



The Tail Mean-Variance Model and Extended Efficient Frontier

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ABSTRACT

In portfolio theory, it is well-known that the distributions of stock returns often have non-Gaussian characteristics. Therefore, we need non-symmetric distributions for modeling and accurate analysis of actuarial data. For this purpose and optimal portfolio selection, we use the Tail Mean-Variance (TMV) model, which focuses on the rare risks but high losses and usually happens in the tail of return distribution. The proposed TMV model is based on two risk measures the Tail Condition Expectation (TCE) and Tail Variance (TV) under Generalized Skew-Elliptical (GSE) distribution. We first apply a convex optimization approach and obtain an explicit and easy solution for the TMV optimization problem, and then derive the TMV efficient frontier. Finally, we provide a practical example of implementing a TMV optimal portfolio selection in the Tehran Stock Exchange and show TCE-TV efficient frontier.

1 Introduction

Investment decision making is one of the key issues in financial management. Selecting the appropriate tools and techniques that can make optimal portfolio is one of the main objectives of the investment world [33]. The main goal of the portfolio selection is the optimal allocation of investments between different assets, which is based on arbitration between the two criteria of return of a portfolio and its risk. So, the most appropriate stocks are determined along with the ratio of each of them. The classical Mean-Variance (MV) model, introduced by Markowitz [25], is a fundamental theory that has developed the theory of portfolio selection. In the MV model, the expected return of the portfolio and the variance of return are considered as investment return and risk, respectively. The MV model is defined as follows:

$$MV(L) = \mathbb{E}(L) + \frac{1}{2}\tau \text{Var}(L) \quad (1)$$

where $\tau > 0$ and L is a random loss at a portfolio. Every investor wants to obtain a portfolio with the highest possible profit at a specified risk or obtain a portfolio with the least possible risk at a specified return. A set of the portfolios obtained by effective methods are located on the curve that is called the Efficient Frontier [26]. Most statistical studies are based on the normal distribution, but when the data set is not symmetric or has a heavy tail, the normal distribution does not correspond to it [29]. Luo et

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al. [24] Found that the real distribution of the returns is associated with the features of the fat tail and high peak, which leads to the attention to the tail risk measurement. A tail risk event occurs when the value of an investment deviates more than three times the standard deviation from its average. The probability of such an event is very low, but it can have serious negative consequences for financial markets and for portfolios. Depending on the side, there may be a left tail risk or right tail. Left tail risk takes place on the left side, and it indicates the negative returns of a portfolio. Right tail risk takes place on the right side, and it is dealing with the positive returns that can be generated. Fat tails often occur in finance but are considered undesirable due to the additional risk involved. Negative risks are when everything can go wrong about a project. Nevertheless, the risks are likely to be positive. Investors tend to think less about positive risk in project management, probably because managers focus more on what goes wrong. The reason we are driven down the negative path and often consider risk as negative is the result of a human condition where we place more emphasis on protecting the loss than achieving a gain. Chow et al. [7] examined the effect of positive and negative tail risk premiums on future returns. On a monthly level, They found that the existence of a premium for bearing positive tail risk today holds no statistically significant power for lower future returns, while its counterpart for bearing negative tail risk has significant predictive power for lower future returns. The Tail Mean-Variance (TMV) model for portfolio selection was introduced by Landsman [19], and is defined as follows:

$$TMV_q(L) = \mathbb{E} \left(L \mid L > VaR_q(L) \right) + \lambda \text{Var} \left(L \mid L > VaR_q(L) \right) \quad (2)$$

where $\lambda > 0$ and L is a random loss with an elliptical distribution at a portfolio. The TMV model, unlike the MV, focuses on the behavior of the tail of returns distribution through the q -quantile defined in the Value at Risk (VaR) measure. The TMV measure can help an investor to know the behavior of risk along the tail of return distribution ($X \geq VaR_q(X)$). This issue is very important for financial managers because they are worried about portfolio performance if there are extreme losses in capital markets. The VaR was introduced by G.P. Morgan [28], at confidence level q is defined as

$$VaR_q(L) = \inf(x \in R : F_L(x) \geq q) \quad (3)$$

$q \in (0,1)$ and $F_L(x)$ is the Cumulative Distribution Function (CDF) of loss L . VaR computation is based on the normal distribution of financial data. Although the normal distribution is the most popular distribution used for modeling, it is not proper for modeling portfolio losses or financial risks [12, 13]. The real distributions of many financial data have non-Gaussian properties, and when the data set is asymmetric, the normal distribution does not fit well [10, 14]. The TMV model in (2) is composed of a weighted sum of two risk measures. The first risk measure is Tail Conditional Expectation (TCE), introduced by Artzner et al. [1], which provides information about the mean of the tail of the loss distribution. Compared to the VaR, the TCE measure offers a more conservative measure of risk for the same level of confidence q . This measure was expanded by Panjer for the multivariate normal family [31], and was introduced by Landsman and Valdez for Elliptical distribution [23]. Landsman et al. [20] obtained the TCE measure for a family of Skew-Elliptical distributions, which named the Generalized Skew-Elliptical (GSE) distributions. The second risk measure is the Tail Variance (TV), proposed by Furman and Landsman [9], which is equal to the deviation of the loss from the mean along the tail of the distribution. Jamshidi and Khaloozadeh [15] derived the TV measure for the generalized skew-elliptical distributions. Wang et al. [36] studied the TMV model, which includes variables and

tail risks, and allocated capital-proportional the asset's risk, using several risk measures, including Value-at-Risk (VaR) and nonlinear weighted (NLW) risk measures. Xu and Mao [37] obtained the optimal capital allocation in the structure of the TMV model for multivariate elliptical distributions and applied the results to various business units for an insurance company. Owadally and Landsman [30] derived a simple and explicit solution for the optimal portfolio on the TMV criterion, which improves on previous work. Jiang et al. [16] derived the explicit solution of the TV for the generalized Laplace distribution and optimization of the TMV portfolio. Kim et al. [18] considered the class of the univariate and multivariate normal mean-variance mixture (NMVM) distributions and derived the conditional tail expectation (CTE) and the Conditional Tail Variance (CTV) for the univariate NMVM family. Bauder et al. [5] presented a Bayesian mean-variance analysis for optimal portfolio selection under parameter uncertainty. They assumed that the parameters of the return on assets, such as mean and covariance, are unknown and used historical data to estimate and solve the problem of optimal portfolio selection. Miryekemam et al. [27] developed several approaches to multi-criteria portfolio optimization. In order to solve the problem of information in the Tehran Stock Exchange in 2017, 45 sample stocks have been identified, and with the assumption of normalization of goals, a genetic algorithm has been used. Darabi and Baghban [8] studied the application of Clayton Copula in Portfolio Optimization and Compared with Markowitz Mean-Variance Analysis. They used copula as an alternative measure to model the dependency structure in research. In this regard, given the weekly data pertaining to early 2002 until late 2013, They used Clayton-copula to generate an optimized portfolio for both copper and gold.

This paper is classified as follows: Next section defines the family of generalized skew-elliptical distributions, section 3 formulates the TCE and TV measures for the generalized skew-elliptical distributions, section 4 presents a simple and explicit formula for the optimal portfolio selection on the TMV model and then derives the TMV efficient frontier, section 5 offers a practical example of implementing a TMV optimal portfolio on the Iran Stock Market, section 6 presents a conclusion to the paper.

2 Generalized Skew-Elliptical Distributions

The skew-elliptical distributions are constructive in many branches of science, such as statistical physics and actuarial science. The elliptical family of distributions was introduced by Kelker [17]. The class of skew-normal distributions was introduced by Azzalini [2] and extended to the multivariate case by Azzalini and DallaValle [4]. The class of Generalized Skew-Elliptical (GSE) distributions was introduced by Azzalini and Capitanio [3]. Branco and Dey [6] extended a general class of multivariate Skew-Elliptical distributions and analyzed various examples such as the skew-normal, skew-Pearson type 2, and skew-student-t. Hu and Kercheval [11] showed that the Student t and Skewed t distributions can be efficiently fitted to the data and is more proportional to the actual returns than the normal distribution. In this paper, we use the generalized skew-elliptical distributions. Consider $Y \sim GSE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma}, g^n, H)$ be an n -variate skew-elliptical random vector including mean vector $\boldsymbol{\mu}$, $n \times n$ positive definite matrix $\boldsymbol{\Sigma}$ and g^n a generator function, where $\boldsymbol{\mu} \in R^n$ and $\boldsymbol{\Sigma} > 0$.

$$f_Y(y) = 2|\boldsymbol{\Sigma}|^{-1/2} g^n \left(\frac{1}{2} (y - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (y - \boldsymbol{\mu}) \right) H \left(\boldsymbol{\gamma}^T \boldsymbol{\Sigma}^{-1/2} (y - \boldsymbol{\mu}) \right) \quad (4)$$

here $|\Sigma|^{-\frac{1}{2}}g^n \left(\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{y} - \boldsymbol{\mu})\right)$ is a Probability Density Function (PDF) of n-variate Elliptical distribution $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, g^n)$. $H(\boldsymbol{\gamma}^T \Sigma^{-1}(\mathbf{y} - \boldsymbol{\mu}))$ is a function of \mathbf{y} , where $H(y)$ is a CDF of a random variable that density is symmetric around zero with a PDF $h(y)$, and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$ is $n \times 1$ vector of shape parameters of the distribution that is constant [21]. The family of GSE distributions is closed under affine transformations, which was proved by Shushi [34]; it means that any linear combination of the skew-elliptical random variables also has a skew-elliptical distribution with the same characteristic generator. Consider A is a $m \times n$ matrix of rank m where $m \leq n$. If $\mathbf{Y} \sim GSE_n(\boldsymbol{\mu}, \Sigma, \boldsymbol{\gamma}, g^n, H)$ then

$$A\mathbf{Y} \sim GSE_m(A\boldsymbol{\mu}, A\Sigma A^T, \bar{\boldsymbol{\gamma}}, g^m, H) \tag{5}$$

where $\bar{\boldsymbol{\gamma}}$ refers to an $m \times 1$ vector of shape parameters, and can be calculated from the characteristic function of X . This feature for the family of scale mixture of skew-normal distributions was proven by Vernic [35]. This basic property has many applications in theoretical and applied probability, especially in the choice of capital. Let \mathbf{w} be a $n \times 1$ weighted vector, so the returns of a portfolio under generalized skew-elliptical distribution is equal to

$$R = \mathbf{w}^T \mathbf{Y} \sim GSE_1(\mathbf{w}^T \boldsymbol{\mu}, \mathbf{w}^T \Sigma \mathbf{w}, \tilde{\gamma}, g^1, H) \tag{6}$$

3 The Risk Measures

The tail conditional expectation for an univariate generalized skew-elliptical distribution, as the risk measure, is introduced by Landsman and Makov [20] and is equal to:

$$TCE_q(Y) = \mathbb{E} \left(L \mid L > VaR_q(L) \right) = \mu + \Lambda_{1,q} \sigma \tag{7}$$

where

$$\Lambda_{1,q} = 2 \frac{\bar{G}^{-1} \left(\frac{1}{2} z_q^2 \right) H(\gamma z_q) + \gamma K(z_q)}{1 - q} \tag{8}$$

$$z_q = VaR_q(z) = \frac{y_q - \mu}{\sigma}$$

$$K(z_q) = \int_{z_q}^{\infty} h(\gamma z) \bar{G}^{-1} \left(\frac{1}{2} z^2 \right) dz$$

The TCE measure is based on the average of the tail of the distribution. This measure alone does not give enough information about the risks on the tail of the distribution. Therefore, we need the deviation of the loss from the mean along the tail of the distribution. We use the explicit formula of the tail variance measure for the GSE distributions, which is introduced by Jamshidi and Khaloozadeh [15] and is equal to:

$$TV_q(Y) = Var \left(L \mid L > VaR_q(L) \right) = \Lambda_{2,q} \sigma^2 \tag{9}$$

where

$$\Lambda_{2,q} = \frac{2z_q \bar{G}^{(1)}\left(\frac{1}{2}z_q^2\right)H(\gamma z_q)}{1-q} + \sigma_{z_1}^2 r(z_q) + \frac{2\gamma h(\gamma z_q)\bar{G}^2\left(\frac{1}{2}z_q^2\right)}{1-q} + \frac{2\gamma}{1-q}\kappa(z_q) - \Lambda_{1,q}^2 \quad (10)$$

$$K(z_q) = \int_{z_q}^{\infty} h(\gamma z)\bar{G}^1\left(\frac{1}{2}z^2\right) dz$$

Now, we use these two risk measures to form the TMV model and derive the extended efficient frontier under the GSE distribution for optimal portfolio selection.

4 The Tail Mean-Variance Optimization Problem

We first introduce some of the notation used. Respectively, \mathbb{R} , \mathbb{R}_+ and \mathbb{R}_{++} indicate the sets of real numbers, non-negative real numbers, and real positive numbers [30]. Consider n risky assets ($n \geq 2$) with mean return vector $\mu \in \mathbb{R}^n$ and variance-covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$. Define $\mathbf{0}$ and $\mathbf{1}$, respectively, as the column vectors of zeros and ones with dimension n . The investment weight vector is denoted by $\mathbf{w} = (w_1, w_2, \dots, w_n)^T \in \mathbb{R}^n$, where w_i is the fraction of wealth invested in asset i .

4.1 The TMV Model under the GSE Distribution

First, consider the classical MV model:

$$g(\mathbf{w}; \tau) = -\mu^T \mathbf{w} + \frac{1}{2} \tau \mathbf{w}^T \Sigma \mathbf{w} \quad (11)$$

that is corresponding to (1). $g(\mathbf{w}; \tau)$ shows a trade-off between expected return $\mu^T \mathbf{w}$ and variance of return $\mathbf{w}^T \Sigma \mathbf{w}$ of the portfolio through a risk-aversion parameter $\tau > 0$. The tail mean-variance risk measure of the portfolio, introduced by Landsman [19], is defined as

$$TMV_q(L) = TCE_q(L) + \lambda TV_q(L) \quad (12)$$

and corresponds to the TMV criterion for portfolio selection. To obtain the TMV model under the GSE distributions, we can substitute (7) and (9) in the above equation. Therefore, it will be as follows

$$f(\mathbf{w}; \lambda, q) = -\mu^T \mathbf{w} + \Lambda_{1,q} \sqrt{\mathbf{w}^T \Sigma \mathbf{w}} + \lambda \Lambda_{2,q} \mathbf{w}^T \Sigma \mathbf{w} \quad (13)$$

where $\Lambda_{1,q}$ and $\Lambda_{2,q}$

$$\Lambda_{1,q} = 2 \frac{\bar{G}^1\left(\frac{1}{2}z_q^2\right)H(\tilde{\gamma}z_q) + \tilde{\gamma}K(z_q)}{1-\alpha}$$

$$\Lambda_{2,q} = \left[\frac{2z_q H(\tilde{\gamma}z_q)\bar{G}^1\left(\frac{1}{2}z_q^2\right)}{1-q} + \sigma_{z_1}^2 r_{1,q}(z_q) + \frac{2\tilde{\gamma}h(\tilde{\gamma}z_q)\bar{G}^2\left(\frac{1}{2}z_q^2\right)}{1-q} + \frac{2\tilde{\gamma}}{1-q}\kappa(z_q) \right] - \Lambda_{1,q}^2 \quad (14)$$

and $\tilde{\gamma}$ refer to an $m \times 1$ vector of shape parameters. The investor's risk preferences in the TMV model are represented by two parameters λ and q . $\lambda \in \mathbb{R}_{++}$ is a risk-aversion parameter and is similar to τ in the classical MV criterion. A low λ means that the manager is risk-averse, while a high λ corresponds to a risk-seeking manager who prioritizes returns over risk. $q \in (0,1)$ defines a certain threshold of loss in the portfolio. An investor is sensitive to losses that occur beyond the q -quantile because such losses are usually rare but large. By minimizing $g(\mathbf{w}; \tau)$ with respect to (w.r.t) $\mathbf{w} \in \mathcal{P}$ subject to a budget constraint $\mathbf{1}^T \mathbf{w} = 1$, the optimal solution result will be equal to, (see, e.g., Ref. [32], p. 382)

$$\bar{w} = w_0 + \frac{1}{\tau} z \tag{15}$$

where $\mathbf{w}_0 = \Sigma^{-1} \mathbf{1}/a$, $\mathbf{z} = \Sigma^{-1}(\boldsymbol{\mu} - \frac{b}{a} \mathbf{1})$, $a = \mathbf{1}^T \Sigma^{-1} \mathbf{1}$, $b = \mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}$, $c = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}$.

$\mathbf{w}_0 \in \mathcal{P}$ is the Minimum Variance portfolio, i.e., the portfolio that minimizes $\mathbf{w}^T \Sigma \mathbf{w}$ subject to $\mathbf{1}^T \mathbf{w} = 1$. $\mathbf{z} \notin \mathcal{P}$ is a self-financing portfolio because it is clear that it satisfies the property $\mathbf{1}^T \mathbf{z} = 1$.

According to (2), we can see that the TMV model under skew-elliptical risks does not satisfy the positive homogeneity of a coherent risk; $TMV(kL) \neq k TMV(L)$ for $k > 0$. Nonetheless, we show that the TMV model reverts to the MV model. First, we express two lemmas.

Lemma 1. Assume $q \in (0,1)$, then $\Lambda_{1,q} > 0$ and $\Lambda_{2,q} > 0$.

Proof. $\Lambda_{1,q} > 0$ is the combination of a positive number and multiplication of a cumulative distribution function by a decumulative distribution function in (8). $\Lambda_{2,q} > 0$ follows directly from (9). \square

Lemma 2. The quartic polynomial $F(x) = (x - l)^2(x^2 + m) - nx^2$ with $l, m, n > 0$ has exactly one non-coincident real zero in $(0, l)$ and exactly one non-coincident real zero in (l, ∞) . (for proof see, e.g., Ref. [21], Appendix A.1.)

Now using a convex optimization method, we minimize the proposed TMV model to obtain optimal portfolio weights.

Proposition 1. Consider $r \sim GSE(\boldsymbol{\mu}, \Sigma, \boldsymbol{\gamma}, g, H)$ is the vector of the rate of asset returns. Then under TMV criterion, the results are as follows when short sales are permitted. The minimum of $f(\mathbf{w}; \lambda, q)$ in (13) w.r.t $\mathbf{w} \in \mathcal{P}$ subject to a budget constraint $\mathbf{1}^T \mathbf{w} = 1$, exists and is unique and is equal to

$$w^* = arg \min_{\mathbf{w} \in \mathcal{P}} f(\mathbf{w}; \lambda, q) = w_0 + \frac{1}{\tau^*} z \tag{16}$$

$\tau^* \in \mathbb{R}$ is the unique root of the quartic equation

$$(\tau - 2\lambda\Lambda_{2,q})^2 \left[\frac{\tau^2}{a} + c - \frac{b^2}{a} \right] - \tau^2 \Lambda_{1,q}^2 = 0 \tag{17}$$

which is located in the range $(2\lambda\Lambda_{2,q}, \infty)$.

Proof. Define the Lagrangian $\mathcal{L}_f(\mathbf{w}, \gamma_f) = f(\mathbf{w}; \lambda, \alpha) - \gamma_f(\mathbf{1}^T \mathbf{w} - 1)$, where $\gamma_f \in \mathbb{R}$ is a Lagrange multiplier. $\partial \mathcal{L}_f / \partial \mathbf{w} = \mathbb{0}$ and $\partial \mathcal{L}_f / \partial \gamma_f = 0$ are written as follows

$$\begin{cases} \mu - \left(\frac{\Lambda_{1,q}}{\sqrt{\mathbf{w}^T \Sigma \mathbf{w}}} + 2\lambda \Lambda_{2,q} \right) \Sigma \mathbf{w} + \gamma_f \mathbf{1} = \mathbb{0} \\ \mathbf{1}^T \mathbf{w} = 1 \end{cases} \quad (18)$$

Suppose a solution for the above equation exists, and show it by $(\mathbf{w}^*, \gamma_f^*)$. Now if we consider the MV optimization problem with $g(\mathbf{w}; \tau)$ defined in (11), the Lagrangian is defined as $\mathcal{L}_g(\mathbf{w}, \gamma_g) = g(\mathbf{w}; \tau) - \gamma_g(\mathbf{1}^T \mathbf{w} - 1)$ where $\gamma_g \in \mathbb{R}$ is another Lagrange multiplier. The optimal solution $(\bar{\mathbf{w}}, \bar{\gamma}_g)$ is equal to the solution

$$\begin{cases} \mu - \tau \Sigma \mathbf{w} + \gamma_g \mathbf{1} = \mathbb{0} \\ \mathbf{1}^T \mathbf{w} = 1 \end{cases} \quad (19)$$

The solution $(\mathbf{w}^*, \gamma_f^*)$ of (18) corresponds with the solution $(\bar{\mathbf{w}}, \bar{\gamma}_g)$ of (19) while that τ gives a specific value, we show by τ^* which is equal to

$$\tau^* = \frac{\Lambda_{1,q}}{\sqrt{\mathbf{w}^{*T} \Sigma \mathbf{w}^*}} + 2\lambda \Lambda_{2,q} = \frac{\Lambda_{1,q}}{\sqrt{\bar{\mathbf{w}}^T \Sigma \bar{\mathbf{w}}}} + 2\lambda \Lambda_{2,q} \quad (20)$$

The existence of \mathbf{w}^* depends on the existence of a solution for $\tau^* \in \mathbb{R}_{++}$ in (20). Landsman [20] collected some statistical properties about the returns of \mathbf{w} , \mathbf{w}_0 , \mathbf{z} and shown which the variance of return on the MV optimal portfolio is:

$$\bar{\mathbf{w}}^T \Sigma \bar{\mathbf{w}} = (\mathbf{1}^T \Sigma^{-1} \mathbf{1})^{-1} + \frac{1}{\tau^2} \boldsymbol{\mu}^T \mathbf{z}$$

by substituting $\bar{\mathbf{w}}^T \Sigma \bar{\mathbf{w}}$ into (20) and rearranging equation, we will have

$$\tau^* - 2\lambda \Lambda_{2,q} = \frac{\tau^* \Lambda_{1,q}}{\sqrt{(\mathbf{1}^T \Sigma^{-1} \mathbf{1})^{-1} \tau^{*2} + \boldsymbol{\mu}^T \mathbf{z}}} \quad (21)$$

that also can be rewritten like (17). According to Lemma 1 and when $\tau^* \in \mathbb{R}_{++}$, the right-hand side of (21) is positive and real therefore the left-hand side will be positive as well, Thus, $\tau^* > 2\lambda \Lambda_{2,q}$. Also, Lemma 2 proves the existence of a unique real root of (17) in the range $(2\lambda \Lambda_{2,q}, \infty)$.

Corollary 1. $\arg \min_{\mathbf{w} \in \mathbb{P}} f(\mathbf{w}; \lambda, q) = \arg \min_{\mathbf{w} \in \mathbb{P}} g(\mathbf{w}, \tau^*)$, where τ^* is given in Proposition 1.

The optimization of the TMV criterion can be converted to the optimization of the classical MV criterion with the parameter τ^* as the risk-aversion parameter, calculated according to Proposition 1. So, a risk manager can easily use the TMV model for leptokurtic asset returns and aversion to tail risk.

4.2 The TMV Efficient Frontier

Each investor selects a portfolio that best meets his needs, which these portfolios located on the efficient frontier. So, investor's preferences play an important role in investment decisions. The efficient frontier is the set of optimal portfolios that present the greatest expected return for a given level of risk or the least risk for a defined level of return. The mean-variance efficient frontier \mathcal{E} is determined by

$$\mathcal{E} = \{\mathbf{w}_0\} \cup \left\{ \mathbf{w} \in \mathbb{R}^n : \mathbf{w} = \underset{\mathbf{w} \in \mathcal{P}}{\text{arg min}} g(\mathbf{w}; \tau) \text{ for each } \tau \in \mathbb{R}_{++} \right\} \quad (22)$$

Proposition 2. The TMV efficient frontier in the $(\sigma_{q,w}^2, \mu_{q,w})$ coordinates system is equal to

$$\mu_{\alpha,w} = \frac{b}{a} + \sqrt{\frac{\Delta}{a} \left(\frac{\sigma_{q,w}^2}{\Lambda_{2,q}} - \frac{1}{a} \right)} + \frac{\Lambda_{1,q}}{\sqrt{\Lambda_{2,q}}} \sigma_{\alpha,w} \quad (23)$$

where $\mu_{q,w} = -TCE_q(X) = \boldsymbol{\mu}^T \mathbf{w} - \Lambda_{1,q} \sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}$, $\sigma_{q,w}^2 = TV_q(X) = \Lambda_{2,q} \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}$.

Proof. The TMV optimal portfolio in the model (13) is obtained by

$$\mathbf{w} = \mathbf{w}_0 + \frac{1}{\tau^*} \mathbf{z}$$

where $\mathbf{w}_0 = \boldsymbol{\Sigma}^{-1} \mathbf{1}/a$, $\mathbf{z} = \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - \frac{b}{a} \mathbf{1})$ and $\tau^* \in \mathbb{R}_{++}$ is the unique root of the quartic equation (18).

If we rewrite the expected portfolio return and variance of portfolio return using \mathbf{w} , will have

$$\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} = \frac{1}{a} + \frac{1}{(\tau^*)^2} \frac{\Delta}{a}, \quad \boldsymbol{\mu}^T \mathbf{w} = \frac{b}{a} + \frac{1}{\tau^*} \frac{\Delta}{a}$$

where $\Delta = ac - b^2$. Now we can obtain the TCM frontier in the $(\sigma_{q,w}^2, \mu_{q,w})$ coordinate systems as the following

$$\mu_{q,w} = \frac{b}{a} + \sqrt{\frac{\Delta}{a} \left(\frac{\sigma_{q,w}^2}{\Lambda_{2,q}} - \frac{1}{a} \right)} + \frac{\Lambda_{1,q}}{\sqrt{\Lambda_{2,q}}} \sigma_{q,w}$$

where $\mu_{q,w} = \boldsymbol{\mu}^T \mathbf{w} - \Lambda_{1,q} \sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}$, $\sigma_{q,w}^2 = \Lambda_{2,q} \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}$. \square

Corollary 2. When $q = 0$ the TMV efficient frontier reduced to

$$\mu_w = \frac{b}{a} + \sqrt{\frac{\Delta}{a} \left(\sigma_w^2 - \frac{1}{a} \right)} \quad (24)$$

the TMV efficient frontier in the (σ_w, μ_w) coordinate system is according to the MV efficient frontier. Since $\Lambda_{1,q} = 0$ and $\Lambda_{2,q} = 1$ when $q = 0$, the TMV frontier in (23) reduces to the MV frontier in (24).

5 Application to Stock Data Returns

Consider a portfolio of 10 various industries, including 50 companies for the period 2017 to 2019 that listed in the Tehran Stock Exchange. Selected companies are from industries including Chemical products, Basic metals, Petroleum products, Automobile and parts, Food products except sugar, Sugar, Supply of electricity and gas, Pharmaceutical products, Cement and plaster, and Ceramic Tile. Stocks respectively show by $\mathbf{S} = (S_1, S_2, \dots, S_n)^T$ and stock daily returns denote by $\mathbf{X} = (X_1, X_2, \dots, X_n)^T$ with $n = 50$. The vector of means, standard deviation, skewness, and kurtosis of daily stock returns are given in Table 1. As we can see, the asset return rates do not follow normal

distributions and show the features of the leptokurtic and fat tail. For this purpose, we use the generalized skew-elliptical distributions for data analysis. We model the asset returns with a multivariate skew student-t-normal distribution with seven degrees of freedom ($\nu = 6$), that skew function has a normal distribution. Therefore captures the fat-tailed feature of asset returns, as also discussed by Landsman and Valdez [22].

Table 1: Descriptive Statistics for Daily Stock Returns.

Companies	Mean	Std	Skewness	Kurtosis	Companies	Mean	Std	Skewness	Kurtosis		
Ind_1	PJMZ1	0.0009	0.0217	-5.2658	66.218	Ind_6	GGAZ1	0.0033	0.0273	0.0258	2.4613
	PKLJ1	0.0015	0.0257	-9.724	179.62		GSBE1	0.0056	0.0352	1.3601	11.796
	PRDZ1	0.0022	0.0201	0.3483	4.9201		GESF1	0.0031	0.0291	0.3386	4.3577
	PTAP1	0.0015	0.0228	-1.6466	22.769		GHEG1	0.004	0.0293	-0.1188	2.5789
	IPTR1	0.0017	0.0223	0.1791	3.3022		SHKR1	0.0045	0.0296	0.062	2.7268
Ind_2	FOLD1	0.0027	0.0244	-6.0581	99.253	Ind_7	BFJR1	0.0014	0.0221	-0.4777	7.7877
	SHZG1	0.0021	0.0235	0.1409	3.1726		BMPN1	0.0031	0.0357	9.3702	165.85
	MSMI1	0.0024	0.0242	-0.6380	22.656		MOBN1	0.0017	0.0215	-1.7332	20.758
	FKHZ1	0.0022	0.0213	-1.5236	18.866		DMVD1	0.0033	0.0234	-0.6085	6.3547
	FKAS1	0.0025	0.0218	8.6188	133.61		VHOR1	0.0034	0.0272	0.1099	5.116
Ind_3	PNBA1	0.0016	0.0302	-7.4014	123	Ind_8	TMVD1	0.003	0.0261	-1.5126	14.617
	PNTB1	0.0038	0.0261	0.2171	3.4691		ABDI1	0.0024	0.0328	-10.032	186.69
	PTEH1	0.002	0.0319	-6.0289	83.319		EXIR1	0.004	0.0261	0.0393	3.6707
	PNES1	0.0017	0.0325	-8.8247	157.48		DALZ1	0.0012	0.0272	-5.2642	81.0698
	SEPP1	0.0023	0.0413	-12.3978	244.79		DSOB1	0.0012	0.0303	-2.3345	22.512
Ind_4	IKCX1	0.0024	0.026	-0.0128	2.6238	Ind_9	SHZG1	0.0021	0.0297	-3.4176	86.821
	SSAP1	0.0016	0.0269	-0.3106	4.1868		SEFH1	0.0026	0.0276	-0.9711	10.779
	PKOD1	0.0013	0.0252	0.1841	2.6362		SKAZ1	0.0025	0.0344	-2.6255	39.604
	BHMN1	0.0012	0.0316	-7.7768	131.83		SSEPI	0.0022	0.0295	-1.1234	14.776
	ZMYD1	0.0019	0.0269	0.0707	2.6914		SFNO1	0.0033	0.0275	-1.4005	13.667
Ind_5	GCOZ1	0.0014	0.0336	-7.5601	128.39	Ind_10	KPRS1	0.0023	0.0301	-0.5555	7.5495
	MINO1	0.0028	0.0237	0.4781	3.4123		KSAD1	0.0035	0.0269	0.0536	4.2847
	TSBE1	0.0021	0.0219	0.1333	5.6158		TKSM1	0.0035	0.0336	0.4414	4.2497
	SHAD1	0.0051	0.0562	13.169	243.69		ALVN1	0.0034	0.0311	0.0438	1.9382
	SLMN1	0.0063	0.0324	8.8032	132.42		SINA1	0.003	0.0274	0.1541	2.3707

The proposed TMV optimal portfolio weights can be obtained quickly and easily using Proposition 1, whose results are presented in Table 2. The risk-aversion parameter is $\lambda = 5$. For comparison, we also present in this table, other optimal portfolios when the vector of skewness parameters $\gamma = 0$, i.e.,

a mean-variance portfolio and a minimum variance portfolio. We can see in Table 2 that the presence of $\gamma = 0$ in the optimization criterion, the range of weights changes.

Table 2: Optimal Portfolio Weights by Minimizing (i) the Tail Mean-Variance Criterion (Min TMV), (ii) the Mean-Variance Criterion (Min MV), (iii) the Variance (Min Variance).

	Optimal Portfolio Wiegths									
Stocks	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10
Min TMV q=0.95	-0.0105	0.0120	0.0804	-0.0004	0.0243	0.0315	0.0108	0.0349	0.0227	0.0498
Min MV	-0.1338	-0.0319	0.1237	-0.0735	-0.0501	0.0520	-0.0116	0.0652	-0.0020	0.0512
Min Variance	0.0412	0.0305	0.0623	0.0302	0.0554	0.0229	0.0202	0.0221	0.0330	0.0492
Stocks	S11	S12	S13	S14	S15	S16	S17	S18	S19	S20
Min TMV q=0.95	0.0255	0.0427	-0.0113	-0.0064	0.0125	0.0133	0.0174	-0.0049	-0.0175	-0.0207
Min MV	0.0058	0.0942	-0.0357	-0.0065	-0.0068	0.0553	0.0000	-0.0676	-0.0643	-0.0620
Min Variance	0.0337	0.0210	-0.0010	-0.0064	0.0205	-0.0043	0.0247	0.0213	0.0021	-0.0034
Stocks	S21	S22	S23	S24	S25	S26	S27	S28	S29	S30
Min TMV q=0.95	-0.0178	0.0129	0.0393	0.0105	0.0696	0.0398	0.0732	0.0220	0.0290	0.0503
Min MV	-0.0788	0.0113	0.0531	0.0198	0.1825	0.0716	0.1533	0.0551	0.0652	0.1193
Min Variance	0.0078	0.0135	0.0335	0.0066	0.0222	0.0265	0.0396	0.0082	0.0138	0.0214
Stocks	S31	S32	S33	S34	S35	S36	S37	S38	S39	S40
Min TMV q=0.95	0.0340	0.0248	0.0293	0.0594	0.0368	0.0354	0.0220	0.0319	0.0139	-0.0231
Min MV	-0.0165	0.0398	-0.0216	0.1168	0.0534	0.0635	0.0244	0.1118	-0.0245	-0.0845
Min Variance	0.0551	0.0185	0.0506	0.0354	0.0298	0.0236	0.0209	-0.0015	0.0300	0.0026
Stocks	S41	S42	S43	S44	S45	S46	S47	S48	S49	S50
Min TMV q=0.95	0.0313	0.0112	-0.0066	-0.0246	0.0279	-0.0023	0.0287	0.0086	0.0068	0.0202
Min MV	0.0156	0.0077	-0.0241	-0.0455	0.0698	-0.0101	0.0684	0.0217	0.0180	0.0619
Min Variance	0.0379	0.0126	0.0007	-0.0158	0.0103	0.0010	0.0121	0.0032	0.0020	0.0027

The purpose of the proposed proposition was to minimize the TMV criterion for the GSE distributions. To evaluate model, we consider two validations. For the first validation, we use the obtained weights from the MV model to calculate the TCE, TV, and TMV values as risk measures, and for the second, we consider that the value of each share is equal, and then calculate these criteria. The evaluation results are presented in Table 3; it shows that the obtained weights of the TMV model are optimal because the calculated values of risks in this method have lower values than the other two methods.

Table 3: Comparison of the TCE, TV and TMV Optimal Value and Validation Models

Risk Validation	TCE	TV	TMV
Optimal Value (Min TMV $q=0.95$)	0.0173	0.0005	0.0200
Validation.1 Value	0.0367	0.0024	0.0488
Validation.2 Value	0.0209	0.0007	0.0246

In Fig. 1, the TMV optimal portfolio, obtained at the confidence level $q = 0.95$, is shown in mean-standard deviation space, and also the MV efficient frontier is displayed. The TMV optimal portfolio is located on the efficient frontier, which consists of the concave segment of the frontier. We compare the TMV optimal portfolio with the corresponding MV optimal portfolio by minimizing their criteria and by minimizing the criteria in (2) and (1), also set $\lambda = \tau/2$ for consistency between (2) and (1). As can be concluded, the TMV portfolio is more conservative than the MV portfolio because its return has a lower variance than that of the MV portfolio, and this is not surprising since avoiding large losses is a major target in the TMV criterion.

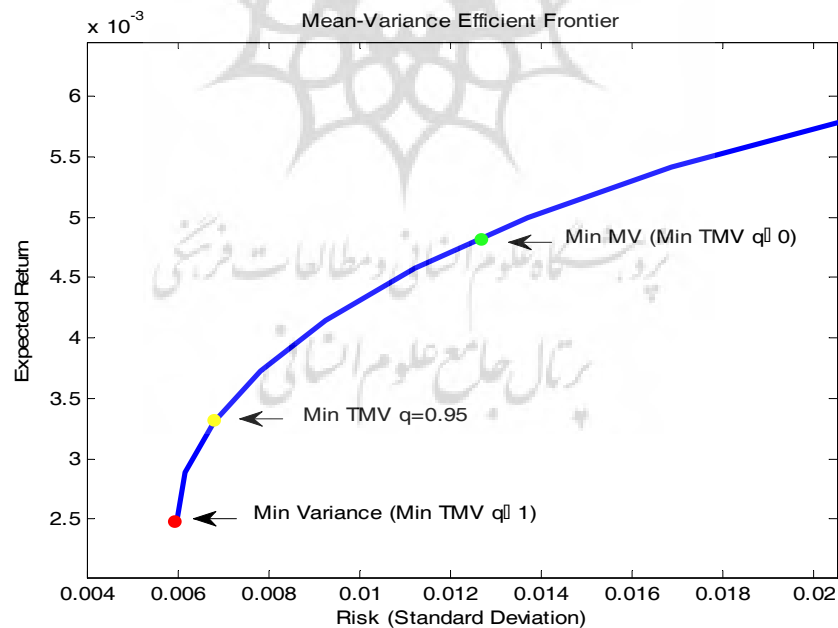


Fig. 1: Efficient Frontier and Optimal Portfolios Obtained by Minimizing (i) the Tail Mean-Variance Criterion (Min TMV), (ii) the Mean-Variance Criterion (Min MV), (iii) the Variance (Min Variance).

As $q \rightarrow 0$, the TMV portfolio tends to the MV portfolio because $\Lambda_{1,q} \rightarrow 0$ and $\Lambda_{2,q} \rightarrow 1$ and as $q \rightarrow 1$, the TMV portfolio tends to the minimum variance portfolio. According to this matter, an investor can quantify the q -quantile of loss beyond that he is sensitive to losses. The higher this threshold, his portfolio becomes more conservative because it is more sensitive to high losses. According to (14), the values of the parameters $(\Lambda_{1,q}, \Lambda_{2,q})$ are calculated as (1.15,2.37), (0.46,1.46), (0.29,1.17), (0.06,1.08) when $q = 0.95, 0.7, 0.5, 0.1$. The TCE and TV risk can be obtained in terms of these values and the covariance matrix estimation. From the empirical results of $(\Lambda_{1,q}, \Lambda_{2,q})$, we can observe that the TCE risk is more sensitive to the choice of the confidence level of q , compared with the TV risk of the portfolio.

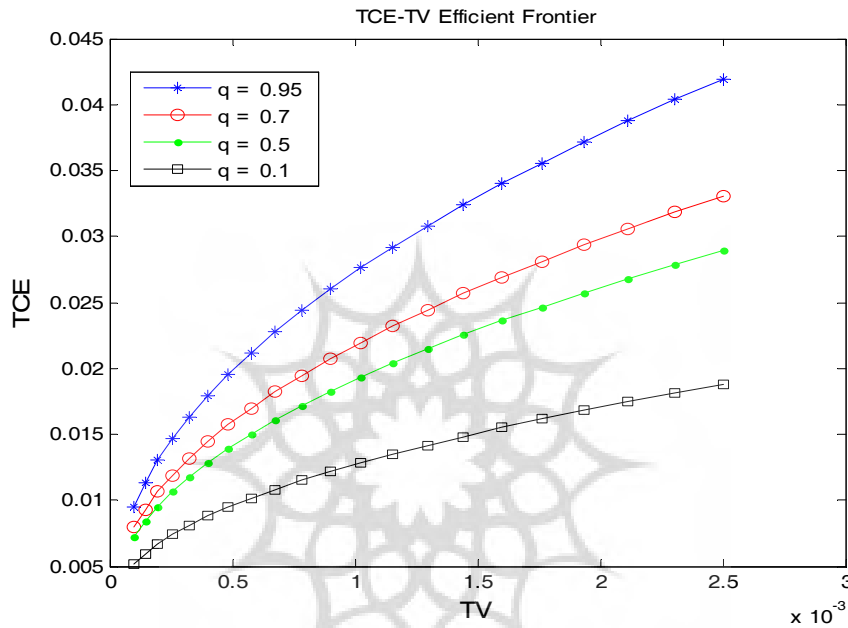


Fig. 2: TCE-TV Frontier.

In Fig. 2, the TCE-TV efficient frontier, calculated at the confidence levels $q = 0.95, 0.7, 0.5, 0.1$, is displayed. As explained, the TCE and TV risk are sensitive to changes in q , so the efficient frontier also changes with changes in the different confidence levels. In particular, when $q = 0$, the TMV portfolio frontier reduced to the MV frontier.

6 Conclusion

In the insurance and financial markets, events of extreme losses happen in the tail of loss distributions, and investors are sensitive to these losses. In this paper, we presented the tail mean-variance model using two risk measures, i.e., the TCE and TV under the generalized skew-elliptical distribution. The family of elliptical dependent structures is appropriate for modeling non-symmetric phenomena. The TMV criterion can help an investor to understand the behavior of risk along the tail of loss distribution ($X \geq VaR_q(X)$) by considering different confidence levels. We also used the TMV model for optimal portfolio selection and obtained an explicit solution by providing a simple computational method, away from the inversion, partition, and concatenation of large matrices. Also, the

extended efficient frontier formula derived from this criterion is presented. The TMV criterion allows investors to choose portfolios that take into account the extreme risk with large losses, and this criterion is proper for leptokurtic and fat-tailed asset returns. By using the TMV criterion and extended efficient frontier, an investor can control his aversion to tail risk and determine the q -quantile of loss beyond that he is sensitive to losses. As the threshold of loss gets high, an investor becomes more sensitive to extreme losses, and his optimal portfolio becomes more conservative. This problem is an essential issue in investing. For further research, we want to investigate the direct effect of the skewness parameter on the optimal weights and efficient frontier.

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