Adv. Math. Fin. App., 2022, **7(**2), P. 379-390

Advances in Mathematical Finance & Applications

www.amfa.iau-arak.ac.ir
Print ISSN: 2538-5569
Online ISSN: 2645-4610
Doi: 10.22034/AMFA.2020.1880449.1308

Research Paper

Measuring the Interval Industry Cost Efficiency Score in DEA

Ghasem Tohidi, Simin Tohidnia*

Department of Mathematics, Faculty of Basic Sciences, Central Tehran Branch, Islamic Azad University, Tehran, Iran

ARTICLE INFO
Article history:
Received 2019-10-06
Accepted 2020-02-04

Keywords:
Data envelopment analysis
Industry cost efficiency
Interval data

ABSTRACT

In this paper we extend the concept of "cost minimizing industry structure" and develop two DEA models for dealing with imprecise data. The main aim of this study is to propose an approach to compute the industry cost efficiency measure in the presence of interval data. We will see that the value obtained by the proposed approach is an interval value. The lower bound and upper bound of the interval industry cost efficiency measure are computed and then decomposed into three components to examine the relationship between them and the lower and upper bounds of the individual interval cost efficiency measures. We also define the cost efficient organization of the industry as a set of DMUs, which minimizes the total cost of producing the interval industry output vector. In fact, this paper determines the optimal number of DMUs and the reallocation of the industry observed outputs among them. We hereby determine the effects of the optimal number of DMUs and the reallocation of outputs among them on the interval industry cost efficiency measure. Finally, a numerical example will be presented to illustrate the proposed approach.

1 Introduction

Data envelopment analysis (DEA) is a non-parametric approach that has many applications in evaluating the performance of decision making units (DMUs). This approach is firstly introduced by Charns et al. [5]. In general, DEA approach has many applications, and many different DEA models have been presented for evaluating the efficiency of DMUs. For example, Peykani et al. [16] presented a novel fuzzy DEA based on a general fuzzy measure. They also proposed a fuzzy DEA approach for ranking of stocks [18]. DEA approach is also extended to study the robust optimization problem [17,19,20]. In addition, network DEA models have been developed to examine DMUs with network structure. See for example Peykani and Mohammadi [14,15]. An important set of DEA models are allocation models that can be applied when the prices of inputs and/or outputs are known. Farrell [8] presented the cost model for evaluating DMUs with known input cost vector. In this framework, Portela et al. [22] developed a cost efficiency model for the case where input quantities and their prices can vary simultaneously. Other DEA models also have been developed to examine the efficiency of DMUs for the case where input/output prices are available. For example, Tagashira and Minami [25], Tohidnia and Tohidi [28]. Other applications of DEA approach can be found in [11,13,22-24,28-31]. Because of the data are not usually

* Corresponding author. Tel.: *+989128069171*
E-mail address: *s_tohidniya@yahoo.com*

known exactly in the real world, DEA models have been also extended for dealing with imprecise data. In this context, Cooper et al. [6] presented some models. After that, two equivalent models were introduced by Despotis and Smirlis [7] to determine the lower and upper bounds for the efficiency score of DMUs in the presence of the interval data, which is a special type of the imprecise data. The result was called the interval efficiency. Hosseinzadeh Lotfi et al. [10] proposed two DEA models for evaluating the cost efficiency score with interval data. They computed an interval for the cost efficiency measure. Tohidi and Tohidnia [27] improved the interval cost efficiency of inefficient DMUs by using ideal points. Furthermore, DEA has been applied for measuring the efficiency score of the industry. The concept of structural efficiency of an industry was first introduced by Farrell [8]. He defined the structural efficiency measure as the output-weighted average of the technical efficiencies of the constituent DMUs of the industry. Førsund and Hjalmarsson [9] extended this concept and considered the technical efficiency of the average firm as a measure of structural efficiency of the industry. Li and Cheng [12] evaluated the industry technical efficiency score based on the relationship between the production technologies of the individual DMUs and that of the industry. Baumol et al. [2] introduced the concept of "cost minimizing industry structure" which is associated with the reallocation of outputs and a variable number of DMUs (see also Baumol and Fisher [3]). Following Baumol et al [2], Cesaroni [4] introduced the concept of the cost efficient organization of an industry and computed the industry optimal structure along with the specific impact on cost efficiency due to organizational inefficiency that stems from a non-optimal number of DMUs.

In this paper we extend the approach presented in Cesaroni [4] for measuring the cost efficiency score of the industry in the presence of imprecise data. The main purpose of this study is to propose an approach to compute the industry cost efficiency measure and generalization of the concept of the cost efficient organization of the industry in the presence of interval data. We will see that the value obtained by the proposed approach is an interval value. We compute the lower bound and upper bound of the interval industry cost efficiency measure and then decompose them into three components to examine the relationship between them and the lower and upper bounds of the individual interval cost efficiency measures. Following Cesaroni [4], we define the cost efficient organization of the industry as a set of DMUs, which minimizes the total cost of producing the interval industry output vector. In fact, this paper determines the optimal number of DMUs and the reallocation of the industry observed outputs among them. We also determine the effects of the optimal number of DMUs and the reallocation of outputs among them on the interval industry cost efficiency measure. As stated above, we use the idea of Cesaroni [4] in developing models to evaluate the cost efficiency score of an industry in the presence of interval data. Thus, similar to the Cesaroni [4] method, in general, three important features can be attributed to the method presented in this paper: 1) the allocation of inputs so that production costs are minimized at the industry level, 2) investigating the relationship between the interval industry cost efficiency score and the individual interval cost efficiency measures when reallocation of outputs is possible, 3) the determination of the optimal number of DMUs. The structure of the paper is as follows. Section 2 outlines the Preliminaries. Section 3 presents an approach to study the interval industry cost efficiency measure. Section 4 shows the application of the proposed approach. Section 5 concludes.

2 Preliminaries

Consider a set of n DMUs where each DMU_j, $j=1,...n$, consumes the input vector $\boldsymbol{x}_j=(x_{1j},x_{2j},\dots,x_{mj})^t \geq 0$ for producing the output vector $\boldsymbol{y}_j=(y_{1j},y_{2j},\dots,y_{sj})^t \geq 0$. All units operate under the same VRS production technology, T, as follows [1]:

$$T=\left\{(\boldsymbol{x},\boldsymbol{y}):\boldsymbol{x}\geq\sum_{j=1}^{n}\lambda_j\boldsymbol{x}_j\ ,\boldsymbol{y}\leq\sum_{j=1}^{n}\lambda_j\boldsymbol{y}_j\ ,\sum_{j=1}^{n}\lambda_j=1\ ,\ \lambda_j\geq 0\ ,\ j=1,\ldots,n\right\}. \tag{1}$$

The technology at the industry level is defined as the sum of the individual production possibility sets shown in (1) and denoted by $T^{IND}=nT$ [4]. Furthermore, consider $\boldsymbol{X}_0=\sum_{j=1}^{n}\boldsymbol{x}_j$ and $\boldsymbol{Y}_0=\sum_{j=1}^{n}\boldsymbol{y}_j$ respectively as the industry-observed input and output vectors. Also assume that the inputs prices are known and $\boldsymbol{c}=(c_1,c_2,\ldots,c_m)^t>0$ is the common input price vector for all DMUs. The cost efficiency measure of DMU_o can be evaluated by the following model (this form of the cost efficiency model was used in Sueyoshi [24]):

$$\begin{aligned}\min\quad & \boldsymbol{c}\sum_{j=1}^{n}\lambda_j\boldsymbol{x}_j\\ \text{s.t.}\quad & \sum_{j=1}^{n}\lambda_j\boldsymbol{y}_j\geq\boldsymbol{y}_o,\\ & \sum_{j=1}^{n}\lambda_j=1,\\ & \lambda_j\geq 0,\quad j=1,\ldots,n.\end{aligned} \tag{2}$$

The model (2) yields the minimum possible cost for producing the output vector $\boldsymbol{y}_o$. Assume that $\boldsymbol{x}_j^*=\sum_{j=1}^{n}\lambda_j^*\boldsymbol{x}_j$ is the optimal solution of model (2). Then the cost efficiency score of DMU_o can be computed using the ratio $\dfrac{\boldsymbol{cx}_j^*}{\boldsymbol{cx}_o}$. Now, we consider $x_{ij}\in\left[x_{ij}^l,x_{ij}^u\right]$ and $y_{rj}\in\left[y_{rj}^l,y_{rj}^u\right]$, $x_{ij}^l>0,\ y_{rj}^l>0$, as the interval input and output vectors of $\mathrm{DMU}_j,\ j=1,\ldots n$. Hosseinzade Lotfi et al. [10] proposed two models for evaluating the interval cost efficiency score of DMUs with interval data. They assume that the *i*th input cost of DMU_j , c_{ij} , is an interval number and $c_{ij}\in\left[c_{ij}^{\min},c_{ij}^{\max}\right]$. Their models are as follows:

$$\begin{aligned}\min\quad & \sum_{i=1}^{m}c_{io}^{\min}x_i\\ \text{s.t.}\quad & \sum_{j=1}^{n}\lambda_j x_{ij}^l=x_i\ ,\quad i=1,\ldots,m,\\ & \sum_{j=1,\neq o}^{n}\lambda_j y_{rj}^u+\lambda_o y_{ro}^l\geq y_{ro}^l,\quad r=1,\ldots,s,\\ & \lambda_j\geq 0,\qquad\qquad j=1,\ldots,n,\\ & x_i\geq 0,\qquad\qquad i=1,\ldots,m,\end{aligned} \tag{3}$$

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m} c_{io}^{\max} x_i \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^{u} = x_i, \quad i = 1,\dots,m, \\
& \sum_{j=1,\neq o}^{n} \lambda_j y_{rj}^{l} + \lambda_o y_{ro}^{u} \geq y_{ro}^{u}, \quad r = 1,\dots,s, \\
& \lambda_j \geq 0, \qquad j = 1,\dots,n, \\
& x_i \geq 0, \qquad i = 1,\dots,m.
\end{aligned} \tag{4}
$$

The lower and upper bounds of the cost efficiency score of DMU_o can be computed by the results of models (5) and (6), respectively as follows:

$$
\underline{CE}_o = \frac{\sum_{i=1}^{m} c_{io}^{\min} \underline{x}_i^{*}}{\sum_{i=1}^{m} c_{io}^{\max} x_{io}^{u}}, \tag{5}
$$

$$
\overline{CE}_o = \frac{\sum_{i=1}^{m} c_{io}^{\max} \overline{x}_i^{*}}{\sum_{i=1}^{m} c_{io}^{\min} x_{io}^{u}}, \tag{6}
$$

where $\underline{x}_i^{*}$ and $\overline{x}_i^{*}$ are respectively the optimal solutions of models (3) and (4) can be treated as a cost efficiency score for DMU_o. Cesaroni [4] introduced the concept of the cost efficient organization of an industry as a set of production possibilities like $\{(\boldsymbol{x}_h, \boldsymbol{y}_h) : (\boldsymbol{x}_h, \boldsymbol{y}_h) \in T, h = 1,\dots,k\}$, which minimizes the total cost of producing the industry output vector $\boldsymbol{Y}_0$. To compute the minimum cost of production based on the above definition, Cesaroni [4] presented some models and then computed the industry cost efficiency (ICE) using the ratio of this minimum cost to the actual cost of production. Also he decomposed the ICE measure into the product of three components. For more details on his approach see Cesaroni [4]. In the next section, by using the idea of Cesaroni [4] we evaluate the industry cost efficiency measure in the presence of interval data.

3 Industry Cost Efficiency Measure in the Presence of Interval Data

Assume that we deal with a set of DMUs with interval inputs and interval outputs. Thus we will have $\boldsymbol{X}_0 \in \left[\boldsymbol{x}_0^l, \boldsymbol{x}_0^u\right]$ and $\boldsymbol{Y}_0 \in \left[\boldsymbol{y}_0^l, \boldsymbol{y}_0^u\right]$ whereand $\boldsymbol{Y}_0^l = \sum_{j=1}^{n} \boldsymbol{y}_j^l$, $\boldsymbol{X}_0^u = \sum_{j=1}^{n} \boldsymbol{x}_j^u$, $\boldsymbol{X}_0^l = \sum_{j=1}^{n} \boldsymbol{x}_j^l$,
Because in this study the input price vector is considered fix and common for all . $\boldsymbol{Y}_0^u = \sum_{j=1}^{n} \boldsymbol{y}_j^u$
DMUs, we set $c_{io}^{\min} = c_{io}^{\max} = c_i$ $(\forall i)$ in the models (3) and (4), where the value of c_i is the price of

*i*th input for each DMU. In this section, we are going to extend the concept of the cost efficient organization of the industry introduced in Cecaroni [4] for dealing with interval data and then evaluate the industry cost efficiency score in the presence of interval data. We first rewrite models (3) and (4) based on model (2), respectively as shown in (7) and (8).

$$\begin{aligned}
\min \quad & \boldsymbol{c}\sum_{j=1}^{n}\lambda_j \boldsymbol{x}_j^l \\
\text{s.t.} \quad & \sum_{j=1,\neq o}^{n}\lambda_j \boldsymbol{y}_j^u + \lambda_o \boldsymbol{y}_o^l \geq \boldsymbol{y}_o^l, \\
& \sum_{j=1}^{n}\lambda_j = 1, \\
& \lambda_j \geq 0,\ j = 1,\dots,n,
\end{aligned} \tag{7}$$

$$\begin{aligned}
\min \quad & \boldsymbol{c}\sum_{j=1}^{n}\lambda_j \boldsymbol{x}_j^u \\
\text{s.t.} \quad & \sum_{j=1,\neq o}^{n}\lambda_j \boldsymbol{y}_j^l + \lambda_o \boldsymbol{y}_o^u \geq \boldsymbol{y}_o^u, \\
& \sum_{j=1}^{n}\lambda_j = 1, \\
& \lambda_j \geq 0,\ j = 1,\dots,n.
\end{aligned} \tag{8}$$

Suppose $\boldsymbol{x}_j^{*l} = \sum_{j=1}^{n}\lambda_j^* \boldsymbol{x}_j^l$ and $\boldsymbol{x}_j^{*u} = \sum_{j=1}^{n}\lambda_j^* \boldsymbol{x}_j^u$ are respectively the results of models (7) and (8). Upper bound and lower bound of the interval cost efficiency of DMU_o can be computed using the optimal values of these models as follows:

$$\underline{CE}_o = \frac{\boldsymbol{c}\boldsymbol{x}_j^{*l}}{\boldsymbol{c}\boldsymbol{x}_o^u}, \tag{9}$$

$$\overline{CE}_o = \frac{\boldsymbol{c}\boldsymbol{x}_j^{*u}}{\boldsymbol{c}\boldsymbol{x}_o^u}. \tag{10}$$

Following Cesaroni [4], we consider $\{(\boldsymbol{x}_h, \boldsymbol{y}_h) : (\boldsymbol{x}_h, \boldsymbol{y}_h) \in T, h = 1,\dots,k\}$ as a set of DMUs that produces the industry output vector $\boldsymbol{Y}_0$ so that the overall cost of production is minimized. The DMUs belonging to this set are denoted by the index h. k is an integer variable and represents the number of DMUs that can produce $\boldsymbol{Y}_0$ at the minimum possible cost. In Cesaroni [4], a model was developed to determine such a set of DMUs. Now, we extend that model for dealing with interval data, and present two models for allocating the observed industry output vector $\boldsymbol{Y}_0$ among a set of efficient DMUs. In this case the intervals of the input and output values corresponding to each DMU are effective in determination of the set of DMUs which minimizes the total cost of producing the given industry output vector. The proposed models are shown in (11) and (12).

$$\begin{aligned} \min_{\{(\boldsymbol{x}_h,\boldsymbol{y}_h)\}} \quad & \sum_{h=1}^{k} \boldsymbol{c}\boldsymbol{x}_h^l \\ \text{s.t.} \quad & \sum_{h=1}^{k} \boldsymbol{y}_h^l \geq \boldsymbol{Y}_0^l, \end{aligned} \tag{11}$$

$$\begin{aligned} \min_{\{(\boldsymbol{x}_h,\boldsymbol{y}_h)\}} \quad & \sum_{h=1}^{k} \boldsymbol{c}\boldsymbol{x}_h^u \\ \text{s.t.} \quad & \sum_{h=1}^{k} \boldsymbol{y}_h^u \geq \boldsymbol{Y}_0^u. \end{aligned} \tag{12}$$

In fact, model (11) determines the set of DMUs where each DMU produces a part of the amount of $\boldsymbol{Y}_0^l$ with minimum possible cost. The constraint $\sum_{h=1}^{k} \boldsymbol{y}_h^l \geq \boldsymbol{Y}_0^l$ ensures that the sum of the lower bounds of the outputs produced by the obtained set of DMUs is not less than the industry- observed output vector $\boldsymbol{Y}_0^l$. In addition, the constraint $\sum_{h=1}^{k} \boldsymbol{x}_h^l \leq \boldsymbol{X}_0^l$ can be added to model (11). This constraint ensures that the sum of the input vectors used by this set of DMUs to produce the industry-observed output vector $\boldsymbol{Y}_0^l$ is less than or equal to the industry-observed input vector $\boldsymbol{X}_0^l$. However, for $k = n$, the industry-observed input vector $\boldsymbol{X}_0 \in \left[\boldsymbol{X}_0^l, \boldsymbol{X}_0^u\right]$ can produce the industry-observed output vector $\boldsymbol{Y}_0 \in \left[\boldsymbol{Y}_0^l, \boldsymbol{Y}_0^u\right]$, so this model is feasible, and from a logical point of view there is no need to add the constraint $\sum_{h=1}^{k} \boldsymbol{x}_h^l \leq \boldsymbol{X}_0^l$ [4]. Similarly, model (12) determines the set where each member of it produces a part of the amount of $\boldsymbol{Y}_0^u$ with minimum possible cost. If we consider k as a parameter, then for any given integer k models (11) and (12) can be rewritten as follows, respectively:

$$\begin{aligned} \min_{(\bar{\boldsymbol{x}}_h,\bar{\boldsymbol{y}}_h)} \quad & k\boldsymbol{c}\bar{\boldsymbol{x}}_h^l \\ \text{s.t.} \quad & k\bar{\boldsymbol{y}}_h^l \geq \boldsymbol{Y}_0^l, \end{aligned} \tag{13}$$

$$\begin{aligned} \min_{(\bar{\boldsymbol{x}}_h,\bar{\boldsymbol{y}}_h)} \quad & k\boldsymbol{c}\bar{\boldsymbol{x}}_h^u \\ \text{s.t.} \quad & k\bar{\boldsymbol{y}}_h^u \geq \boldsymbol{Y}_0^u, \end{aligned} \tag{14}$$

where $\bar{\boldsymbol{y}}_h^u = \frac{1}{k}\sum_{h=1}^{k} \boldsymbol{y}_h^u$ $\bar{\boldsymbol{y}}_h^l = \frac{1}{k}\sum_{h=1}^{k} \boldsymbol{y}_h^l$, $\bar{\boldsymbol{x}}_h^u = \frac{1}{k}\sum_{h=1}^{k} \boldsymbol{x}_h^u$, $\bar{\boldsymbol{x}}_h^l = \frac{1}{k}\sum_{h=1}^{k} \boldsymbol{x}_h^l$

Cesaroni [4] interpreted the parameter k as the "cost minimizing number of firms" [2,3]. It is essential that the production technology is convex to formulate models (11) and (12) respectively as (13) and

(14). Because each of models (13) and (14) finds a single optimal scale size being replicated k times. So in general, we can reformulate models (11) and (12) respectively as (15) and (16).

$$\begin{aligned} \min_{\lambda_j, k} \quad & k\boldsymbol{c}\sum_{j=1}^{n} \lambda_j \boldsymbol{x}_j^l \\ \text{s.t.} \quad & k\sum_{j=1}^{n} \lambda_j \boldsymbol{y}_j^l \geq \boldsymbol{Y}_0^l, \end{aligned} \tag{15}$$

$$\begin{aligned} \min_{\lambda_j, k} \quad & k\boldsymbol{c}\sum_{j=1}^{n} \lambda_j \boldsymbol{x}_j^u \\ \text{s.t.} \quad & k\sum_{j=1}^{n} \lambda_j \boldsymbol{y}_j^u \geq \boldsymbol{Y}_0^u. \end{aligned} \tag{16}$$

$(\bar{\boldsymbol{x}}^{*l}, \bar{\boldsymbol{y}}^{*l}, k_l^*)$ is an optimal solution to model (15) and $(\bar{\boldsymbol{x}}^{*u}, \bar{\boldsymbol{y}}^{*u}, k_u^*)$ is an optimal solution to model (16), where $\bar{\boldsymbol{x}}^{*l} = \sum_{j=1}^{n} \lambda_j^* \boldsymbol{x}_j^l$ and $\bar{\boldsymbol{y}}^{*l} = \sum_{j=1}^{n} \lambda_j^* \boldsymbol{y}_j^l$. The value of $\bar{\boldsymbol{x}}^{*u}$ and $\bar{\boldsymbol{y}}^{*u}$ can be computed similarly. In addition, since $\boldsymbol{c}$ and $\boldsymbol{x}_j$ are assumed to be nonnegative, models (15) and (16) will be unbounded if k is negative. Therefore, these models will have an optimal solution when k is positive, which is guaranteed by defining k as the "cost minimizing number of firms".

Now, we calculate the lower bound and upper bound of the interval industry cost efficiency score (IICE) as follows:

$$\underline{IICE} = \frac{k^* \boldsymbol{c}\bar{\boldsymbol{x}}^{*l}}{\boldsymbol{c}\boldsymbol{X}_0^u}, \tag{17}$$

$$\overline{IICE} = \frac{k^* \boldsymbol{c}\bar{\boldsymbol{x}}^{*u}}{\boldsymbol{c}\boldsymbol{X}_0^u}, \tag{18}$$

We used the results of models (15) and (16) to calculate the values of $\underline{IICE}$ and $\overline{IICE}$. In fact, the industry production possibility $(\bar{\boldsymbol{X}}_0, \bar{\boldsymbol{Y}}_0)$ where $\bar{\boldsymbol{X}}_0 \in \left[k^* \bar{\boldsymbol{x}}^{*l}, k^* \bar{\boldsymbol{x}}^{*u}\right]$ and $\bar{\boldsymbol{Y}}_0 \in \left[k^* \bar{\boldsymbol{y}}^{*l}, k^* \bar{\boldsymbol{y}}^{*u}\right]$, is obtained from solving models (15) and (16). The industry production possibility $(\bar{\boldsymbol{X}}_0, \bar{\boldsymbol{Y}}_0)$ produces the output vector $\bar{\boldsymbol{Y}}_0 (\bar{\boldsymbol{Y}}_0 \geq \boldsymbol{Y}_0)$ at the minimum possible cost at the industry level. Therefore, the minimum possible cost of producing the industry-observed output vector $\boldsymbol{Y}_0$, belong to the interval $\left[k^* \boldsymbol{c}\bar{\boldsymbol{x}}^{*l}, k^* \boldsymbol{c}\bar{\boldsymbol{x}}^{*u}\right]$. The lower and upper bounds of this interval are used to calculate the values of $\underline{IICE}$ and $\overline{IICE}$ defined in (17) and (18).

3.1 Decomposition of the Interval Industry Cost Efficiency Score

Sometimes it may occur that the lower bound and/or upper bound of the interval cost efficiency scores of all DMUs is equal to one, while the lower bound and/or upper bound of the interval industry cost efficiency score is not equal to one. We explain this issue by decomposing $\underline{IICE}$ and $\overline{IICE}$ as shown in (19) and (20).

$$\underline{IICE}=\frac{k^{*}\boldsymbol{c\bar{x}}^{*l}}{\boldsymbol{cX}_0^{u}}=\frac{\sum_{j=1}^{n}\boldsymbol{cx}_j^{*l}}{\boldsymbol{cX}_0^{u}}\times\frac{n\boldsymbol{c\bar{x}}_n^{*l}}{\sum_{j=1}^{n}\boldsymbol{cx}_j^{*l}}\times\frac{k^{*}\boldsymbol{c\bar{x}}^{*l}}{n\boldsymbol{c\bar{x}}_n^{*l}}=\underline{WACE}\times\underline{RE}_{n}\times\underline{RE}_{k^*}. \tag{19}$$

$$\overline{IICE}=\frac{k^{*}\boldsymbol{c\bar{x}}^{*u}}{\boldsymbol{cX}_0^{u}}=\frac{\sum_{j=1}^{n}\boldsymbol{cx}_j^{*u}}{\boldsymbol{cX}_0^{u}}\times\frac{n\boldsymbol{c\bar{x}}_n^{*u}}{\sum_{j=1}^{n}\boldsymbol{cx}_j^{*u}}\times\frac{k^{*}\boldsymbol{c\bar{x}}^{*u}}{n\boldsymbol{c\bar{x}}_n^{*u}}=\overline{WACE}\times\overline{RE}_{n}\times\overline{RE}_{k^*}. \tag{20}$$

Where $\boldsymbol{\bar{x}}_n^{*l}$ and $\boldsymbol{\bar{x}}_n^{*u}$ are the optimal solutions of (15) and (16) for $k=n$, and also $\boldsymbol{x}_j^{*l}$ and $\boldsymbol{x}_j^{*u}$ are the optimal solutions of models (7) and (8), respectively. According to the presented models, we will have the following inequalities:

$$k^{*}\boldsymbol{c\bar{x}}^{*l}\le n\boldsymbol{c\bar{x}}_n^{*l}\le\sum_{j=1}^{n}\boldsymbol{cx}_j^{*l}\le\boldsymbol{cX}_0^{u}, \tag{21}$$

$$k^{*}\boldsymbol{c\bar{x}}^{*u}\le n\boldsymbol{c\bar{x}}_n^{*u}\le\sum_{j=1}^{n}\boldsymbol{cx}_j^{*u}\le\boldsymbol{cX}_0^{u}. \tag{22}$$

Now, we examine the components of $\underline{IICE}$. Based on the inequalities shown in (21), the value of each component is less than or equal to one. The first component of $\underline{IICE}$ can be rewritten as follows:

$$\underline{WACE}=\frac{\sum_{j=1}^{n}\boldsymbol{cx}_j^{*l}}{\boldsymbol{cX}_0^{u}}=\sum_{j=1}^{n}\frac{\boldsymbol{cx}_j^{u}}{\boldsymbol{cX}_0^{u}}\cdot\frac{\boldsymbol{cx}_j^{*l}}{\boldsymbol{cx}_j^{u}}. \tag{23}$$

In fact, the component $\underline{WACE}$ can be shown as the weighted average of the individual interval cost efficiency scores of *n* DMUs. If the lower bounds of the interval cost efficiency scores of all DMUs are equal to one, then all DMUs will be cost efficient in the worst situation. Thus we will have

$$\frac{\sum_{j=1}^{n}\boldsymbol{cx}_j^{*l}}{\boldsymbol{cX}_0^{u}}=\sum_{j=1}^{n}\frac{\boldsymbol{cx}_j^{u}}{\boldsymbol{cX}_0^{u}}\times1=\frac{\sum_{j=1}^{n}\boldsymbol{cx}_j^{u}}{\boldsymbol{cX}_0^{u}}=\frac{\boldsymbol{cX}_0^{u}}{\boldsymbol{cX}_0^{u}}=1, \tag{24}$$

It is clear that the value of $\underline{IICE}$ will be equal to one only if its three components are equal to one. Therefore, in the case where all DMUs are cost efficient in the worst situation, we cannot conclude that the interval industry cost efficiency score is also equal to one in the worst situation. In fact, it is possible that the allocation of the observed industry output vector among the set of cost efficient point is not determined properly, and thus the value of the second and/or third component is not equal to one. The second component (reallocative efficiency) captures the impact of the reallocation of the output vector $\boldsymbol{Y}_0^l$ among the existing number of DMUs on $\underline{IICE}$, and the third component examine the impact of the determination of the optimal number of DMUs that produce the output vector. The Components $\boldsymbol{Y}_0^l$ resulting from the decomposition of $\overline{IICE}$ are interpreted in the same manner as the components of $\underline{IICE}$.

4 Illustration of Proposed Approach

Suppose 10 DMUs in Table 1 consume two interval inputs x_1 and x_2 to produce an interval output y_1, and all DMUs operate under the fix and common input price vector $(c_1, c_2) = (4, 2)$. We measure the interval cost efficiency of DMUs by using models (7) and (8). Table 1 shows the results for 10 DMUs.

Table 1: Interval data and the individual interval cost efficiency scores

DMU	x_{1j}^l	x_{1j}^u	x_{2j}^l	x_{2j}^u	y_{1j}^l	y_{1j}^u	$\underline{CE}$	$\overline{CE}$
DMU1	1	1.5	1.5	5	3	8	0.219	1
DMU2	0.8	2	1.75	4.5	2.5	9	0.206	1
DMU3	0.5	1.75	0.75	3	5.25	10.5	0.269	1
DMU4	2	3.5	4	7.5	4	12	0.121	1
DMU5	1.5	3	3	5	5	10	0.159	1
DMU6	1.25	2.75	3.25	4.75	7.5	15	0.171	1
DMU7	1.5	2.5	5	8.25	7	13	0.132	1
DMU8	5	6.25	4.5	7	5.5	10.5	0.090	1
DMU9	3.5	5	2	5.14	3.25	9	0.116	1
DMU10	6	7.75	6	9.2	1	6	0.226	0.314

We first evaluate the lower bound and upper bound of the interval industry cost efficiency measure by models (15) and (16) for the special case, $k = n$. In this case, the output vectoris produced by all $\boldsymbol{Y}_0^l$ existing DMUs. The results are summarized in Table 2. Now, we solve models (15) and (16) again, for measuring the values of $\underline{IICE}$ and $\overline{IICE}$ at the optimal number of DMUs, k_l^* and k_u^*, respectively. It can be seen that from Tables 2 and 3, how the number of DMUs affect the amounts of the interval industry cost efficiency score. By comparing the values of $\underline{IICE}$ in Tables 2 and 3, we find out that when the optimal value of k is determined by models (15) and (16), the value of the interval industry cost efficiency score will be evaluated more accurately.

Table 2: Results of models (15) and (16) for $k = n$

$\overline{x}_1^{*l}$	$\overline{x}_1^{*u}$	$\overline{x}_2^{*l}$	$\overline{x}_2^{*u}$	$\underline{IICE}$	$\overline{IICE}$
0.5	1.75	0.75	3	0.133	0.495

Table 3: Results of models (15) and (16)

$\overline{x}_1^{*l}$	$\overline{x}_1^{*u}$	$\overline{x}_2^{*l}$	$\overline{x}_2^{*u}$	$\underline{IICE}$	$\overline{IICE}$	k_l^*	k_u^*
0.5	1.75	0.75	3	0.120	0.495	9	10

The results obtained from the decomposition of $\underline{IICE}$ and $\overline{IICE}$ are respectively shown in Tables 4 and 5. The value of 0.9 corresponding to $\underline{RE}_{k^*}$ represents the portion of the interval industry cost inefficiency that has been created because of the non-optimal number of DMUs.

Table 4: Lower bound decomposition

$\underline{WACE}$	$\underline{RE}_n$	$\underline{RE}_{k^*}$	$\underline{IICE}$
0.133	1	0.9	0.120

Table 5: Upper bound decomposition

$\overline{WACE}$	$\overline{RE}_n$	$\overline{RE}_{k^*}$	$\overline{IICE}$
0.871	0.568	1	0.495

5 Conclusions

The input and output data are not usually known exactly in the real world, so we decided to appropriate this paper to examining the cost efficiency score at the industry level in the presence of imprecise data. For this purpose, we extended the approach presented in Cesaroni [4] to deal with the case where DMUs consume interval inputs to produce interval outputs. We computed the cost efficiency score at the industry level and called the result the interval industry cost efficiency score. We calculated the lower bound and upper bound of the interval industry cost efficiency score, and then decomposed each of them into three components. In addition, we determined the optimal number of DMUs that can produce the interval industry observed output vector and also reallocation of the output vector among them in the industry. In practical applications, it may occur that the output prices are known instead of the input prices. The idea used in this paper can be applied in such situation. In this case, we can compute the lower and upper bounds of the revenue efficiency score at the industry level, and then examine the relationship between them and the lower and upper bounds of the individual interval revenue efficiency measures. As another idea for future research, the models proposed in this study can be extended to use in cases where the industry is composed of a set of DMUs with a network structure.

References

[1] Banker, R.D., Charnes, A., Cooper, W.W., *Some models for estimating technical and scale inefficiencies in data envelopment analysis*, Management Science, 1984, **30**, P. 1078-1092. Doi:10.1287/mnsc.30.9.1078.

[2] Baumol, W.J., Panzar, J.C., Willig, R.D., *Contestable markets and the theory of industry structure*, New York: Harcourt Brace Jovanovich 1982.

[3] Baumol, W.J., Fisher, D., *Cost-minimizing number of firms and determination of industry structure*, The Quarterly Journal of Economics, 1978, **92**, P. 439-468. Doi:10.2307/1883153.

[4] Cesaroni G., *Industry cost efficiency in data envelopment analysis*, Socio-Economic Planning Sciences, 2017, **16**, P.30041-30046. Doi: 10.1016/j.seps.2017.01.001.

[5] Charnes, A., Cooper, W.W., Rhodes, E., *Measuring the efficiency of decision making units*, European Journal of Operational Research, 1978, **2**(6), P.429–444. Doi: 10.1016/0377-2217(78)90138-8.

[6] Cooper, W.W., Park, K.S., Yu, G., *IDEA and AR-IDEA: models for dealing with imprecise data in DEA*. Management Science, 1999, **45**, P.597–607. Doi: 10.1287/mnsc.45.4.597.

[7] Despotis, D.K., Smirlis, Y.G., *Data envelopment analysis with imprecise data*, European Journal of Operational Research, 2002, **140**, P.24–36. Doi: 10.1016/S0377-2217(01)00200-4.

[8] Farrell, M.G., *The measurement of productive efficiency*, Journal of the Royal Statistical Society, Series A, 1957, **120**, P. 253-281. Doi: 10.2307/2343100.

[9] Førsund, F., Hjalmarsson, L., *Generalized Farrell measures of efficiency: An application to milk processing in Swedish dairy plants*, The Economic Journal, 1979, **89**, P. 294-315. Doi: 10.2307/2231603.

[10] Hosseinzade Lotfi, F., Jahanshahloo, G.R., Shahverdi, R., Rostamy-Malkhalifeh, M., *Cost Efficiency and Cost Malmquist Productivity Index with Interval Data*, International Mathematical Forum, 2007, **2**(9), P. 441 – 453. Doi: 10.12988/imf.2007.07040.

[11] Joulaei, M., Mirbolouki, M., *Reduction of DEA-Performance Factors Using Rough Set Theory: An Application of Companies in the Iranian Stock Exchange*, Advances in Mathematical Finance and Applications, 2020, **5**(1), P.53-67. Doi: 10.22034/amfa.2019.1868389.1223.

[12] Li, S.K, Cheng, Y.S., *Solving the puzzles of structural efficiency*, European Journal of Operational Research, 2007, **180**, P. 713-722. Doi: 10.1016/j.ejor.2006.05.010.

[13] Navidi, S., Rostamy-Malkhalifeh, M., Banihashemi, S., *Using MODEA and MODM with Different Risk Measures for Portfolio Optimization*, Advances in Mathematical Finance and Applications, 2020, **5**(1), P. 29-51. Doi: 10.22034/amfa.2019.1864620.1200.

[14] Peykani, P., Mohammadi E., *Interval network data envelopment analysis model for classification of investment companies in the presence of uncertain data*, Journal of Industrial and Systems Engineering, 2018, **11**(Special issue: 14th International Industrial Engineering Conference), P. 63-72.

[15] Peykani, P., Mohammadi E., *Window network data envelopment analysis: an application to investment companies*, International Journal of Industrial Mathematics, 2020, **12**(1), P. 89-99.

[16] Peykani, P., Mohammadi E., Emrouznejad, A., Pishvaee, M.S., Rostamy-Malkhalifeh, M., *Fuzzy data envelopment analysis: An adjustable approach*, Expert Systems with Applications, 2019, **136**, P. 439-452 Doi: 10.1016/j.eswa.2019.06.039.

[17] Peykani, P., Mohammadi E., Emrouznejad, A., Pishvaee, M.S., Rostamy-Malkhalifeh, M., Jabbarzadeh, A., *A novel fuzzy data envelopment analysis based on robust possibilistic programming: possibility, necessity and credibility-based approaches*, RAIRO-Operations Research, 2018, **52**(4), P. 1445-1463. Doi: 10.1051/ro/2018019.

[18] Peykani, P., Mohammadi E., Rostamy-Malkhalifeh, M., Hosseinzade Lotfi, F., *Fuzzy Data Envelopment Analysis Approach for Ranking of Stocks with an Application to Tehran Stock Exchange*, Advances in Mathematical Finance and Applications, 2019, **4**(1), P. 31-43. Doi:10.22034/amfa.2019.581412.1155.

[19] Peykani, P., Mohammadi E., Jabbarzadeh, A., Jandaghian, A., *Utilizing Robust Data Envelopment Analysis Model for Measuring Efficiency of Stock, A case study: Tehran Stock Exchange*, Journal of New Research in Mathematics, 2016, **1**(4), P. 15-24.

[20] Peykani, P., Mohammadi E., Seyed Esmaeili, F.S., *Stock Evaluation under Mixed Uncertainties Using Robust DEA Model*, Journal of Quality Engineering and Production Optimization, 2019, **4**(1), P. 73-84. Doi: 10.22070/jqepo.2019.3652.1080.

[21] Poordavoodi, A., Moazami Goudarzi, M.R., Haj Seyyed Javadi, H., Rahmani, A.M., Izadikhah, M., *Toward a More Accurate Web Service Selection Using Modified Interval DEA Models with Undesirable Outputs*, Computer Modeling in Engineering & Sciences, 2020, **123**(2), P. 525-570, Doi: 10.32604/cmes.2020.08854

[22] Portela, M.C.A.S. and Thanassoulis, E., *Economic efficiency when prices are not fixed: disentangling quantity and price efficiency*, Omega, 2014, **47**, P. 36-44. Doi: 10.1016/j.omega.2014.03.005.

[23] Shahhosseini, M., Hu, G., Pham, H., *Optimizing Ensemble Weights for Machine Learning Models: A Case Study for Housing Price Prediction*, In INFORMS International Conference on Service Science, Springer, Cham, 2019, P. 87-97.

[24] Shahhosseini, M., Martinez-Feria, R.A., Hu, G., Archontoulis, S.V., *Maize yield and nitrate loss prediction with machine learning algorithms*, Environmental Research Letters, 2019. Doi: 10.1088/1748-9326/ab5268.

[25] Sueyoshi, T., *Measuring efficiencies and returns to scale of Nippon Telegraph & Telephone in production and cost analyses*, Management Science, 1997, **43**, P. 779-796. Doi: 10.1287/mnsc.43.6.779.

[26] Tagashira, T., Minami C., *The Effect of Cross-Channel Integration on Cost Efficiency*, Journal of Interactive Marketing, 2019, **47**, P. 68-83. Doi: 10.1016/j.intmar.2019.03.002.

[27] Tohidi, G., Tohidnia, S., *A non-radial approach to improve the interval cost efficiency score in DEA*, International Journal of Industrial and Systems Engineering, 2021, **39**(2), P. 151-161.

[28] Tohidnia, S., Tohidi, G., *Measuring productivity change in DEA-R: A ratio-based profit efficiency model*, Journal of the Operational Research Society, 2019, **70**, P. 1511–1521. Doi: 10.1080/01605682.2018.1506561.

[29] Tavana, M., Izadikhah, M., Farzipoor Saen, R., Zare, R., *An integrated data envelopment analysis and life cycle assessment method for performance measurement in green construction management*. Environ Sci Pollut Res, *2021*, **28**, P. 664–682. Doi: 10.1007/s11356-020-10353-7

[30] Walheer, B., *Disaggreation of the Cost Malmquist Index with joint and output-specific inputs*, Omega, 2018, **75**, P. 1-12. Doi: 10.1016/j.omega.2017.01.012.

[31] Wang, Q., Wu, Z., Chen, X., *Decomposition weights and overall efficiency in a two-stage DEA model with shared resources*, Computers and Industrial Engineering, 2019, **136**, P. 135-148. Doi: 10.1016/j.cie.2019.07.014.