Advances in mathematical finance & applications, 5(4), (2020), 419-435
DOI: 10.22034/amfa.2020.1897346.1407

Published by IA University of Arak, Iran
Homepage: www.amfa.iau-arak.ac.ir

Overview of Portfolio Optimization Models

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ARTICLE INFO

Article history:
Received 13 April 2020
Accepted 09 July 2020

Keywords:
Portfolio Optimization Models
Mathematical Models
Meta-Heuristic Models.

ABSTRACT

Finding the best way to optimize the portfolio after Markowitz's 1952 article has always been and will continue to be one of the concerns of activists in the investment management industry. Researchers have come up with different solutions to overcome this problem. The introduction of mathematical models and meta-heuristic models is one of the activities that has influenced portfolio optimization in recent decades. Along with the growing use of portfolios and despite its rich literature, there are still many unanswered issues and questions in this area. Also, Iranian capital markets, as emerging markets, require native research to answer these questions and issues. The purpose of this study is to provide a useful and effective tool to assist professionals and researchers in portfolio selection theory. This study, while comprehensively reviewing the literature on the subject and the developments and expansions made in the area of portfolio selection and optimization, reviews the types of problems and optimization methods.

1 Introduction

Portfolio selection is one of the most common issues faced by different investors with varying levels of capital, and yet one of the most complex in the financial world [44]. The issue of portfolio selection is a model of balancing risk and return. This involves a set of securities that attempt to determine the proportion of investment in each in order to minimize investment risk and maximize return on investment [24]. However, high yields are usually high risk [8].

Investors typically hold several securities in an investment portfolio [5]. In 1952, Harry Markowitz used mathematical programming and variance to evaluate portfolio, mean and return, portfolio selection by optimizing two conflicting criteria of risk and return. His mathematical modeling was a long way from the real world, but it had a profound effect on improving the portfolio selection procedure, with many researchers thereafter refining his theory, but so far a comprehensive model that investors can choose for the optimal portfolio of investments, to use it has not been introduced. The Markowitz model had two important drawbacks. First, its risk assessment criterion was not a suitable criterion for portfolio risk assessment, and second, the aforementioned model was not appropriate for the long-term horizon [22]. On the other hand, the issue of portfolio selection in the real world involves transaction costs or at least the amount of transactions that make it a complex mathematical problem [25].

After averaging the variance models, the researchers focused on other models such as discrete-time models, continuous-time models, and random programming models. Each of these methods has its disadvantages and advantages that can be selected according to the investor decision making conditions. Continuous and discrete multi-period models are solved by dynamic scheduling and optimal control methods. Due to the large dimensions of the optimal portfolio selection problem, the men-

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tioned models face serious challenges. To this end, simplifications were introduced that led to distancing from the real world and limited the application of these methods [22].
The concepts of portfolio optimization and diversification have been instrumental in developing and understanding financial markets and financial decision making. The publication of Harry Markowitz's portfolio theory was the main and most important success in this regard [11]. Since its launch, Markowitz has made many changes to people's attitudes to investing and portfolio and has been used as an efficient tool for portfolio optimization [27]. Markowitz suggested that investors take risk and return together and select the amount of capital allocation between different investment opportunities based on the interaction between the two [11]. One of the most important issues in capital markets that should be of interest to investors, whether natural or legal, is the choice of the optimal portfolio of investment and in this regard, investing in investing in choosing the best portfolio of investments. Pay attention to the amount of risk and its returns. It is usually assumed that investors do not like risk aversion and avoid it and are always looking to invest in asset items that have the highest return and the least risk. In other words, investors return on equity. Investments are seen as a desirable factor and they consider risk variance as an undesirable element. In portfolio optimization, the main issue is the optimal selection of assets and securities that can be prepared with a certain amount of capital, although minimizing the risk. And maximizing return on investment seems simple, but in practice there are many ways It is used to form an optimal portfolio: the problem of Markowitz optimization and the determination of the effective investment boundary, when mathematical models are solvable when the number of marketable assets and market constraints are low, but when the conditions and Given the limitations of the real world, it will be a complex and difficult task, for years to solve such complex problems of advanced mathematics and computers with the help of human beings to help him to overcome environmental uncertainty and ambiguity more and more. One of the approaches that have been used in recent years to solve many optimization problems has been to unravel human ambiguities and to succeed in addressing complex issues, also known as heuristic methods and algorithms. Innovative methods introduced to address the shortcomings of classical optimization methods by comprehensive and random search, they greatly guarantee the likelihood of better results, including algorithms such as genetic algorithm, bird migration, metal refrigeration, Forbidden search, ant colony, bee dance, colonial competition, etc. [9, 10]. So far, no research has been done on portfolio optimization models in Iran. Therefore, this study aimed to review portfolio optimization models.

2 Theoretical Basics and Background of the Research

One of the most important issues in portfolio optimization is risk measurement. The problem of portfolio optimization is basically the determination of the amount of each share in the portfolio with the two objectives of minimizing risk and maximizing returns. Risk quantification is one of the very old challenges that have plagued many researchers. To this end, a variety of risk measures have been introduced so far. But risk is almost always defined in the financial field on the basis of figures from returns records.

2.1 Optimization

To get the best possible result for an issue according to the conditions governing it is called optimization. The intrinsic characteristic of humans and other beings is the tendency to do things with the least amount of time and to maximize profits, which is the main reason for human concerns to increase the efficiency and effectiveness of their activities against relatively limited resources. Optimization can be defined as the process of finding conditions that give the maximum or minimum value of a function [19, 20]. Since there is no one-size-fits-all approach to solving all the optimization problems, a variety of optimization methods have emerged to solve different optimization problems [2].

The two most important components in deciding on investment are the amount of risk and return on capital assets. Optimal asset selection is often driven by the trade-off between risk and return, and the higher the asset risk, investors expect higher returns. Identifying the asset boundary of the portfolio enables investors to derive the most expected return on their investment based on their utility and degree of risk aversion and risk-taking. Each investor selects a point on the effective boundary based on their risk-taking and risk aversion and sets their portfolio composition to maximize returns and minimize risk [46]. Portfolio optimization is selection of the best combination of financial assets in a way that maximizes return on investment and minimizes portfolio risk as much as possible. The basic idea of modern portfolio theory is that if invested in assets that are not fully correlated, the risk of those assets neutralizing each other and a fixed return on less risk can be achieved [34].

2.2 Portfolio Optimization Models

The algorithms that exist to solve the optimization problems can be divided into two categories: precision algorithms and approximate algorithms. Exact algorithms are able to find optimal solutions accurately, but approximate algorithms are able to find near optimal solutions for difficult optimization problems and are divided into three categories of heuristic, meta-heuristic, and hyper-heuristic. The two main problems of the heuristic algorithms are their local optimality, and their inability to apply them to various problems. The meta-heuristic algorithms presented to solve the heuristic algorithms are a variety of approximate optimization algorithms that have local optimization solutions that are applicable to wide range of problems.

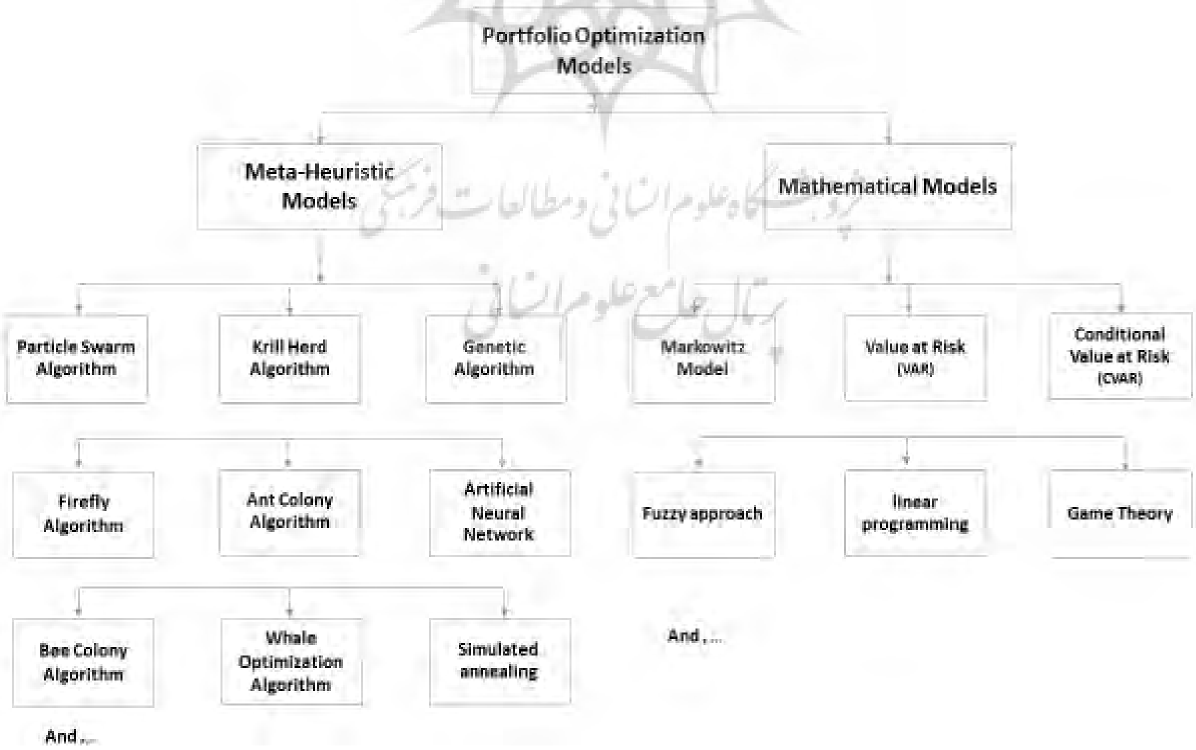

Table 1: Comparison of Recent Researches in Portfolio Optimization [50]

Researcher (s)	Investment horizon	Solution Method	Theory Framework	Risk Criterion	Number of goals	The type of data used	Consider the cost of transactions	Allocate part of capital to risk-free assets	Limits the minimum degree of diversification
Yao et al. [57]	Multi-period	Dynamic planning	Probability Theory	Variance	Single Purpose	certain	∂	∂	∂
Mehlawat [36]	Multi-period	Ideal Planning	Credit Theory	Fuzzy Entropy	multi-purpose	Fuzzy	¬	∂	∂
Guo et al. [17]	Multi-period	Genetic based fuzzy simulation	Credit Theory	Fuzzy variance	Single Purpose	Fuzzy	¬	¬	∂
DeMiguel et al. [7]	Multi-period	-	Probability Theory	Variance	Single Purpose	certain	¬	∂	∂
Cong [6]	Multi-period	Monte Carlo simulation	Probability Theory	Variance	Single Purpose	certain	∂	¬	∂
Vercher and Bermúdez [56]	Single period	Genetic Algorithm	Credit Theory	Half an absolute deviation	multi-purpose	Fuzzy	∂	∂	∂
Liu et al. [32]	Multi-period	PSO hybrid algorithm	Credit Theory	Absolute deviations below procedure	multi-purpose	Fuzzy	¬	∂	∂
Liu and Zhang [33]	Multi-period	Genetic Algorithm	Theory of Possibility	Half-fuzzy variance	multi-purpose	Fuzzy	¬	∂	¬
Gupta et al. [18]	Multi-period	Discrete approximate iteration method	Theory of Possibility	Absolute mean deviations	Single Purpose	Fuzzy	¬	¬	∂
Zhang et al. [59]	Multi-period	Fuzzy Evolutionary Algorithm	Theory of Possibility	Half an absolute deviation	Single Purpose	Fuzzy	¬	∂	¬
Liu et al. [30]	Multi-period	PSO Algorithm	Theory of Possibility	Variance	Single Purpose	Fuzzy	¬	∂	¬
Liu et al. [31]	Multi-period	TOPSIS	Theory of Possibility	Variance	multi-purpose	Fuzzy	¬	∂	∂

2.3 Markowitz Mean-Variance Model

Markowitz was the one who introduced and developed the concept of portfolio diversification. He showed in general how the diversification of the portfolio reduces the risk for the investor. Investors can obtain efficient portfolio for a given return by minimizing portfolio risk. Further, the above process can lead to the formation of efficient baskets called the mean-variance efficient boundary [35].

To use the Markowitz model requires the following data:

1) Expected return related to share i, denoted by E (R_i).

2) The standard deviation of expected return for share i, which is shown as a measure of the risk per share considered with Si.

3) Covariance, as a measure of the association and correlation between the various stock returns rates represented by the symbol δ_{ij}.

The reason a company's stock is a risky asset is that its overall (weekly, monthly, annual) rate of return is not (random). As these rates change with time, they can be subdivided into probability distributions and obtain the required criteria for the Markowitz model, such as mean, standard deviation, covariance, and so on (etc…).

The Markowitz model is based on the following assumptions: Investors are risk averse and have expected incremental utility, and the final utility curve of their wealth is declining. Investors choose their portfolio based on expected mean-variance returns. Therefore, their indifference curves are a function of the expected rate of return and variance. Every investment option is infinitely divisible. Investors have a time horizon and this is the same for all investors. Investors prefer a higher return on a given level of risk, and on the other hand, investors consider two factors in their choice [24].

A) "High expected return" which is the desirable factor.
B) "Uncertainty of Return" which is an undesirable factor.

To obtain the optimal portfolio choice in the Markowitz method that has the least variance for a particular level of return, we have the following linear programming model [35].

$$
\begin{aligned}
&Min\ z = \delta_p^2 \\
&St{:}\ \overline{r_p} = \sum_{i=1}^{n} w_i.\overline{r_i} \\
&\sum_{i=1}^{n} w_i = 1 \\
&w_i > 0
\end{aligned}
\tag{1}
$$

In relation (1) we have:

w_i=Weight related to i's share in the portfolio

$\overline{r_p}$ = Expected return on portfolio

$\overline{r_j}$ = Stock returns

δ_p^2= Portfolio Return Variance

The portfolio return variance is calculated as follows:

$$\delta_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \, cov(\overline{r_i}.\overline{r_j}) \quad (2)$$

2.4 Value at Risk (VaR)

The value at risk Criterion was introduced in 1993 as a benchmark for risk identification. This criterion is defined as the maximum expected loss on a target horizon with a confidence interval and under normal market conditions [27]. Generally, the value at risk for the purchase position measures the potential loss in value of a risk asset or portfolio over a specific time period at a given confidence level. A similar definition can also be applied to the value at risk models of the sales position, in which case the value at risk for a portfolio is the potential profit of a holding in a given time period and Specific confidence level.

There are three main components in assessing the value of assets at risk: a certain level of loss, a fixed period of time in which to assess the risk, and the level of confidence required. In addition, value at risk can be held for an asset or for a subset of assets or for the entire financial institution. VaR is a concept before it is a risk measure and it is precisely for this reason that numerous methods and approaches have been developed to calculate and measure it. Each of these approaches in turn seeks to satisfy the concept of value at risk. The value at risk is the maximum loss that the depreciation of the portfolio for a given period in the future does not exceed with a certain confidence factor. In other words, VaR measures the worst expected loss under normal market conditions over a specified period of time and at a certain confidence level. VaR answers the question of how much of an asset's value or portfolio of assets is at risk with an x% probability over a specified time horizon.

Mathematically the value at risk can be expressed as follows:

$$Pr\{P_0 - P_1 \geq VaR\} \leq \alpha \quad (3)$$

Where P_0 is the portfolio value at time zero and P_1 is the portfolio value at time 1 and δ is the statistical error level P_1. The above equation states that the probability that the portfolio's depreciation in the future period is more than the value at risk is equal to the maximum δ ,in other word , the probability that the portfolio loss is less than the value at risk in the future period is 1-δ if We show the cumulative distribution function of portfolio value in the next period with F (P); its inverse means $\boldsymbol{F_P^{-1}}(\delta)$ represents the percentages of portfolio value in the forthcoming period, thus the value at risk of portfolio The following relationship is obtained [45]:

$$VaR = P_0 - F_P^{-1}(\alpha) \quad (4)$$

2.5 Conditional Value at Risk (Cvar)

Different criteria have been proposed for measuring undesirable risk, the most important of which is the value at risk that does not have variance defects, that is, it does not rely on a particular distribution and only considers undesirable fluctuations. Display the value at risk with the VaR symbol. By definition, value at risk is the maximum value that the portfolio for a given period in the future does not exceed, with a certain confidence coefficient [48]. But this criterion, with all its acceptance, still could not be a complete risk measure. One of the main problems with value at risk is that the risk of an asset

portfolio, on the basis of this criterion, may be greater than the total risk of individual assets [47]. However, portfolio diversification allows for the loss of one share by the profit of the other and therefore it is reasonable to assume that the risk of an asset portfolio is lower than the total risk of each asset. Some research has shown that the use of a risk-weighted measure may prevent portfolio diversification [55]. Therefore, in recent years, "conditional risk value" has been introduced for the development of value at risk, while it has the disadvantages of criterion value at risk [1]. The value at conditional risk indicates that if the situation is unfavorable, we expect to suffer some loss [21].
Artzner by introducing the measure of conditional value at risk, that this criterion would cover the value at risk deficiencies. This model and criterion, also known as expected risk and comet variance, all the features that VaR was faced with the deficiencies, as well as cover. This criterion is defined as follows:
Averages of risks that are large and beyond value at risk. In other words, α percent of the mean distribution of random variable returns is greater than the value at risk.Value at risk is measured by the following relationship for short-term periods:

$$CVaR = \frac{e^{-\frac{z_{\alpha}^{2}}{2}}}{\alpha\sqrt{2\pi}}\,\delta_p - \overline{r_p} \tag{5}$$

Where α is the maximum error considered, Z_{α} is the corresponding value of α probability in the normal distribution and δ_p of the variance of the basket is considered.
For long periods of the following formula is used: the same as value at risk for long periods $\mu \neq 0$ to consider. So we have:

$$CVaR = \frac{e^{-\frac{z_{\alpha}^{2}}{2}}}{\alpha\sqrt{2\pi}}\,\delta_p - \mu w_i = \frac{e^{-\frac{z_{\alpha}^{2}}{2}}}{\alpha\sqrt{2\pi}}\delta_p - \overline{r_p} \tag{6}$$

This value indicates that the probability of loss in a T-day period is greater than the probable risk value, α%.

2.6 Game Theory

Game theory studies decision making in environments where there is a combination of interaction and cooperation. John Van Neumann first introduced game theory. In a 1928 book, Theory of Game and Economic Behavior, which he published in collaboration with economist Oscar Mongenstern, he expanded on the basic concepts of this theory and outlined its application in economics. Since then, this theory has been applied in various sciences including sociology, psychology, political science, evolutionary science, and the environment [58].
A game is a technical presentation of a situation where the outcome of each person's action depends not only on his own action but also on the actions of others. So the optimal decision he makes is related to his expectation of what others will do. The basic characteristic of in-game decision-making is that each player must analyze the reaction of others to his / her choice and decision-making before making a decision and then making the decision that is best for him / her. The main purpose of game theory is to develop sensible criteria for choosing a policy or strategy. In the game theory framework, policies refer to the set of decisions that an actor can make at any given decision point. The development of criteria for choosing optimal policy in game theory is based on two assumptions: first, which all players are rational and second, that they use their full potential in adversity to achieve the best

possible outcome [28, 29]. To behave wisely is to think deeply about a person before taking action and to consider his purpose, preferences, and limitations, then to choose and act in such a way that they do not suffer.

In game theory, games have different types and different categories are considered, the main of which is to classify games into two types of zero-sum and zero-sum games. A zero sum game means that one player's win equals the loss of the other, and one player's gain is associated with the other's loss. The game is played in conditions of competition and conflict [53, 54]. However, in non-zero sum games, the two players may win or lose at the same time and the profit or loss is not necessarily equal. Non-zero-sum games are more complex than zero-sum games and are categorized into cooperative and non-cooperative games. In cooperative games, the possibility of a mutually agreeable agreement between the two players is added to the policies that improve the interests of both. Other types of game categories such as symmetrical and asymmetric games, simultaneous and sequential games, complete gameplay and incomplete information, specify different game conditions. The ultimatum game, the dictator game, is one of the popular games [39, 40].

2.7 Meta-Heuristic Algorithms

There are basically many ways to solve a portfolio optimization problem, the most famous of which is the Markowitz optimization problem. But if more constraints are used to solve the Markowitz problem, it can no longer be solved by the quadratic method. In this case, meta-algorithms should be used. meta-heuristic algorithms are algorithms that are generally inspired by nature and can be used to solve nonlinear problems with constraints. Metabolism algorithms are axial search algorithms that come close to the optimal solution in each cycle of the algorithm. Using these algorithms there is no guarantee that we will certainly get the optimal solution, but when the limitations of the problem are very high, these algorithms can be a good way to solve such problems. There are also many algorithms and methods to predict the data. The meta-heuristic algorithms have three different modes: 1- single-member algorithms, 2- group members search algorithms, 3- hybrid algorithms [10].

The solution to the optimization problem using meta-heuristic algorithms was first developed by Plia [41]. Unlike precision solving methods, meta-heuristic algorithms can be used to large-scale problems by providing satisfactory solutions in a reasonable amount of time. The problem size and its structure must be taken into account when using meta-heuristic algorithms or precise methods to solve a problem [49]. Single-member search algorithms are algorithms that solve the problem by using a member defined in the algorithm and ultimately optimize the problem.

For example, the banned search algorithm, simulated refrigeration and hill climbing algorithms are single-member algorithms. Group members search algorithms are algorithms that solve the problem by using the group defined for the algorithm. These algorithms work better than single-member algorithms. For example, genetic algorithms, particle swarm, bee colonies are group search algorithms. Combined algorithms are algorithms that combine the other two algorithms. These algorithms are generally slower but more accurate than the other two. The important thing about these algorithms is that if any algorithm is merged with the genetic algorithm, an algorithm is called the memetic algorithm. This algorithm has received much attention in recent years from researchers. For example, the frog mutation algorithm is a memetic algorithm.

Table 2: Advantages and disadvantages of meta-heuristic algorithms

Disadvantages	Advantages	Year	Innovators	Algorithm
Its high running cost - requires a lot of memory and computing - has poor math support	Parallel System - Flexibility - Limitation - Selecting the Best From the Population - High Chance to Achieve Global Optimization - Easy Implementation	1975	Holland	Genetic Algorithm
Difficult theory-non-independence-iteration of changes based on uncertain convergence time-probability	Parallel System-Positive Feedback-Quick Finding-Avoiding Early Convergence of Dynamic Issues	1992	Dorigo	Ant Colony Algorithm
Getting caught in local optimum-reducing population diversity	Zero order - no need for complex mathematical operations, high flexibility - easy implementation - fast convergence - memory - information sharing - no deletion	1995	Kennedy-Eberhart	Particle Swarm Optimization Algorithm
Trapped in local optimal-inefficiency in discrete problems	Relatively Optimized-Easy Implementation - Harmony Contribution -Simple Computing-Simple Concept-Less Mathematical Requirements- High Flexibility to Search for Better Space	2001	Kim	Searching for Harmony
Coordinate Number of Variables-Quantitative Parameters-Population Dependence-Randomization-Multiple Parameters-Need to Adjust Parameters-Using Probability Method	Efficiency of many optimal solutions - Control parameters - High convergence speed - Local minimum output - High flexibility - Multidimensional problems - Global optimization - Easy detection - Global and local search - High probability of finding an answer	2005	Karaboga	Bee Colony Algorithm
No precise method for determining parameters - trapping in local optimum - no instantaneous change - not remembering the optimum	Problems of maximization and unconditional minimization -Easy adjustment-Low parameters-Very fast convergence of member independence-Transition from local optimization-Parallel implementation-Automated whole population segmentation-Multi-quality optimization-Variability in solutions	2006	Yang	Firefly Algorithm
Time-to-non-formulation calculations of a problem	It solves any formulation problem — no need for evolution-based problem knowledge	2008	Simon	Biogeography-Based Optimization
Can't converge all the time with satisfactory performance - lack of access to optimal point in multi-quality issues	Parallel system - including both global and local strategies - widespread application in optimization problem solving - speed of implementation	2012	Gandomi - Alawi	Krill Herd Algorithm

2.8 Genetic Algorithm

The genetic algorithm first proposed by John Holland is a kind of search algorithm based on the mechanism of natural selection and genetic science. Genetic algorithm is a comprehensive probabilistic search method that follows the process of natural biological evolution [15]. The work of the genetic algorithm is deceptively simple, very easy to understand and, in simple terms, the simplest way that humans believe that animals have evolved accordingly [13]. This algorithm combines the robustness of survival of the best string structure with the random information exchange operation and forms a very powerful search algorithm. To solve the problem with this algorithm, at first the response must be encoded so that the algorithm can be evaluated and implemented by various operators. The implementation of this algorithm begins with a primitive set of random solutions called the initial population. Each member in the population is called a chromosome that represents a solution to the existing problem. A chromosome is a string of numbers called a gene. During each iteration of the algorithm, a new set of chromosomes is generated. The population at a given time is called generation. During each generation, the amount of chromosomes fits with the fitting function that estimates one chromosome according to the problem's objective function. During the reproduction process, genetic operators, such as the offspring and the mutation operator, are applied to the chromosomes. The chromosomes that are produced there are called the baby. At this stage, after evaluating the neonates' fitness, the better chromosomes will be selected by one of the selection procedures and transferred to the next generation. For each of the genetic operators a probability parameter is defined that the operators are likely to apply to the chromosomes. This process will continue until the algorithm's stopping condition is reached [38].

2.9 Simulated Annealing Algorithm (SA)

This algorithm is derived from the gradual cooling method, which involves placing the material at high temperature and then gradually lowering the temperature. This method is a simple and effective meta-heuristic search for combinatorial optimization problems [14]. This optimization algorithm is based on the science of physics and was first developed by Metropolis et al. [37] and then by Kirkpatrick et al. In 1983 and 1985 [26].

Material engineers use gradual refrigeration to achieve a state in which the solid material is well organized and its energy is minimized. This method involves placing the material at high temperature and gradually lowering the temperature. The SA algorithm is a simple and effective meta-heuristic search that solves compositional optimization problems. The simulated annealing method simulates the gradual cooling process to solve the optimization problem. The objective function of the problem is similar to the energy of matter which must be minimized by defining the virtual temperature. Temperature in this case is a parameter in the algorithm that can be controlled. The SA algorithm simulates energy changes in the annealing process until the system converges to equilibrium. The plan was introduced by Metropolis in 1953. The objective function of the problem is the energy level of the system. Each solution to the optimization problem is assigned to a system state. The decision variables are related to the molecular structure. The optimal is the steady state of the system. Finding the local optimal solution means that a quasi-steady state is obtained. The SA algorithm is a probabilistic algorithm in which a mechanism is derived from the local optimal algorithm [14].

2.10 Ant Colony Algorithm

Ant Algorithm Optimization (ACO) is a population-based meta-heuristic technique developed by Marco Dorigo in 1992. This algorithm combines random search technique and learning mechanism [16]. This algorithm is based on ants' living systems and imitating their behavior for searching food [7]. The initial algorithm aimed to find an optimal path in a graph based on the ant colony behavior of searching for a path between the nest and the food source [43]. Ants exchange food information with pheromones that they sour along the way. An ant returns to the nest by finding a food source. When the ants return to the nest on a shorter path, more pheromones and a shorter path will remain. [7]

To use the Ant Colony Algorithm to solve the problem, we first need to form a graph in which the probability of moving to the next path is determined by the following qualitative function and then the pheromone value or weight of each path is updated as follows:

$$p_{\ddot{U}}(t) = \frac{\tau_{\ddot{U}}(t)^{\alpha}\left(\frac{1}{d_{\ddot{U}}}\right)^{\beta}}{\sum K\varepsilon allallowedvertices \tau_{ik}\,(t)^{\alpha}\left(\frac{1}{d_{\ddot{U}}}\right)^{\beta}} \quad (7)$$

$$\tau_{ij}(t+1) = (1-p)\tau_{ij}(t) + \sum_{k\varepsilon ant.thatchoseedge(i.j)} \frac{Q}{L_k}$$

$P_{ij(t)}$ The probability of the ant passing along the path i and j، $\tau_{ij}(t)$ The amount of pheromone bound to the path is i and j، d_{ij} distance between node i and j ,α andβ Parameters for impact control p،d_{ij} Pheromone evaporation coefficient، L_K ,Cost of route K nth Ant and Q It is also constant [43].

2.11 Particle Swarm Optimization Algorithm

Particle Swarm Optimization is a meta-heuristic algorithm that was originally developed by a social psychologist and an electrical engineer [4]. This algorithm was first proposed by Kennedy and Eberhart. In formulating this method, it is inspired by the group flight of birds, the group swimming of fish and their social life, which is formulated using a series of simple mathematical relationships. Like any other meta-simulation method, this algorithm begins with the creation of an initial random population, here called a group of particles. The properties of each particle in the group are determined on the basis of a set of parameters that must determine their optimal values. In this way, each particle would represent a point in the solution space of the problem. Each particle has a memory, meaning it remembers the best position that it can find in the search space. So each particle moves in two directions, first to the best position that it has ever, then to the best position that all particles have ever had. Thus, in this method, the change of position of each particle in the search space will be influenced by its own experience and knowledge as well as its neighbors [38].

2.12 Bee Colony Algorithm

Artificial Bee Colony, or ABC, is a meta-heuristic algorithm based on bee social life introduced by Karaboga in 2005 to optimize numerical problems [42]. This algorithm performs better than other mass intelligence algorithms [23].The bee colonization algorithm consists of three basic components: worker bees, non-worker bees, and food sources. There are two behavioral modes in this model, which include employing the new probe bee for rich food sources, receiving positive feedback, and leaving the poor feeder bees with negative feedback. In the artificial bee algorithm, a colony consists of three groups of bees: worker bees associated with specific food sources, observant bees that supervise the selection of food sources, and scout bees randomly looking for food sources. Observer and forerunner bees are both called non-working bees. At first, all food resources are discovered by scout bees, then nectar is extracted by working bees and observers. This continuous exploitation results in the depletion of that resource. Then worker bees are transformed into scout bees to search for more food supplies again. In artificial bee colonies, the position of the food source is a possible answer to the optimization problem, and the amount of nectar of the food source corresponds to the fitting of the corresponding answer. The number of worker bees or observer bees equals the number of population solutions, so each bee is associated with only one food source. In the first step, the algorithm generates an initial randomly distributed population of SN answers (SNs), the number of bees working or observing. Each answer xi is a vector D. Here D is the number of optimization parameters. In the artificial bee colony algorithm, each round of the main loop contains three major steps. Initially, worker bees are sent to food sources and nectar levels are estimated. After sharing food resource information, food source areas are selected by observers and the nectar of the new food source is extracted and evaluated. Then the scout bees are identified and sent to potential food sources randomly. These steps are repeated a certain number, which is called the maximum number of MCN loops. The artificial observer bee selects a food source according to the probable amount of PI associated with that food source. This probability is calculated by the following equation:

$$P_I = \frac{Fitness(S_i)}{\sum_{n=1}^{SN} Fitness(S_n)} \tag{8}$$

In this respect, Fitness (Si) is the value of the answer S_i, which corresponds to the amount of nectar in the i nth place, and Sn is the number of food sources equal to the number of bees working or observing. In order to position the new candidate food, given the location in memory, the artificial bee colony uses the following statement:

$$V_{ij} = \left(X_{ij} - X_{kj}\right)\emptyset_{ij} + X_{ij} \tag{9}$$

In the above relation j∈ {1 and 2, ..., D} and K∈ {1 and 2, ..., sn} are indices that are randomly selected. Although K is selected at random, its value varies with i. ι_{ij} is a random number between negative one and positive one and is a parameter that controls the production of neighboring food sources around X_{ij}. From this relation we find that when the difference between X_{ij} and X_{jk} decreases, it means that the deviation from the position of the first point is reduced. Therefore, as the set approaches the optimal solution, the length of the glam is also reduced. If the parameter value exceeds the maze limit, it is set to an acceptable value. In this technique, the scout bee can find a new food source using the following relationship:

$$X_j^i = X_j^{min} + rand[0.1](X_{max}^j - X_{min}^j) \tag{10}$$

After the new candidate source location, V_{ij}, was created and evaluated by the artificial bee, its performance was evaluated relative to the previous one, and if the new food had more or as much nectar as the previous one, it would replace the previous one, otherwise The former remains in memory. In other words, a greedy choice is made in the operation between the previous location and the new location. In general, the bee colony employs the following three different selection processes:

A: The global selection process that calculates the probability value at this stage and is used by observer bees to discover probable areas.

B: The greedy choice to choose a superior source is made.

A: A random selection is made by scout bees [42].

2.13 Krill Herd Algorithm

In recent years, many researchers have resorted to the use of different meta-heuristic algorithms for portfolio optimization. The shrimp cluster optimization algorithm proposed by Gandomi and Alawi in

2012 is and a biologically based algorithm that is inspired by the collective movements of shrimp clusters. In this algorithm, the minimum distance of each single shrimp from the feed and its minimum distance from their compaction point is considered as the objective function. This algorithm comprises two global strategies and two local strategies that work in parallel to increase the efficiency of the algorithm [12]. Antarctic shrimp is a marine species and one of its main features is life in large groups. Many studies have so far explored the mechanisms underlying the formation of clusters in different marine species and also developed mathematical models to evaluate the role of these mechanisms [12].

Attacking of predators such as seals, penguins, or seabirds can reduce shrimp density. Re-forming shrimp bunches is a process that has two primary goals: to increase shrimp density and to obtain food. In this process, the shrimp moves to the best solution when searching for the highest density and food, and the best solution, the lowest distance to the high density and high food, which means less objective function. To use multi-objective group behavior instead of a single objective, coefficients must be determined. Gandomi and Alawi [12] determined these coefficients based on a specialized literature review on empirical observations of shrimp behavior and after an experimental study [51, 52].

3 Discussion and Conclusion

Today, one of the major concerns of investment and financial managers is optimal decision making at high speed and amid a large volume of information and data on stocks and capital markets. Especially when investment diversification in investment portfolios increases, optimal decision-making is very important given the limits of expected returns and the level of risk and liquidity of assets and other variables [22]. It is difficult and difficult to choose which portfolio to invest in, and which stock is better off than other stocks, which is worthy of being selected and placed in the portfolio, and how to allocate capital between these securities, the topics are complicated. Theoretically, the issue of portfolio selection in the case of risk minimization in the form of constant considering returns using mathematical formulas can be solved by a quadratic equation, but in practice and in the real world, given the number of choices. Much like in the capital markets, the mathematical approach used to solve this model requires extensive computation and planning. Since stock market behavior does not follow a linear pattern, therefore, common linear methods cannot be used to describe this behavior and may be useful [3]. Portfolio optimization allows for the attraction of more investors, because with the emergence of an appropriate investment process, fixed capital is absorbed by the community. In choosing the optimal portfolio, the main issue is choosing the size of the asset purchase and sale considering the budget constraint. Nowadays, many researches due to inefficiencies in the financial markets, the reliance on traditional methods based on historical data reduces the need to take new approaches to investment are recommended.

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