i} - (t} \tag{28}$$

Exponential distribution

Similar to what was done above, when time follows exponential distribution with λ (replenishments/period) the $Q_{1si}, Q_{2si}, \overline{B}_{si}, \overline{L}_{si}, \overline{I}_{si}$, $Q_{1ki}, Q_{2ki}, \overline{B}_{ki}, \overline{L}_{ki}, \overline{I}_{ki}, Q_{1i}, Q_{2i}, \overline{B}_i, \overline{L}_i, \overline{I}_i$ in total cost saving are computed as below:

$$Q_{1si} = \frac{D_i}{\theta_i}e^{-\lambda_i\frac{R_{si}}{D_i}}\left(\frac{\alpha_i + \theta_i}{\lambda_i} - \frac{\theta_i}{\theta_i + \lambda_i}\right) + \frac{D_i}{\theta_i}e^{\theta_i\frac{R_{si}}{D_i}}\left(\frac{\theta_i}{\theta_i + \lambda_i}\right) \tag{29}$$

$$Q_{2si} = \frac{D_i}{\lambda_i} e^{-\lambda_i \frac{R_{si}}{D_i}} (\alpha_i - 1) + \frac{D_i}{\lambda_i} \quad (30)$$

$$\overline{B}_{si} = \alpha_i \frac{D_i}{\lambda_i} e^{-\lambda_i \frac{R_{si}}{D_i}} \quad (31)$$

$$\overline{L}_{si} = (1 - \alpha_i) \frac{D_i}{\lambda_i} e^{-\lambda_i \frac{R_{si}}{D_i}} \quad (32)$$

$$\overline{I}_{si} = \frac{e^{\theta_i \frac{R_{si}}{D_i}} D_i}{\theta_i (\theta_i + \lambda_i)} + \frac{D_i e^{-\lambda_i \frac{R_{si}}{D_i}}}{\lambda_i (\theta_i + \lambda_i)} - \frac{D_i}{\lambda_i \theta_i} \quad (33)$$

$$Q_{1ki} = \frac{D_i}{\theta_i} e^{-\lambda_i \frac{R_{ki}}{D_i}} \left(\frac{\alpha_i + \theta_i}{\lambda_i} - \frac{\theta_i}{\theta_i + \lambda_i} \right) + \frac{D_i}{\theta_i} e^{\theta_i \frac{R_{ki}}{D_i}} \left(\frac{\theta_i}{\theta_i + \lambda_i} \right) \quad (34)$$

$$Q_{2ki} = \frac{D_i}{\lambda_i} e^{-\lambda_i \frac{R_{ki}}{D_i}} (\alpha_i - 1) + \frac{D_i}{\lambda_i} \quad (35)$$

$$\overline{B}_{ki} = \alpha_i \frac{D_i}{\lambda_i} e^{-\lambda_i \frac{R_{ki}}{D_i}} \quad (36)$$

$$\overline{L}_{ki} = (1 - \alpha_i) \frac{D_i}{\lambda_i} e^{-\lambda_i l_{Rki}} \quad (37)$$

$$\overline{I}_{ki} = \frac{e^{\theta_i l_{ki}} D_i}{\theta_i (\theta_i + \lambda_i)} + \frac{D_i e^{-\lambda_i \frac{R_{ki}}{D_i}}}{\lambda_i (\theta_i + \lambda_i)} - \frac{D_i}{\lambda_i \theta_i} \quad (38)$$

$$Q_{1i} = \frac{D_i}{\theta_i} e^{-\lambda_i \frac{R_i}{D_i}} \left(\frac{\alpha_i + \theta_i}{\lambda_i} - \frac{\theta_i}{\theta_i + \lambda_i} \right) + \frac{D_i}{\theta_i} e^{\theta_i \frac{R_i}{D_i}} \left(\frac{\theta_i}{\theta_i + \lambda_i} \right) \quad (39)$$

$$Q_{2i} = \frac{D_i}{\lambda_i} e^{-\lambda_i \frac{R_i}{D_i}} (\alpha_i - 1) + \frac{D_i}{\lambda_i} \quad (40)$$

$$\overline{B}_i = \alpha_i \frac{D_i}{\lambda_i} e^{-\lambda_i \frac{R_i}{D_i}} \quad (41)$$

$$\overline{L}_i = (1 - \alpha_i) \frac{D_i}{\lambda_i} e^{-\lambda_i \frac{R_i}{D_i}} \quad (42)$$

$$\overline{I}_i = \frac{e^{\theta_i \frac{R_i}{D_i}} D_i}{\theta_i (\theta_i + \lambda_i)} + \frac{D_i e^{-\lambda_i \frac{R_i}{D_i}}}{\lambda_i (\theta_i + \lambda_i)} - \frac{D_i}{\lambda_i \theta_i} \quad (43)$$

$$TS = \sum_{i=1}^{n} [(\frac{D_i}{\lambda_i} e^{-\lambda_i \frac{R_{si}}{D_i}} (\alpha_i - 1) + \frac{D_i}{\lambda_i}) \varphi_{1i} - (\frac{D_i}{\theta_i} e^{-\lambda_i \frac{R_{si}}{D_i}} \left(\frac{\alpha_i + \theta_i}{\lambda_i} - \frac{\theta_i}{\theta_i + \lambda_i} \right) \quad (44)$$

$$+ \frac{D_i}{\theta_i} e^{\theta_i l_{Rsi}} \left(\frac{\theta_i}{\theta_i + \lambda_i} \right)) \varphi_{2i} - (\frac{e^{\theta_i \frac{R_{si}}{D_i}} D_i}{\theta_i (\theta_i + \lambda_i)} + \frac{D_i e^{-\lambda_i \frac{R_{si}}{D_i}}}{\lambda_i (\theta_i + \lambda_i)} - \frac{D_i}{\lambda_i \theta_i}) \varphi_{3i}$$

$$- (\alpha_i \frac{D_i}{\lambda_i} e^{-\lambda_i \frac{R_{si}}{D_i}}) \varphi_{4i} - ((1 - \alpha_i) \frac{D_i}{\lambda_i} e^{-\lambda_i \frac{R_{si}}{D_i}}) \varphi_{5i} - (e^{-\lambda_i \frac{R_{si}}{D_i}} \left(\frac{D_i \theta_i}{\lambda_i (\theta_i + \lambda_i)} \right)$$

$$- A_i + \varphi_{6i}]$$

Similar to a uniform distribution, the $\varphi_{1,}\varphi_{2,}\varphi_{3,}\varphi_{4,}\varphi_{5,}\varphi_{6,}$ are constant and do not have any effect on the concavity. It is proved that this function is concave (see appendix). Therefore, the optimal value of the function is given by the first derivative equal to zero.

$$R_{si}^{*} = D_i\{\ln\left(\left(-\frac{\varphi_{2i}\theta_i}{\theta_i+\lambda_i}\right)+\left(\frac{\varphi_{3i}}{(\theta_i+\lambda_i)}\right)\right) -\ln\left(\left(\frac{-\varphi_{2i}\lambda_i}{\theta_i}\left(\frac{\alpha_i\theta_i}{\lambda_i}-\frac{\theta_i}{\theta_i+\lambda_i}\right)\right)+\left(\frac{\varphi_{3i}}{(\theta_i+\lambda_i)}\right)+\varphi_{5i}(1-\alpha_i)+\varphi_{4i}\alpha_i - \varphi_{1i}(\alpha_i-1)\right)\}/(-\theta_i-\lambda_i) \quad (45)$$

Solution method

Similar to the approach of Taleizadeh, Zarei, & Sarker (2016), Karimi-Nasab & Wee (2015), the following lemma is used to obtain decision variables:

(i) $R_i,\ R_{ki}$ for uniform and exponential distribution (see Appendix)

(ii) calculate $Q_{1ki}, Q_{2ki}, \overline{B}_{ki}, \overline{L}_{ki}, \overline{I}_{ki}, Q_{1i}, Q_{2i}, \overline{B}_i, \overline{L}_i, \overline{I}_i$ for two distributions

(iii) calculate $\varphi_{1,}\varphi_{2,}\varphi_{3,}\varphi_{4,}\varphi_{5,}\varphi_{6,}$

(iv) calculate an optimal value for R_{si}

Constrained model

As the total available warehouse space is M, the space required for each unit of product is M_i, and the upper limits for inventory in various periods are $R_i,\ R_{ki},\ and\ R_{si}$. In summary, the complete mathematical model would be:

Maximize TS (46)

$$\sum_{i=1}^{n} R_i m_i \le M \quad (47)$$

$$\sum_{i=1}^{n} \mathrm{R}_{ki} m_i \le M \quad (49)$$

$$\sum_{i=1}^{n} \mathrm{R}_{si} m_i \le M \quad (50)$$

Note that the TS (above) varies in two distributions and follows equations (17) and (44) for uniform and exponential distributions.

Solution method

Constrained non-linear optimization is a key form of the problem in the fields of economics, management, and engineering. Mathematical programming and meta-heuristic methods are two common ways to solve these types of problems. Mathematical programming can obtain solutions with higher accuracy as compared to the meta-heuristic method but in large scale and NP-hard problems, consume a lot of time So during the last decades, a wide variety of meta-heuristic algorithms have been designed and applied to solve the constrained non-linear optimization problems. GA and PSO are typical examples of these algorithms that have strengths and weaknesses (Talezadeh et al., 2013b; Alejo-Reyes, 2020). In this paper, GA and PSO algorithms are used to solve the problem, and then, their accuracy, performance, and time-consuming are compared.

GA initial population

GA chromosomes or candidate solutions for the i[th] product are the maximum inventory levels in three periods, i.e., normal price, the announcement of price increase after the price increased. Therefore, one chromosome or string for 10 items is a10*3 matrixes. The positive real numbers are randomly generated in each matrix to meet constraints. Hence, N chromosomes are generated for the initial population:

i=1	R_1	R_{S1}	R_{k1}
i=2	R_2	R_{S2}	R_{k2}
i=3	.	.	.
i=4	.	.	.
i=5	.	.	.
i=6	.	.	.
i=7			.
i=8			
i=9			
i=10			

Fig 2- The structure of a chromosome

GA crossover operation

To perform crossover operation, there are two common algorithms including single-point crossover and multiple crossovers point like in real organisms. Therefore, a sub-matrix from the parent (1) is randomly selected and then, the permutation is copied from the chromosome of the first paren. Finally, the chromosome of the second parent is scanned and if the number is not yet in the offspring, it would be added. Repeatedly, the second child is also made by using the same procedure just as for the first child. (Fig. 3)

PARENT(1)			PARNET(2)		
R_1	R_{S1}	R_{k1}	R_1	R_{S1}	R_{k1}
R_2	R_{S2}	R_{k2}	R_2	R_{S2}	R_{k2}
R_3	R_{S3}	R_{K3}	R_3	R_{S3}	R_{K3}
R_4	R_{S4}	R_{K4}	R_4	R_{S4}	R_{K4}
R_5	R_{S5}	R_{K5}	R_5	R_{S5}	R_{K5}
R_6	R_{S6}	R_{K6}	R_6	R_{S6}	R_{K6}
R_7	R_{S7}	R_{K7}	R_7	R_{S7}	R_{K7}
R_8	R_{S8}	R_{K8}	R_8	R_{S8}	R_{K8}
R_9	R_{S9}	R_{K9}	R_9	R_{S9}	R_{K9}
R_{10}	R_{S10}	R_{K10}	R_{10}	R_{S10}	R_{K10}

SPRING(1)			SPRING(2)		
R_1	R_{S1}	R_{k1}	R_1	R_{S1}	R_{k1}
R_2	R_{S2}	R_{k2}	R_2	R_{S2}	R_{k2}
R_3	R_{S3}	R_{K3}	R_3	R_{S3}	R_{K3}
R_4	R_{S4}	R_{K4}	R_4	R_{S4}	R_{K4}
R_5	R_{S5}	R_{K5}	R_5	R_{S5}	R_{K5}
R_6	R_{S6}	R_{K6}	R_6	R_{S6}	R_{K6}
R_7	R_{S7}	R_{K7}	R_7	R_{S7}	R_{K7}
R_8	R_{S8}	R_{K8}	R_8	R_{S8}	R_{K8}
R_9	R_{S9}	R_{K9}	R_9	R_{S9}	R_{K9}
R_{10}	R_{S10}	R_{K10}	R_{10}	R_{S10}	R_{K10}

Fig 3- The crossover operation

GA mutation operation

A mutation is performed by making minor changes in the mutated chromosomes. To do this, a random number RN between (0,1) is generated for each gene. If RN is less than a predetermined mutation probability Pm, then the mutation occurs in the gene. Otherwise, it does not. In this research, 0 and10, are chosen as the values of Pm. Note that infeasible chromosomes resulting from this operation do not move to the new population.

GA

Objective function: *f(*x*)*

1. Define Fitness F
2. Initialize population
3. Initialize probabilities of crossover (*pc*) and mutation (*pm*)
4. Do
5. Generate new solutions by crossover and mutation
6. Apply mutation and cross-over to each chromosome
7. Accept the new solution if its fitness increases
8. Select the current best for the next generation
9. While maximum iterations or minimum error criteria are not attained*

*Note that the algorithm stops until a maximum number of 500 iterations is reached.

Particle Swarm Optimization (PSO)

PSO is a technique based on swarm (population) and particles. In this method, each particle is a possible solution to the problem; and moves around in a multidimensional search space. Each particle changes its position according to its location and the position of its neighboring particles. PSO tries to find the optimal solution by moving particles and evaluating the fitness of their new position.

Initial population

The initial population in PSO is generated by creating a particle. In this research, the particles are similar to chromosomes in the GA algorithm. It means that they are from a 3*10martix for 10 products. Every particle has its position and velocity.

PSO fitness function

Fitness functions in PSO algorithms are considered objective functions.

Velocity and Position

After the initialization stage, every particle must be updated by its best local position and also its best global position:

$$v_i^{t+1} = \omega . v_i^t + c_1 . r_1 . (Pbest_i - x_i^t) + c_2 . r_2 . (Gbest_i - x_i^t) \quad (51)$$

$$x_i^{t+1} = x_i^t + v_i^{t+1} \quad (52)$$

Where ‘t’ is the previous iteration, c_1 and c_2 are the individuals and global learning rates, r_1 and r_2 are uniformly random numbers in ranges U= [0 1], and ω is the inertia weight. Commonly, the values of c_1 and c_2 are set equal to 2. Then, two multiples by r_1, and r2 contribute to the social and personal experience equal to each particle. Intertie weight ω is a mechanism for controlling exploration and exploitation, i.e., the contribution of the previous velocity. At the first stage, its value is close to 1 form more exploration and other iterations reduced for more exploitation: $\omega = 0.9\omega$

PSO algorithm

1. Objective function: $f(x)$, $x = (x_1, x_2, ..., x_n)$;
2. Initialize particle position and velocity for each particle and set k = 1.
3. Initialize the particle’s best-known position to its initial position
4. Do
5. Update the best-known position of each particle and swarm’s best-known position
6. Calculate particle velocity according to the velocity equation
7. Update particle position according to the position equation
8. While maximum iterations or minimum error criteria are not attained*

*Note that the algorithm stops until a maximum number of 500 iterations is reached.

5. Numerical example

Unconstraint problem

To illustrate the application of the above-mentioned solution procedure, we will use numerical examples. The parameters of examples are addressed in Table 1.

Table 1- Parameters of the model

i	D	H	π	"	C	C_K	$"_K$	H_K	P	α	l_{min}	l_{max}	λ
1	70	8.00	1.01	29.00	40.00	50.000	19.000	10.000	68.00	0.7	0.1	0.4	5
2	76	7.22	1.03	26.27	36.10	45.125	17.245	9.025	61.37	0.8	0.2	0.4	5
3	82	6.44	1.05	23.54	32.20	40.250	15.490	8.050	54.74	0.8	0.2	0.4	5
4	88	5.66	1.07	20.81	28.30	35.375	13.735	7.075	48.11	0.7	0.2	0.4	10
5	94	4.88	1.09	18.08	24.40	30.500	11.980	6.100	41.48	0.7	0.2	0.4	5
6	100	4.10	1.11	15.35	20.50	25.625	10.225	5.125	34.85	0.5	0.1	0.4	10
7	106	3.32	1.13	12.62	16.60	20.750	8.470	4.150	28.22	0.7	0.1	0.2	5
8	112	2.54	1.15	9.89	12.70	15.875	6.715	3.175	21.59	0.6	0.1	0.2	10
9	118	1.76	1.17	7.16	8.80	11.000	4.960	2.200	14.96	0.7	0.1	0.2	5
10	124	0.98	1.19	4.43	4.90	6.125	3.205	1.225	8.33	0.5	0.1	0.3	10

***oo te taat: θ=0.1 ‘A=50**

The optimal values for uniform and exponential distributions are addressed in Table 2.

Table 2- The optimal solutions of the unconstrained model

	Uniform			Exponential		
	R_S	Q_{1S}	TS	R_S	Q_{1S}	T_S
1	25.8518	17.7291	210.1595	40.8695	20.7984	181.3497
2	28.8247	23.0648	267.1384	44.4957	35.3571	215.7841
3	31.106	24.8912	255.525	48.0696	49.923	206.0894
4	33.366	26.8079	214.727	31.6774	31.8496	73.3584
5	35.6519	28.6414	197.0898	55.2122	79.0533	145.7149
6	36.9305	25.4855	138.0554	35.9423	46.2197	52.5992
7	20.6394	16.0665	69.8089	62.7084	108.1936	109.4345
8	21.8083	16.9869	55.6206	40.7645	60.6251	38.0602
9	23.0089	17.893	40.897	71.1347	137.3088	61.4744
10	35.4927	25.2401	32.0431	46.7342	75.0152	15.2866

Constrained problem

In this section, the same parameters in the previous section are considered and both GA and PSO methods are used to obtain the optimal solution. The specific parameters of those algorithms are presented in Table 3. All of these parameters are obtained by trial and error.

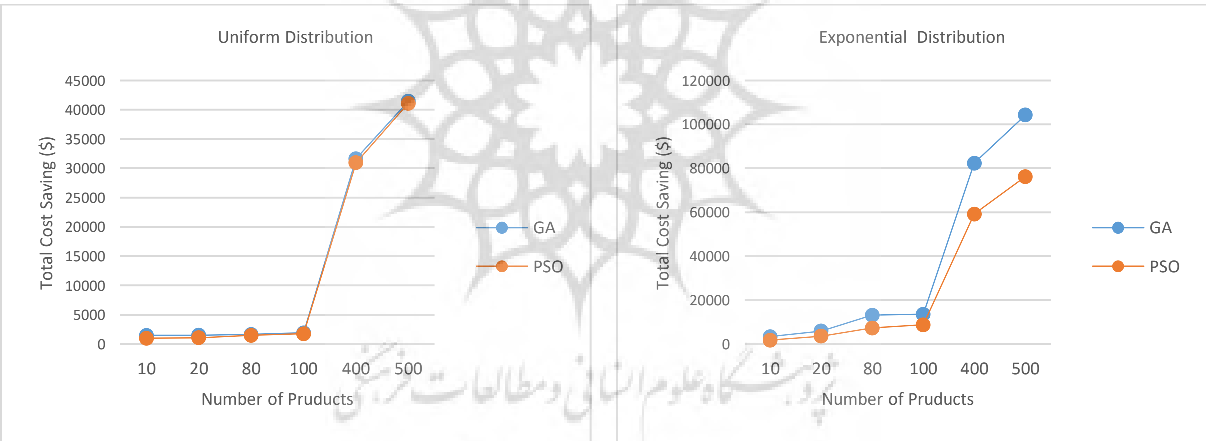
Table 3- parameters of the algorithm

GA Parameters	Values	PSO Parameters	Values
Initial population size	100	Number of initial particles	100
Pc	0.9	ω	The decrease from 0.9 to 0.3
Pm	0.1	C1	2.0
Stop criteria	Reach maximum iteration	C2	2.0
		Stop criteria	Reach maximum iteration

In this section, multiple problems are designed and solved in different sizes, which can be classified into small, medium, and large categories. The first category includes 10 and 20 products; the second category includes 80 and 100 products, and the third category includes 400 and 500 products. The following table shows the differences between time and cost for both uniform and exponential distributions. It should be noted that all values in the tables are written after 10 iterations of each problem (each problem has 500 iteration loops) and its best mode are displayed. In Table 4 and Figure 4, the results are compared.

Table 4- The optimal solutions of the constrained model

Number of products	GA TS($)	GA Time(s)	PSO TS($)	PSO Time(s)
Uniform distribution				
10	1478.9625	14.0776	994.9977	35.7438
20	1509.8552	15.3666	1071.4022	36.0521
80	1658.9988	17.2976	1481.4763	39.0854
100	1924.2439	17.5966	1744.0593	39.3118
400	31649.2948	26.0723	30956.226	49.4003
500	41441.7108	28.5222	41099.5035	54.1311
Exponential distribution				
10	3348.9605	16.1923	1750.0724	36.3512
20	5938.2014	16.9589	3604.1214	36.9368
80	13166.0403	21.6333	7319.1813	42.1105
100	13536.1344	23.4927	8768.8806	44.1671
400	82371.4125	46.2719	59179.1205	70.2324
500	104280.3503	53.6850	76239.5825	77.8652



(a) (b)

Fig. 4- Comparison of optimal solutions by GA and PSO for (a)uniform distribution and (b)exponential distribution

6. Discussion and sensitivity analysis

The selection of parameters is a significant issue in the decision-making context. Thus, to analyze the effects of changes on the maximum value of inventory levels, the order quantity, and total profit, some sensitivity analysis is performed and the results are shown in Tables 4 and 5. The values of each parameter are changed from +75% to -75% for a single product, regardless of the space constraints. Better displaying parameter changes and their effects are shown in Figures 5 and 6.

Table 5- The results of sensitivity analysis for uniform distribution

Parameters	Change (%)	Values			Change (%)		
		Q_S	R_S	$T(R^*_S)$	Q_S	R_S	$T(R^*_S)$
D	75	18.6	23.8	195.8	0.754716981	0.75	0.7497766
	50	15.9	20.4	167.8	0.500000000	0.50	0.4995532
	25	13.2	17.0	139.9	0.245283019	0.25	0.2502234
	0	10.6	13.6	111.9	0.000000000	0.00	0.0000000
	-25	7.9	10.2	83.9	-0.254716981	-0.25	-0.2502230
	-50	5.3	6.8	55.9	-0.500000000	-0.50	-0.5004470
	-75	2.6	3.4	27.9	-0.754716981	-0.75	-0.750670
P	75	10.6	13.6	108.7	0.000000000	0.00	-0.0285970
	50	10.6	13.6	108.8	0.000000000	0.00	-0.0277030
	25	10.6	13.6	109.1	0.000000000	0.00	-0.0250220
	0	10.6	13.6	111.9	0.000000000	0.00	0.0000000
	-25	-	-	-	-	-	-
	-50	-	-	-	-	-	-
	-75	-	-	-	-	-	-
h_k	75	10.6	13.6	125.1	0	0	0.1179625
	50	10.6	13.6	120.6	0	0	0.0777480
	25	10.6	13.6	116.2	0	0	0.0384272
	0	10.6	13.6	111.9	0	0	0.0000000
	-25	10.6	13.6	107.6	0	0	-0.0384270
	-50	10.6	13.6	103.5	0	0	-0.0750670
	-75	10.6	13.6	99.5	0	0	-0.1108130
[lmin lmax]	75	13.3	18.3	146.4	0.254716981	0.3455882	0.3083110
	50	12.4	16.7	134.5	0.169811321	0.2279412	0.2019660
	25	11.5	15.2	123	0.084905660	0.1176471	0.0991957
	0	10.6	13.6	111.9	0.000000000	0.0000000	0.0000000
	-25	9.7	11.9	101.1	-0.084905660	-0.1250000	-0.0965150
	-50	8.8	10.3	90.7	-0.169811321	-0.2426470	-0.1894550
	-75	7.9	8.6	80.7	-0.254716981	-0.3676470	-0.2788200
θ	75	10.7	13.5	114	0.009433962	-0.0073530	0.0187668
	50	10.6	13.5	113.3	0.000000000	-0.0073530	0.0125112
	25	10.6	13.5	112.6	0.000000000	-0.0073530	0.0062556
	0	10.6	13.6	111.9	0.000000000	0.0000000	0.0000000
	-25	10.5	13.6	111.2	-0.009433962	0.0000000	-0.0062560
	-50	10.5	13.6	110.5	-0.009433962	0.0000000	-0.0125110
	-75	10.5	13.7	109.9	-0.009433962	0.0073529	-0.0178730

In figure 5, a comparison is made for the sensitivity results of changing demand rates in uniform distribution.

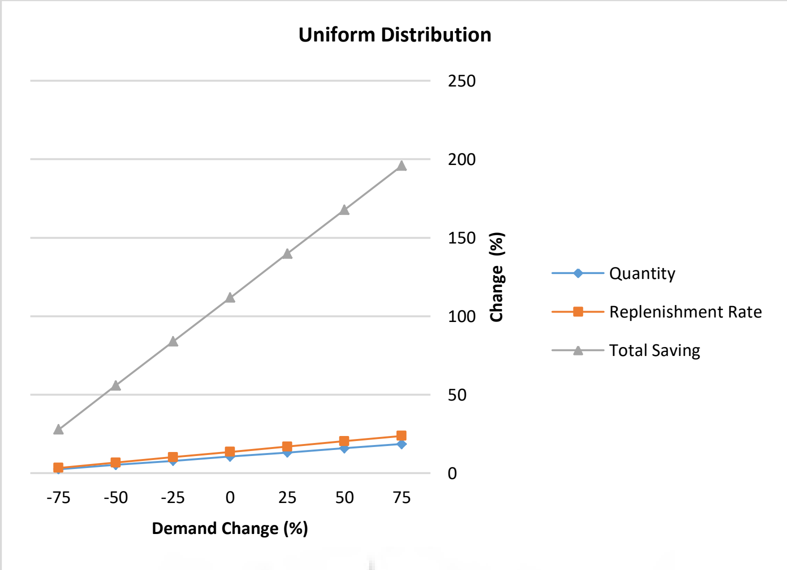


Fig. 5- The sensitivity results in changes in the Demand rate

Table 6- The results of sensitivity analysis for exponential distribution

Parameters	Change (%)	Values			Change (%)		
		Q_S	R_S	$T(R^*_S)$	Q_S	R_S	$T(R^*_S)$
D	75	25.1	71.4	317.50	0.755245	0.750000	0.751241
	50	21.5	61.2	272.10	0.503497	0.500000	0.500827
	25	17.9	51.0	226.70	0.251748	0.250000	0.250414
	0	14.3	40.8	181.30	0.000000	0.000000	0.000000
	-25	10.7	30.6	135.90	-0.251750	-0.250000	-0.250410
	-50	7.1	20.4	90.50	-0.503500	-0.500000	-0.500830
	-75	3.5	10.2	45.10	-0.755240	-0.750000	-0.751240
p	75	14.3	40.8	135.60	0.000000	0.000000	-0.252070
	50	14.3	40.9	141.20	0.000000	0.002451	-0.221180
	25	14.3	40.9	152.10	0.000000	0.002451	-0.161060
	0	14.3	40.8	181.30	0.000000	0.000000	0.000000
	-25	-	-	-	-	-	-
	-50	-	-	-	-	-	-
	-75	-	-	-	-	-	-
h_k	75	14.3	40.9	225.30	0.000000	0.002451	0.242692
	50	14.3	40.9	211.70	0.000000	0.002451	0.167678
	25	14.3	40.9	197.10	0.000000	0.002451	0.087148
	0	14.3	40.8	181.30	0.000000	0.000000	0.000000
	-25	14.3	40.8	164.20	0.000000	0.000000	-0.094320

Parameters	Change (%)	Values			Change (%)		
		Q_S	R_S	$T(R^*_S)$	Q_S	R_S	$T(R^*_S)$
	-50	14.3	40.7	145.70	0.000000	-0.002450	-0.196360
	-75	14.3	40.6	125.40	0.000000	-0.004900	-0.308330
λ	75	8.1	27.6	96.560	-0.433570	-0.323530	-0.467400
	50	9.5	30.8	114.81	-0.335660	-0.245100	-0.366740
	25	11.4	35.0	140.99	-0.202800	-0.142160	-0.222340
	0	14.3	40.8	181.35	0.000000	0.000000	0.000276
	-25	19.1	49.4	250.46	0.335664	0.210784	0.381467
	-50	28.7	63.6	390.73	1.006993	0.558824	1.155157
	-75	56.9	93.6	784.20	2.979021	1.294118	3.325427
θ	75	14.5	37.2	185.30	0.013986	-0.088240	0.022063
	50	14.5	38.3	184.10	0.013986	-0.061270	0.015444
	25	14.4	39.5	182.80	0.006993	-0.031860	0.008274
	0	14.3	40.8	181.30	0.000000	0.000000	0.000000
	-25	14.2	42.3	179.70	-0.006990	0.036765	-0.008830
	-50	14.1	44.0	178.00	-0.013990	0.078431	-0.018200
	-75	14	45.9	176.20	-0.020980	0.125000	-0.028130

In Figure 6, a comparison is made for the sensitivity results of changing λ in the exponential distribution.

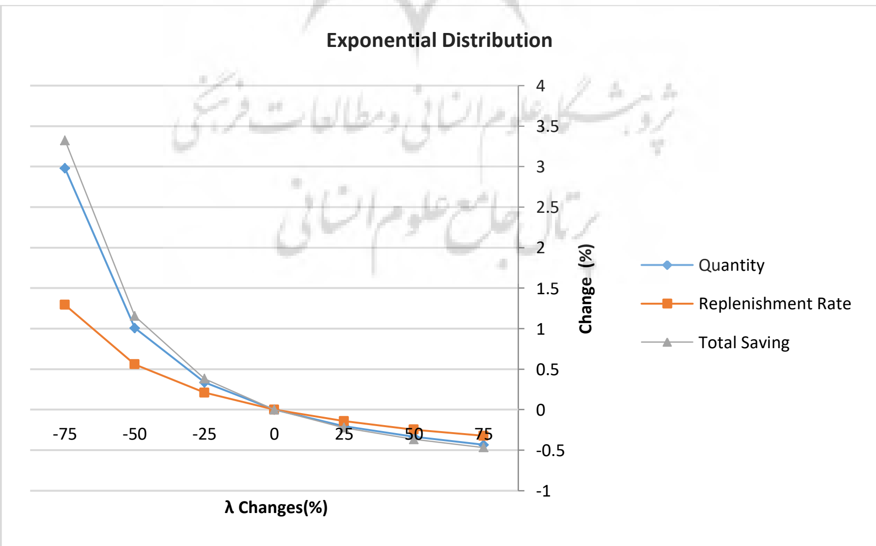


Fig. 6- Tee ssss ttvtty eesssss aa aaaaaλλ λλ λ

According to tables 5, and 6, when the demand rate (D) increases, the replenishment level (R_s), total saving $T(R_s)$, and order quantity (Q_s) increase too. In other words, the replenishment level, order quantity, and total saving are highly sensitive to the demand rate. It means that it is more profitable to place an order when the demand increases. From tables 5 and 6 we can understand that when the price increases, the replenishment level, and order quantity doesn't change but saving cost decreases slightly, so, total saving and price are to some extent sensitive to each other. It is clear that after a price increase, the holding cost does not change replenishment level and order quantity, but directly affects the total saving. It means that the replenishment level and order quantity are not sanative but the total saving value is slightly sensitive to the changes in holding cost. According to Table 6, when the λ (mean number/year) in exponential distribution increases the replenishment level, order quantity, and total saving decrease. In other words, all three items are moderately sensitive to λ. Table 5 shows direct but little interaction between the maximum and the minimum amounts of allowable time in uniform distribution and replenishment level, order quantity, and total saving. Finally, according to Table 5,6, we can see that there are positive effects of deteriorating rate on profit and order quantity level but negative effects on replenishment level. It is worth noting that in all cases, the effects are low. Therefore, it must be said that the customer should use a special-order policy when the orders include high-deterioration rate products.

6.1 Theoretical implications

According to previous research, including Taleizadeh et al. (2013c), Yang et al. (2015), Karimi-Nasab & Wee (2015), and Taleizadeh, Zarei, & Sarker (2016), it can be found that they did not include real-world constraints and more attention has been paid to the increase in price over time due to a random delivery period and a motivational policy. In several studies, there are no limitations and the problem is modeled and solved with integer decision variables and linear programming (Zeballos, Seifert & Protopappa-Sieke, 2013; Sarkar & Moon, 2014; Giri & Sharma, 2016; Braglia, Castellano & Frosolini, 2016; Braglia, Castellano, & Song, 2017). Therefore, in studies that focus exclusively on known price increases, the gap is quite evident when a study is aimed at getting a better understanding of the real-world conditions (Cimen & Kirkbride, 2017). Therefore, this research seeks to consider goods that do not have a stable lifespan, as well as storage space limitations and problem-solving by using meta-heuristics algorithms.

Meta-hubristic algorithms such as Genetic Algorithm (GA) and Particle Swarm Optimization (PSO) can be used in inventory control to obtain optimal reorder points (Dye, 2012; Mousavi et al., 2014; Buhnia, Shaikh & Gupta, 2015; Bhunia & Shaikh, 2015;

Vandani, Niaki, & Aslanzade, 2017; Akbari Kaasgari, Imani, & Mahmoodjanloo, 2017; Azadeh, 2017; Hiassat, Diabat & Rahwan, 2017; Tiwari, 2017). In general, scholars suggest that hybrid meta-heuristic algorithms have gained considerable attention for their capability to solve difficult problems in different fields of science especially solving inventory problems. Due to the non-linearity of the proposed model of this study, particle swarm optimization (PSO) and genetic algorithm (GA), were implemented as optimizing solvers instead of analytical methods (Talezadeh et al., 2013b; Alejo-Reyes et al., 2020).

7. Conclusions

The models presented in this study were solved with consideration of the relevant assumptions along with numerical examples. The derivation method was used to solve the unconstrained problem, and both the genetic algorithm and particle swarm optimization algorithm were used for the constrained problem. Accordingly, the optimum value of the model was calculated. Changes in profits, replenishment level, and order quantities were examined for some of the parameters, including demand, purchase price, and holding cost. Sensitivity analysis indicated that an increase in deterioration and demand rates leads to increased total profits. Also, the fewer the number of replenishment periods in one year, the more is cost-effective it. It has also been shown that the genetic algorithm has the best ability to converge in comparison with the particle swarm algorithm, and in less time, it becomes more desirable.

In general, based on the findings of this research, managers and retailers would be able to have more effective plans for their inventory and replenishment levels under changing circumstances. The research model is a real-world inventory control problem that has been observed in many cases, such as small supermarkets, pharmacies, grocery stores, and so on. This model helps the suppliers decide when to visit and replenish the retail inventory. Hence, suppliers can visit retailers at irregular intervals. The purpose of this study was to determine the retailer’s optimal order quantity and maximize the benefits.

7.1 Research limitations and future study agenda

In this research, some parameters such as delay in payment, pre-payment policies, and also financial constraints were not considered effective variables, hence they should be noted as the limitations of this study. The proposed model of this study could be extended in several ways. It may deal with the demand rate as a function of price, time, stock, etc., considering the delayed payment and advanced payment policy. It is also possible to use other meta-heuristic algorithms, hybrid algorithms, considering fuzzy parameters, and adding more constraints to the model including limitations on order quantities and also financial constraints.

References

Abad, P.L. (2008).Optimal price and order size under partial backordering incorporating shortage, backorder and lost sale costs, *International Journal of Production Economics*, 114, 179-186.

Akbari Kaasgari, M., Imani, D.M., & Mahmoodjanloo, M. (2017). Optimizing a vendor-managed inventory (VMI) supply chain for perishable products by considering discount: Two calibrated meta-heuristic algorithms. *Computers & Industrial Engineering*, 103, 227-241.

Alejo-Reyes.A., Olivares-Benitez.E., Mendoza, A., & Rodriguez, A. (2020). Inventory replenishment decision model for the supplier selection problem using metaheuristic algorithms. *Mathematical Biosciences & Engineering*, 17(3), 2016-2036.

Amorim, P., Costa, A., Almada-Lobo, M. B. (2013). Innocence of consumer purchasing behavior on the production planning of perishable food. *OR Spectrum*, 36(3), 669-692.

Asif, I. M. &, Biswajit .S (2018). Optimizing a Multi-Product Continuous-Review Inventory Model With Uncertain Demand, Quality Improvement, Setup Cost Reduction, and Variation Control in Lead Time. *IEEE Access*,6, 36176-36187.

Azadeh, A., (2017). A genetic algorithm-Taguchi-based approach to inventory routing problem of a single perishable product with transshipment. *Computers & Industrial Engineering*, 104, 124-133.

Babangida, B & Baraya, Y. (2020). An inventory model for non-instantaneous deteriorating items with time-dependent quadratic demand, *International Journal of Modelling in Operations Management*, 8(1),1-44

Babangida, B. & Baraya, Y. (2020). An inventory model for non-instantaneous deteriorating items with time-dependent quadratic demand, two storage facilities and shortages under trade credit policy. *International Journal of Modelling in Operations Management*, 8, 1-44.

Banerjee, S. & S. Agrawal. (2017). Inventory model for deteriorating items with freshness and price dependent demand: Optimal discounting and ordering policies. *Applied Mathematical Modelling*, 52, 53-64.

Bhunia, A.K. & Shaikh, A.A. (2015), An application of PSO in a two-warehouse inventory model for the deteriorating item under permissible delay in payment with different inventory policies. *Applied Mathematics & Computation*,256 , 831-850.

Bhunia, A.K., Shaikh, A.A., & Gupta, R.K. (2015). A study on two-warehouse partially backlogged deteriorating inventory models under inflation via particle swarm optimization. *International Journal of Systems Science*, 46 (6), 1036-1050.

Bischak, D.P., Robb, D.J., Silver, E.A., & Blackburn, J.D. (2014). The analysis and management of periodic review, order-up-to-level inventory systems with order crossover. *Production & Operations Management*, 23, 762-772.

Bounkhel, M.; Tadj, L.; Benhadid, Y & Hedjar, R. (2019). Optimal Control of Nonsmooth Production Systems with Deteriorating Items Stock-Dependent Demand, with or without Backorders. *Symmetry*, 11(2), 183.

Bradley, Stephen P., Hax, Arnoldo C., Magnanti, & Thomas, L. (1977). *Applied Mathematical Programming*, Addison-Wesley, Chapter 13.

Braglia, M., Castellano, D., & Frosolini, M. (2016). Joint-replenishment problem under stochastic demands with backorders-lost sales mixtures, controllable lead times, and investment to reduce the major ordering cost. *Journal of Operational Research Society*, 67(8), 1108-1120.

Braglia, M., Castellano, D., & Song .D. (2017). Distribution-free approach for a stochastic joint-replenishment problem with backorders-lost sales mixtures, and controllable major ordering cost and lead times. *Compute & Operation Research*, 79, 161-173.

Caloieroa, G., Strozzia, F., Comenges, J.-M.Z. (2008). A supply chain as a series of filters or amplifiers of the bullwhip effect. *International Journal of Production Economics*, 114 (2), 631-645.

Chen, W., J. Li, & X. Jin (2016). The replenishment policy of agri-products with stochastic demand in integrated agricultural supply chains. *Expert Systems with Applications*, 48, 55-66.

Chiang, C. (2013). A note on periodic review inventory models with stochastic supplier's visit intervals and fixed ordering cost, *International Journal of Production Economics*, 146, 662- 666.

Çimen, M., & Kirkbride, C. (2017). Approximate dynamic programming algorithms for multidimensional flexible production-inventory problems. *International Journal of Production Research*, 55(7), 2034-2050.

Diabat, A., Taleizadeh, A.A., & Lashgari .M. (2017). A lot-sizing model with partial downstream delayed payment, partial upstream advance payment, and partial backordering for deteriorating items. *Journal of Manufacturing Systems*, 45, 322-342.

Dye, C.-Y., A (2012). finite horizon deteriorating inventory model with two-phase pricing and time-varying demand and cost under trade credit financing using particle swarm optimization. *Swarm & Evolutionary Computation*, 5, 37-53.

Ghosh, A.K. (2003). On some inventory models involving shortages under an announced price increase, *International Journal of Systems Science*, 34(2), 129-137.

Giri, B. C. & Sharma, S (2016). Optimal ordering policy for an inventory system with linearly increasing demand and allowable shortages under two levels trade credit financing. *Operational Research*., 16(1), 25-50.

Goyal, S.K., Giri, B.C. (2001). Recent trends in modeling of deteriorating inventory. *European Journal of Operational Research*, 134(1), 1–6.

Herbon, A (2017). A non-cooperative game model for managing a multiple-aged expiring inventory under consumers' heterogeneity to price and time. *Applied Mathematical Modelling*, 51, 38-57.

Herbon, A. & Khmelnitsky, E. (2017). Optimal dynamic pricing and ordering of a perishable product under additive effects of price and time on demand. *European Journal of Operational Research*, 260 (2), 546-556.

Hiassat, A, Diabat A, Rahwan I (2017) A genetic algorithm approach for location-inventory-routing problem with perishable products. *Journal of Manufacturing Systems*, 42, 93-103.

Hong, S., Kim, Y.H. (2009). A genetic algorithm for joint replenishment based on the exact inventory cost. *Compute & Operation Research*, 36,167–175.

Hsu, W.K. & Yu, H.F. (2011). An EOQ model with imperfective quality items under an announced price increase. *Journal of the Chinese Institute of Industrial Engineers*, 28, 34-44.

Huang, W., Kulkarni, V.G. & Swaminathan, J.M. (2003). Optimal EOQ for announced price increase in Infinite Horizon. *Operations Research*, 51, 336-339.

Jaggi, C.K., Tiwari, S., & Goel, S.K. (2017). Credit financing in economic ordering policies for non-instantaneous deteriorating items with price dependent demand and two storage facilities. *Annals of Operations Research*,248 (1-2), 253-280.

Janssen, L. (2018a), A stochastic micro-periodic age-based inventory replenishment policy for perishable goods, *Transportation Research Part E-Logistics & Transportation Review*, 118, 445-465.

Janssen, L. (2018b). Development and simulation analysis of a new perishable inventory model with a closing days constraint under non-stationary stochastic demand. *Computers & Industrial Engineering*, 118, 9-22.

Karimi-Nasab, M., & Wee, H.M. (2015). An inventory model with truncated exponential replenishment intervals and special sale offer. *Journal of Manufacturing Systems*, 35, 246-250.

Kaya, O. & Ghahroodi S.R. (2018). Inventory control and pricing for perishable products under age and price dependent stochastic demand. *Mathematical Methods of Operations Research*, 88 (1), 1-35.

Lashgari, M., Taleizadeh A.A., & Sadjadi, S.J. (2018). Ordering policies for non-instantaneous deteriorating items under hybrid partial prepayment, partial trade credit, and partial back-ordering. *Journal of the Operational Research Society,* 69 (8), 1167-1196.

Li, R. & Teng, J.T. (2018). Pricing and lot-sizing decisions for perishable goods when demand depends on selling price, reference price, product freshness, and displayed stocks. *European Journal of Operational Research,* 270 (3), 1099-1108.

Maihami, R., Karimi, B. (2014). Optimizing the pricing and replenishment policy for non-instantaneous deteriorating items with stochastic demand and promotional efforts. *Computers & Operations Research*, 51, 302-312.

Malik, A.I.; Sarkar, B (2018). Optimizing a multi-product continuous-review inventory model with uncertain demand, quality improvement, setup cost reduction, and variation control in lead time. IEEE Access , 6, 36176–36187.

Moon, I., Giri, B.C., Ko, B. (2005). Economic order quantity models for ameliorating/deteriorating items under inflation and time discounting. *European Journal of Operational Research,* 162(3), 773–785.

Mousavi, S. M., Sadeghi, J., Niaki, S. T., Alika .A., Bahreininejad .N., & Metselaar .H. S. C. (2014). Two parameter-tuned meta-heuristics for a discounted inventory control problem in a fuzzy environment. *Information Sciences,* 276, 42-62.

Muthuraman, K., Seshadri, S., & Wu, Q. (2014). Inventory management with stochastic lead times. *Mathematics of Operations Research*, 40, 302-32.

Naddor, E. (1966), *Inventory systems*. New York: John Wiley & sons (Chapter 1).

Neeraj K. & Kumar.S (2017). An inventory model for deteriorating items with partial backlogging using linear demand in fuzzy environment. *Cogent Business & Management*, 4(1), 1307687, https://doi.org/10.1080/23311975.2017.1307687.

Ouyang, L. Y. Teng, J. T., Goyal, S. K. & Yang, C. T (2008). An economic order quantity model for deteriorating items with partially permissible delay in payments linked to order quantity. *European Journal of Operational Research*, 194(2), 418-431.

Ouyang, L.Y (2016). Optimal order policy in response to an announced price increase for deteriorating items with limited special-order quantity, *International Journal of Systems Science*, 47 (3), 718-729.

Pal, H., Bardhan .S., & Giri, B.C (2018). Optimal replenishment policy for non-instantaneously perishable items with preservation technology and random deterioration start time. *International Journal of Management Science & Engineering Management*, 13 (3), 188-199.

Palak, G., Sioglu, S.D. & Geunes, J. (2018). Heuristic algorithms for inventory replenishment with perishable products and multiple transportation modes. *IISE Transactions*, 50 (4), 345-365.

Palanivel, M., Uthayakumar & Finite, R. (2015). Horizon EOQ model for non-instantaneous deteriorating items with price and advertisement dependent demand and partial backlogging under inflation. *International Journal of Systems Science*,46 (10), 1762-1773.

Pan, W. (2017). Medical resource inventory model for emergency preparation with uncertain demand and stochastic occurrence time under considering different risk preferences at the airport. *Plos One*, 12(9), 16.

Prekopa, A. (2006). On the Hungarian inventory control model. *European Journal of Operational Research*, 171 (3), 894-914.

Rabbani, M., Pourmohammad, N., & Rafiei, H. (2016). Joint optimal dynamic pricing and replenishment policies for items with simultaneous quality and physical quantity deterioration. *Applied Mathematics & Computation*, 287-288, 149-160.

Sarkar B. & Moon, I. (2014). Improved quality, setup cost reduction, and variable backorder costs in an imperfect production process. *International Journal of Production Economics*, 155, 204-213.

Sarker, B.R., & Kindi, M. (2006). Optimal ordering policies in response to a discount offer. *International Journal of Production Economics*, 100, 195-211.

Sazavar, Z., Mirzapour Al-e-hashem .S.M.J., Govindan,K and Bahli, B (2016). A novel mathematical model for a multi-period, multi-product optimal ordering problem considering expiry dates in a FEFO system. *Transportation Research Part E-Logistics & Transportation Review*, 93, 232-261.

Sharma, S. (2009). On price increase and temporary price reduction with partial back-ordering. *European Journal of Industrial Engineering*, 3, 70-89.

Shu, M.H., Huang, J.C., & Fu, Y.C. (2015). A production-delivery lot-sizing policy with stochastic delivery time and in consideration of transportation cost. *Applied Mathematical Modeling*, 39, 2981-2993.

Singh, S. R., Kumar, N. and Kumari, R (2011). Two-ware house fuzzy inventory model under the conditions of permissible delay in payments. *International Journal of Operations Research*, 11, 78-99.

Soni, N.H. & Suthar, D.N. (2020). Optimal pricing and replenishment policy for non-instantaneous deteriorating items with varying rate of demand and partial backlogging. *Operational Research*, 57, 986-1021.

Taleizadeh A. A., Akhavan Niaki. S. T and, Makui. A (2012). Multiproduct multiple-buyer single-vendor supply chain problem with stochastic demand, variable lead-time, and multi-chance constraint, *Expert Systems with Applications*, 39(5), 5338-5348.

Taleizadeh A.A., Pentico.D. , Jabalameli .M & Aryanezhada, M. (2013a), An economic order quantity model with multiple partial prepayments and partial back-ordering, *Mathematical & Computer Modelling*, 57,311–323.

Taleizadeh A.A., Pentico D.W., Jabalameli M.S. & Aryanezhad M.B. (2013b). An Economic Order Quantity Model with Multiple Partial Prepayments and Partial Backordering, *Mathematical & Computer Modeling*, 57 (3-4), 311–323.

Taleizadeh, A.A (2014). An EOQ model with partial back-ordering and advance payments for an evaporating item. *International Journal of Production Economics*. 155, 185–193.

Taleizadeh, A.A., Akhavan Niaki, S.T., & Jalali Naini, G., (2013c). Optimizing multiproduct multi-constraint inventory control systems with stochastic period length and emergency order, *Journal of Uncertain Systems*, 7, 58-71.

Taleizadeh, A.A., Khanbaglo, M.P.S. & Cárdenas-Barrón, L.E (2020), Replenishment of imperfect items in an EOQ inventory model with partial backordering, *Rairo Operations Research*, 54, 413–434.

Taleizadeh, A.A., Zarei, & Sarker, B.R. (2016). An optimal control of inventory under probabilistic replenishment intervals and known price increase. *European Journal of Operational Research*, 257(3), 777-791.

Taleizadeh, A.A., Pentico D.W., Aryanezhad M.B & Jabalameli M.S (2013). An EOQ Problem under Partial Delayed Payment and Partial Backordering. *Omega*, 41(2), 354-368

Tashakkor, N., Mirmohammadi, S.H., & Iranpoor, M. (2018). Joint optimization of dynamic pricing and replenishment cycle considering variable non-instantaneous deterioration and stock-dependent demand. *Computers & Industrial Engineering*, 123, 232-241.

Tiwari, S. (2017). Two-warehouse inventory model for non-instantaneous deteriorating items with stock dependent demand and inflation using particle swarm optimization. *Annals of Operations Research*, 254 (1-2), 401-423.

Tiwari, S., Cárdenas-Barrón, L.E., Goh, M & Shaikh, A.A (2018). Joint pricing and inventory model for deteriorating items with expiration dates and partial backlogging under two-level partial trade credits in the supply chain. *International Journal of Production Economics*, 200, 16–36.

Tsao, Y.C. & Linh, V.T. (2016). Supply Chain Network Designs Developed for Deteriorating Items Under Conditions of Trade Credit and Partial Backordering. *Networks & Spatial Economics*, 16 (3), 933- 956.

Vandani, B., Niaki, S.T.A. & Aslanzade, S. (2017). Production-inventory-routing coordination with capacity and time window constraints for perishable products: Heuristic and meta-heuristic algorithms. *Journal of Cleaner Production*, 161, 598-618.

Wang, W.T., Wee, H.M., Cheng, Y.L., Wen, C.L. & Cárdenas-Barrón, L.E. (2015). EOQ model for imperfect quality items with partial backorders & screening constraint. *European Journal of Industrial Engineering*, 9, 744.

Yang, H.L. (2006). Two-warehouse partial backlogging inventory models for deteriorating items under inflation, *International Journal of Production Economics*, 103, 362-370.

Yang, C. T., C. Y. Dye, and J.F. Ding (2015). Optimal dynamic trade credit and preservation technology allocation for a deteriorating inventory model. *Computers & Industrial Engineering*, 87, 356-369.

Yang, P. C., Wee, H. M. (2003). An integrated multi-lot-size production inventory model for the deteriorating item. Computers& Operations Research, 30(5), 671-682.

Yu, Y., Wang, Z. and Liang, L. A (2012). vendor managed inventory supply chain with deteriorating raw materials and products, *International Journal of Production Economics*, 136(2), 266-274.

Yao M.J, Chu W.M. (2008), A genetic algorithm for determining optimal replenishment cycles to minimize maximum warehouse space requirements, *Omega*, 36, 619–631.

Zeballos, A., Seifert, R. & Protopappa-Sieke, M. (2013). Single product, finite horizon, periodic review inventory model with working capital requirements and short-term debt. *Computers & Operations Research*, 40(12), 2940-2949.

Zhang, H.N., X.L. Chao, & Shi.C. (2018). perishable inventory systems: convexity results for base-stock policies and learning algorithms under censored demand. *Operations Research*, 66 (5), 1276-1286.

Appendix:

1- Calculating the value of R_i (optimal decision variables before special order)

(a) Uniform distribution:

$$TS_i = \sum_{i=1}^{n} (P_i Q_{2i} - [C_i Q_{1i} + h_i \bar{I}_i + \pi'_i \bar{L}_i + \pi_i \bar{B}_i + A_i]) \tag{A1}$$

Where $Q_{1i}, Q_{2i}, \overline{B}_i, \overline{L}_i, \overline{I}_i$ are defined in Equations 23-27.

By replacing and simplifying such equations, the profit function is rewritten as follows:

$$\begin{aligned} TS(R_i) = [\{ & \frac{P_i}{t_{maxi} - t_{mini}} \left\{ (1-\alpha_i) R_i t_{maxi} + \frac{D_i}{2} (\alpha_i t_{maxi}^2 - t_{mini}^2) - \frac{R_i^2}{2D_i} (1-\alpha_i) \right\} \} \\ & \frac{-C_i}{t_{maxi} - t_{mini}} \left\{ (1-\alpha_i) R_i t_{maxi} + \frac{D_i}{2} (\alpha_i t_{maxi}^2 - t_{mini}^2) + \frac{R_{si}^2}{2D_i} \theta_i (t_{maxi} - t_{mini}) - \frac{R_i^2}{2D_i} (1-\alpha_i) \right\} \\ & - \frac{h_i}{t_{maxi} - t_{mini}} \left\{ \frac{R_i^2}{2D_i} (t_{maxi} - t_{mini}) \right\} \\ & - \frac{(1-\alpha_i)\pi_i'}{t_{maxi} - t_{mini}} \left\{ \frac{D_i t_{maxi}^2}{2} + \frac{R_{si}^2}{2D_i} - R_i t_{maxi} \right\} \\ & - \frac{\alpha_i \pi_i}{t_{maxi} - t_{mini}} \left\{ \frac{D_i t_{maxi}^2}{2} + \frac{R_{si}^2}{2D_i} - R_i t_{maxi} \right\} - A_i] \end{aligned} \tag{A2}$$

The first and second derivatives of the above term are calculated for obtaining R_i as follows:

$$\begin{aligned} \frac{dTS(R_i)}{dR_i} = [& \frac{P_i}{t_{maxi} - t_{mini}} ((1-\alpha_i) t_{maxi} - (1-\alpha_i) \frac{R_i}{D_i}) \\ & \frac{-C_i}{t_{maxi} - t_{mini}} \left((1-\alpha_i) t_{maxi} - \frac{R_i}{D_i} (1-\alpha_i) + \frac{R_i}{D_i} \theta_i (t_{maxi} - t_{mini}) \right) \\ & - \frac{h_i}{t_{maxi} - t_{mini}} \left(\frac{R_i}{D_i} (t_{maxi} - t_{mini}) \right) - \frac{(1-\alpha_i)\pi_i'}{t_{maxi} - t_{mini}} \left(\frac{R_i}{D_i} - t_{maxi} \right) \\ & - \frac{\alpha_i \pi_i}{t_{maxi} - t_{mini}} \left(\frac{R_i}{D_i} - t_{maxi} \right) \end{aligned} \tag{A3}$$

It is obvious that to determine the optimal value of R, the concavity of the function must be proved in the second derivation, as follows:

$$\frac{d^2TS(R_i)}{dR_i} = \frac{-P_i(1-\alpha_i)}{(t_{maxi}-t_{mini})D_i} + \frac{C_i(1-\alpha_i)}{(t_{maxi}-t_{mini})D_i} - \frac{\theta_i\, C_i}{D_i} - \frac{h_i}{D_i} - \frac{(1-\alpha_i)\pi_i{}'}{(t_{maxi}-t_{mini})D_i} - \frac{\alpha_i\pi_i}{(t_{maxi}-t_{mini})D_i} \tag{A4}$$

Given that $Pi > Ci$, the concavity of the function is proved and the optimal value for the function is obtained as follows:

$$R_i^* = \frac{-(1-\alpha_i)t_{maxi}(P_i-C_i) - t_{maxi}(1-\alpha_i)\pi_i{}' - t_{maxi}\alpha_i\pi_i}{\frac{1}{D_i}\{(1-\alpha_i)t_{maxi}(P_i-C_i) - \theta_i\,(t_{maxi}-t_{mini})(C_i) - (1-\alpha_i)\pi_i' - \alpha_i\pi_i - (t_{maxi}} \tag{A5}$$

(b) Exponential distribution:

Similarly, after some simplifications by using $Q_{1i}, Q_{2i}, \overline{B}_i, \overline{L}_i, \overline{I}_i$ that are defined in Eq39 to 43 $TS(R_i)$ it transforms to:

$$TS(R_i) = P_i\left\{\frac{D_i}{\lambda_i}e^{-\lambda_i\frac{R_i}{D_i}}(\alpha_i-1) + \frac{D_i}{\lambda_i}\right\} - C_i\left\{\frac{D_i}{\theta_i}e^{-\lambda_i\frac{R_i}{D_i}}\left(\frac{\alpha_i\theta_i}{\lambda_i} - \frac{\theta_i}{\theta_i+\lambda_i}\right) + \frac{D_i}{\theta_i}e^{\theta_i\frac{R_i}{D_i}}\left(\frac{\theta_i}{\theta_i+\lambda_i}\right)\right\} - h_i\left\{\frac{e^{\theta_i\frac{R_i}{D_i}}D_i}{\theta_i\,(\theta_i+\lambda_i)} + \frac{D_i e^{-\lambda_i\frac{R_i}{D_i}}}{\lambda_i\,(\theta_i+\lambda_i)} - \frac{D_i}{\lambda_i\,\theta_i}\right\} - \pi_i'\left\{(1-\alpha_i)\frac{D_i}{\lambda_i}e^{-\lambda_i\frac{R_i}{D_i}}\right\} - \pi_i\left\{\alpha_i\,\frac{D_i}{\lambda_i}e^{-\lambda_i\frac{R_i}{D_i}}\right\} \tag{A6}$$

The first derivative is equal to:

$$\frac{dTS(R_i)}{dR_i} = P_i\left\{-e^{-\lambda_i\frac{R_i}{D_i}}(\alpha_i-1)\right\} - C_i\left\{\frac{-\lambda_i}{\theta_i}e^{-\lambda_i\frac{R_i}{D_i}}\left(\frac{\alpha_i\theta_i}{\lambda_i} - \frac{\theta_i}{\theta_i+\lambda_i}\right) + e^{\theta_i\frac{R_i}{D_i}}\left(\frac{\theta_i}{\theta_i+\lambda_i}\right)\right\} - H_i\left\{\frac{e^{\theta_i\frac{R_i}{D_i}}}{(\theta_i+\lambda_i)} - \frac{e^{-\lambda_i\frac{R_i}{D_i}}}{(\theta_i+\lambda_i)}\right\} - \pi_i'\left\{-(1-\alpha_i)e^{-\lambda_i\frac{R_i}{D_i}}\right\} - \pi_i\left\{-(\alpha_i)e^{-\lambda_i\frac{R_i}{D_i}}\right\} \tag{A7}$$

and the second derivative is as follows:

$$\frac{d^2TS(R_i)}{dR_i} = P_i\left\{e^{-\lambda_i\frac{R_i}{D_i}}(\alpha_i - 1)\frac{\lambda_i}{D_i}\right\} - C_i\left\{\frac{{\lambda_i}^2}{D_i\theta_i}e^{-\lambda_i\frac{R_i}{D_i}}\left(\frac{\alpha_i\theta_i}{\lambda_i} - \frac{\theta_i}{\theta_i+\lambda_i}\right) + e^{\theta_i\frac{R_i}{D_i}}\left(\frac{{\theta_i}^2}{D_i(\theta_i+\lambda_i)}\right)\right\}$$

$$-\frac{h_i}{D_i}\left\{\frac{\theta_i e^{\theta_i\frac{R_i}{D_i}}}{(\theta_i+\lambda_i)} + \frac{\lambda_i e^{-\lambda_i\frac{R_i}{D_i}}}{(\theta_i+\lambda_i)}\right\} - \frac{\lambda_i\pi'_i}{D_i}\left\{(1-\alpha_i)e^{-\lambda_i\frac{R_i}{D_i}}\right\} - \frac{\lambda_i\pi_i}{D_i}\left\{(\alpha_i)e^{-\lambda_i\frac{R_i}{D_i}}\right\} \quad \text{(A8)}$$

According to the definition and type of setting to problem parameters, the above function is always negative. Therefore, the optimal value of the function that is the root of the first derivative is calculated as:

$$R_i^* = \frac{D_i\{ln\left(\left(\frac{(C_i)\theta_i}{\theta_i+\lambda_i}\right) + \left(\frac{h_i}{(\theta_i+\lambda_i)}\right)\right) - ln\left(\left(\frac{(C_i)\lambda_i}{\theta_i}\left(\frac{\alpha_i\theta_i}{\lambda_i} - \frac{\theta_i}{\theta_i+\lambda_i}\right)\right) + \left(\frac{h_i}{(\theta_i+\lambda_i)}\right) + \pi'_i(1-\alpha_i) + \pi_i\alpha_i - P_i(\alpha_i\right.}{(-\theta_i-\lambda_i)} \quad \text{(A9)}$$

2- Calculating the value of R_{ki} (optimal decision variables after special order)

(a) *Uniform distribution*

In this situation the profit is:

$$\mathrm{T}S = \sum_{i=1}^{n}(P_iQ_{2ki} - [C_{ki}Q_{1ki} + \overline{I_{ki}}h_{ki} + \pi'_{ki}\overline{L_{ki}} + \pi_i\overline{B_{ki}} + A_i) \quad \text{(A10)}$$

The same as the upper concavity of the function is proved, the maximum value of the optimal inventory level is as follows:

$$R_{ki}^* = \frac{-(1-\alpha_i)t_{maxi}(P_i - C_{ki}) - t_{maxi}(1-\alpha_i){\pi_i}' - t_{maxi}\alpha_i\pi_i}{\frac{1}{D_i}\{(1-\alpha_i)t_{maxi}(P_i - C_{ki}) - \theta_i(t_{maxi} - t_{mini})(C_{ki}) - (1-\alpha_i)\pi'_{ki} - \alpha_i\pi_i - (t_{maxi} - t_{mini})h_{ki}\}} \quad \text{(A11)}$$

(b) Exponential distribution:

$$R_{ki}^* = \frac{D_i\{ln\left(\left(\frac{(C_{ki})\theta_i}{\theta_i+\lambda_i}\right) + \left(\frac{h_{ki}}{(\theta_i+\lambda_i)}\right)\right) - ln\left(\left(\frac{(C_{ki})\lambda_i}{\theta_i}\left(\frac{\alpha_i\theta_i}{\lambda_i} - \frac{\theta_i}{\theta_i+\lambda_i}\right)\right) + \left(\frac{h_{ki}}{(\theta_i+\lambda_i)}\right) + \pi'_{ki}(1-\alpha_i) + \pi_i\alpha_i -\right.}{(-\theta_i-\lambda_i)} \quad \text{(A12)}$$

3- Calculating the value of R_{si} (optimal decision variables after special order)

(a) Uniform distribution

The profit function is as follows:

$$TS = \sum_{i=1}^{n} [Q_{2si}\ \varphi_{1i} - Q_{1si}\ \varphi_{2i} - \overline{I_{Si}}\ \varphi_{3i} - \overline{B_{Si}}\ \varphi_{4i}\ - \overline{L_{si}}\ \varphi_{5i}\ - A_i + \varphi_{6i}] \qquad (A13)$$

The values of $Q_{1si}, \overline{L}_{is}, \overline{B}_{is}, \overline{I}_{is}, Q_{2si}$ are replaced with equations 12 to 16:

$$TS = \sum_{i=1}^{n} [(\frac{\varphi_{1i}}{t_{maxi} - t_{mini}} \left\{ (1-\alpha_i) R_{si} t_{maxi} + \frac{D_i}{2}(\alpha_i t_{maxi}^2 - t_{mini}^2) - \frac{R_{si}^2}{2D_i}(1-\alpha_i) \right\}) \qquad (A14)$$

$$-(\frac{\varphi_{2i}}{t_{maxi} - t_{mini}} \left\{ (1-\alpha_i) R_{si} t_{maxi} + \frac{D_i}{2}(\alpha_i t_{maxi}^2 - t_{mini}^2) + \frac{R_{si}^2}{2D_i}\theta_i (t_{maxi} - t_{mini}) - \frac{R_{si}^2}{2D_i}(1-\alpha_i) \right\})$$

$$-(\frac{\varphi_{3i}}{t_{maxi} - t_{mini}} \left\{ \frac{R_{si}^2}{2D_i}(t_{maxi} - t_{mini}) \right\})$$

$$-(\frac{\varphi_{4i}\ \alpha_i}{t_{maxi} - t_{mini}} \left\{ \frac{D_i t_{maxi}^2}{2} + \frac{R_{si}^2}{2D_i} - R_{si} t_{maxi} \right\})$$

$$-(\frac{\varphi_{5i}\ (1-\alpha_i)}{t_{maxi} - t_{mini}} \left\{ \frac{D_i t_{maxi}^2}{2} + \frac{R_{si}^2}{2D_i} - R_{si} t_{maxi} \right\})\ - A_i + \varphi_{6i}]$$

By obtaining the values of the variables R_i and R_{ki} from A5 and A11 for uniform distribution, A9, A11 for exponential distribution and replacing them in the internal variable we can obtain φ_1 to φ_6. It can be seen that φ_1 to φ_6 are constant, hence the above function is quadratic and concave as previously proved, therefore the optimal values are as follows:

$$R_{si}^* = \frac{-(1-\alpha_i) t_{maxi}(\varphi_{1i} - \varphi_{2i}) - t_{maxi}(1-\alpha_i)\varphi_{5i} - t_{maxi}\alpha_i\varphi_{4i}}{\frac{1}{D_i}\{(1-\alpha_i)t_{maxi}(\varphi_{1i} - \varphi_{2i}) - \theta_i\,(t_{maxi} - t_{mini})(\varphi_{2i}) - (1-\alpha_i)\varphi_{5i} - \alpha_i\varphi_{4i} - (t_{maxi} - t_{mini})\varphi} \qquad (A15)$$

(b) Exponential distribution:

$$R_{si}^* = \frac{D_i\{ln\left(\left(-\frac{\varphi_{2i}\theta_i}{\theta_i+\lambda_i}\right) + \left(\frac{\varphi_{3i}}{(\theta_i+\lambda_i)}\right)\right) - ln\left(\left(\frac{-\varphi_{2i}\lambda_i}{\theta_i}\left(\frac{\alpha_i\theta_i}{\lambda_i} - \frac{\theta_i}{\theta_i+\lambda_i}\right)\right) + \left(\frac{\varphi_{3i}}{(\theta_i+\lambda_i)}\right) + \varphi_{5i}(1-\alpha_i) + \varphi_{4i}\alpha_i}{(-\theta_i - \lambda_i)} \qquad (A16)$$