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By following his instructions, one can easily trace all the conics entering Khayyām's solutions of his equation-like problems of the third class, and thus pass from their description, as given in Khayyām's treatise, to their effective construction. Such a construction is, from Khayyām's point of view, not only very easy, but also a thoroughly standard one, just as that of the segment which provides a solution for the problem considered in the first of the previous examples.

There is thus a very simple reason one could evoke in order to justify Khayyām's choice to leave to the reader the constructions providing the effective solution of his problems: once these problems have been reduced to other ones by means of an appropriate trans-configurational analysis, such a construction is so easy and so thoroughly standard that there would have been no interest in detailing it. Still, it seems to me that this reason might be neither the only nor the main one. Another and more important reason could be that Khayyām held that such a construction did not belong to algebra, as long as algebra was not concerned properly with the solution of equationlike problems, but rather with their reduction to other problems to be solved in a standard way. The third clause entering my previous preliminary characterization of Khayyām's algebra should thus be taken literally: to show how equation-like problems can be systematically solved was not properly to solve them. This leads me to transform such a preliminary characterization 21 into a more precise one: Khayyām's algebra was a mathematical art aiming to: *i*) express the common form of equation-like problems both numerical and geometric; *ii*) classify these problems; *iii*) reduce them, by means of a configurational analysis, to other problems that one knew how to solve.

As well as in the previous case, Khayyām solution is thus a reduction justified by a hidden trans-configurational analysis which relies on simple substitutions and well-known results of classical geometry, used as rules of inference, and, without referring to any diagram, transforms the condition of the given problem to other equivalent conditions that is easy to satisfy by constructing two conics, whose actual construction is left to the reader. Such a trans-configurational analysis is once again composed by two parts. Steps A.1-3 reduce the problem of searching for a segment x satisfying the condition (11) to the problem of searching for a segment x satisfying the condition $P(k, k, h + x) = P(x, x, x - a)$; steps A.4-7 reduce this latter problem to the problem of constructing the point of intersection of two hyperbola.

5. Conclusions

The previous three examples should be enough to illustrate the nature and role of trans-configurational analysis in Khayyām's algebra. There is however an essential difference between the first example and the other two. It concerns the nature of the construction that Khayyām leaves to the reader, that is properly, the synthesis. While in the first example this construction can be easily performed by rule and compass, and is thus a quite standard one in the context of Euclid's geometry, this is not the case for the second and the third example. Thus a question arises: is Khayyām justified in avoiding any consideration concerned with the actual construction of the effective solution of his problems?

During his treatment of equation-like problems of species 21, Khayyām mentions en passant a mathematician who had lived in the final part of the 10th century: Abū al-Sahl al-Qūhī. He is the author of a *Treatise on the Perfect Compass*, probably the first one of a number of treatises composed some time before Khayyām's and devoted to the description and study of a mechanical tool to be used to trace continuously any sort of conics[29]. This is a three-dimensional compass with a sliding drawing point which is possible to regulate in different ways. In his treatise, al-Qūhī shows how to do it in order to trace a conic which has been previously univocally characterized.

[29]. Cf. [13], [10], LXXXI-LXXXII, [11], and [1], 158-178.

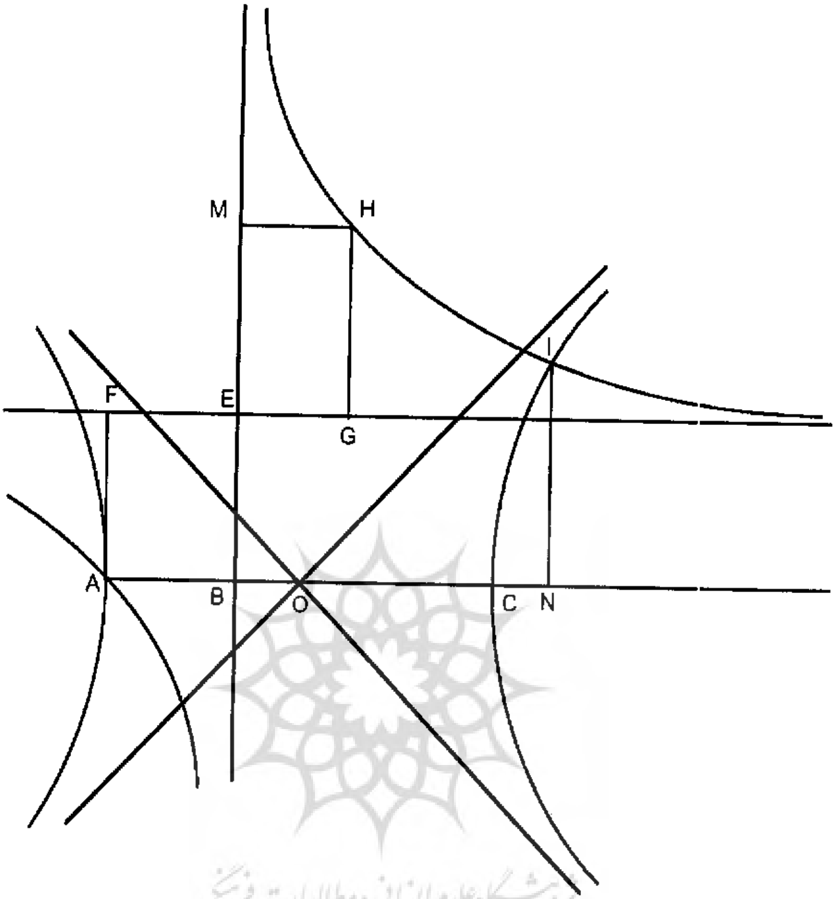

fig. 3

On the segments **AB** = h and **BE** = k, and takes the segment **BC** = a on the straight line **AB**; then, he constructs on the straight line **FE** and **BE** a rectangle **EGHM**, equal to **ABEF** and opposite to it in position; and finally, he states that the first hyperbola is the equilateral one (of center **O**, such that **AO** = **OC**) whose vertex are the points **A** and **C**, and the second is the (equilateral) one which is circumscribed to the rectangle **EGHM** (and whose center is **E**). These hyperbola intersect in the point **A** and in another point I, which is of course the only one Khayyām considers, and whose projection **N** on the axis **AC** gives the side x = **BN** which was sought.

As the segments a and h are given and the segment x has been
supposed to be given, this would be also the case of the A.5
segment y such that $Q(y) : Q(h + x) = x - a : h + x$.

The comparison of the two proportions occurring in A.4 and A.5
provides the new proportion $Q(k) : Q(x) = Q(y) : Q(h + x)$, A.6
which easily reduces first to $k : x = y : h + x$ and then to
$k : y - k = x : h$.

The proportions occurring in A.5 and A.6 define respectively
two equilateral hyperbolas, the first having both its
latus rectus and latus traversus equal to $h + a$, and the A.7
second being circumscribed to the rectangle $R(h, k)$.

In modern formalism, these hyperbola are respectively defined by
the 19 equations $y^2 - x^2 = x\,(h - a)$ - ha and $x\,(y - k) = kh$, (12)
which easily derive from the proportions
$y^2 : (h + x)^2 = x - a : h + x$ and $k : y - k = x : h$. (13)

To solve the problem we cannot but construct these hyperbola in an appropriate relative position and to project their point of intersection on to a common axe. Thus Khayyām first constructs the rectangle **ABEF** (fig. 3) looking for. For short, I limit myself to expose Khayyām's hidden analysis, which is easy to reconstruct from the second part of this argument.

Both Khayyām's explicit argument and this hidden analysis make use of the second of the three lemmas which he proves before passing from problems of the second class to problems of the third one. This[27] is lemma 2, and consists in the exposition of the construction of a parallelepiped whose base is a given square that is supposed to be equal to a given parallelepiped whose base is also a square. As it reduces to the successive construction of two fourth proportionals it does not present any diffculty[28]. Then analysis 18 goes as follows:

Consider the condition (11), and replace in it
the parallelepiped $P(b, u, x)$ with another parallelepiped
$P(k, k, x)$ (i.e. the rectangle $R(b, u)$ with the square $Q(k)$),
and, by lemma 2, the parallelepiped $P(c, u, u)$ with A.1
another parallelepiped $P(k, k, h)$, to get the new equality
$P(k, k, h) + P(k, k, x) + P(a, x, x) = C(x)$,
where the segments a, k, and h are given.

Suppose that x were given. It would be such that $a \leq x$, since $P(a, x, x) \leq C(x)$. Suppose that $x = a + (x - a)$,
where $x - a$ is a segment that is supposed to be given, and
split the cube $C(x)$ into the two parallelepipeds $P(a, x, x)$ A.2
and $P(x, x, x - a)$, in order to get the equality
$P(k, k, h) + P(k, k, x) + P(a, x, x) = P(x, x, a) + P(x, x, x - a)$,
whose all terms would be given.

This equality reduces to the other one
$P(k, k, h) + P(k, k, x) = P(x, x, x - a)$, where the A.3
parallelepipeds can be easily added to get the new equality
$P(k, k, h + x) = P(x, x, x - a)$, whose two terms would be given.
This equality is equivalent to the proportion
$Q(k) : Q(x) = x - a : h + x$. A.4

---

27. Cf. [12], 156-159.

28. This is how Khayyām reasons. Supposing that both a parallelepiped $P(\alpha, \alpha, \beta)$ and a square $Q(\gamma)$ are given, one can construct a segment μ such that $\alpha : \gamma = \gamma : \mu$ and then a segment ν such that $\mu : \alpha = \beta : \nu$. If this is done, ν and β are inversely proportional to $Q(\gamma)$ and $Q(\alpha)$, and so $P(\gamma, \gamma, \nu) = P(\alpha, \alpha, \beta)$.

intersection of two conics whose construction is not detailed[25]. By passing from problems whose condition is given by a three-term equality to problems whose condition is given by a four-term equality, the argument becomes a little more complicated, and its internal structure slightly changes. This will be clear once we will have taken into account the next example.

4. Third example: "A number plus some sides plus some square are equal to a cube"

This[26] is also a species of problems of the third class that Khayyām only considers under a geometric interpretation. Thus, the terms "a cube", "some squares", and "a number" refer to the same geometric objects they referred to in the previous example, while the term "some sides" refers to a parallelepiped whose altitude is the side and whose base is obtained by taking a certain number of times, say p, a square constructed on the unitary segment.

If we use the previous notation, we thus have the condition:

$$P(c, u,u) + P(b, u, x) + P(a, x, x) = C(x) \; ; \qquad \begin{cases} [a = qu] \\ [b = pu] \\ [c = nu] \end{cases} , \qquad (11)$$

Even though in this case Khayyām does not distinguish explicitly the solution of the problem from the proof of its correctness, as he does in other cases (as in the one considered in the previous section), the argument he advances in order to solve such a problem is clearly composed of two parts. The first one describes a construction of the point of intersection of two conics 17 that are perfectly characterized but not explicitly constructed as such, while the second one shows that the segment obtained by projecting this point of intersection on a given axis, on which an origin has been fixed, is the side he was

25. In certain cases, Khayyām explicitly proves that his conics actually intersect for any choice of the given segment entering the condition of the given problem; in other cases, he indicates under which conditions these conics actually intersect. In other cases, he does not face this question. A complete treatment of it will be done some time later by Sharaf al-Dīn al Tūsī [cf. [12], 26 and [2], t. I].

26. Cf. [12], 198-201.

Suppose that x were given. The cube $C(x)$ and the
parallelepiped $P(a, x, x)$, could then be added A.2
to get the new equality $P(a + x, x, x) = C(h)$,
whose two terms would be given.

This equality is equivalent to the continuous proportion
$x : h = h : y = y : a + x$, where y is a fourth proportional A.3
one can easily construct.

Such a continuous proportion can be split up, providing
the two proportions $x : h = h : y$ and $h : y = y : a + x$. A.4

These proportions respectively define an hyperbola
circumscribed to the square $Q(h)$, and a parabola A.5
of latus rectus h and vertex translated of a.

This argument is a composition of two trans-configurational analysis. The first one is composed by steps *A.1-2* and reduces the problem of searching for a segment x satisfying the condition (7) to the problem of searching for a segment x satisfying the condition $P(a + x, x, x) = C(h)$. The second one is composed by steps A.3-5 and reduces this latter problem to the problem of constructing the point of intersection of two conics. Taken as a whole, this argument is thus a trans-configurational analysis reducing the given problem to the problem of constructing such a point of intersection. It relies on simple substitutions and well-known results of classical geometry, used as rules of inference, and, without referring to any diagram, transforms the condition of the given problem to another equivalent condition that is easy to satisfy by construction. The argument Khayyām explicitly exposes as being a solution to the given problem is then a description of this latter construction, but as no mention is made to the procedure to be followed in order to construct the conics, it is in its turn a reduction. And his proof is nothing but a proof of the correctness of such a reduction, that is, of the 16 equivalence of the given problem and the problem obtained by means of it.

This is the general scheme of Khayyām's solution of equation-like problems of the 3rd class: such a solution is actually a reduction justified by a hidden trans-configurational analysis and followed by a proof of its correctness; moreover, the new problem obtained by means of this reduction is invariably that of constructing a point of

One then gets the continuous proportion:
AG : EG = EG : BC = BC : BG. P.3

And from here the equality:
C (**BC**) = *P* (**AG, BG, BG**) = *C* (**BG**) + *P* (**AB, BG, BG**). P.4

By posing **BG** = x, this equality just reduces to
C (h) = C (x) + P (a, x, x),
which, for Sol.2, is equivalent to condition (7). P.5

We should notice that Khayyām says nothing either about the construction of the two conics entering his solution or about the construction of the two parabola mentioned in lemma 1. I shall come back later on the reason for this choice. For the time being, I limit myself to remark that as long as Khayyām does not make explicit how to construct his conics, his solution is not properly a synthesis. From this point of view, it is quite similar to the solution he advances for geometric equation-like problems of species 7, which I have considered in the previous section. In both cases, Khayyām hints at how to construct the sought segment, but as a matter of fact he does not actually construct it. Thus, in both cases, Khayyām's solution is actually a reduction of the given problem to another one, which is left to the reader to solve. However, while in the first case such a reduction is obtained by means of an explicit trans-configurational analysis, in the present case this stage is lacking. All Khayyām does is to describe how it would be possible to obtain a certain segment, supposing that one were able to construct four particular conics, and to prove that this segment is just the one being sought. Nevertheless, one could wonder: how is it possible to pass from the given problem to such a description, which is actually the description of the solution of another problem (a problem which is proved to be equivalent to the given one only afterwards)? The answer seems to me quite obvious: this is done by means of a trans-configurational analysis 15 that, according to a classical habit, Khayyām does not make explicit. It is still easy to reconstruct it by turning one attention to the proof. It goes as follows:

Consider the condition (7), and using lemma 1,
replace in it the parallelepiped *P(c, u, u)* with a cube *C(h)*,
so to get the new equality *C(x) + P(a, x, x) = C(h)*, A.1
where the segments a and h are both given.

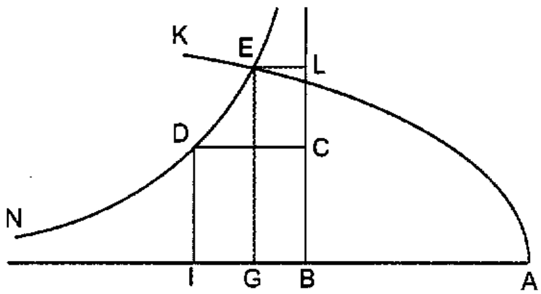

fig. 2

By lemma 1 construct a segment h,
such that *C(h) = P(c, u, u)*. Sol.2

Thus h is given. On the straight line **AB**, take **BI** equal to *h*, and construct on it the square **BCDI**. Sol.3

Construct the hyperbola **NDE** circumscribed to such a square. Sol.4

Construct the parabola **AEK** with *latus rectus* **BC** and vertex **A**. Sol.5

From the point of intersection **E** of these conics trace the perpendicular **EG** to the straight line **AB**. Sol.6

The segment **BG** is the sought side. Sol.7

The exposition of this solution is followed by a proof[24]. It goes as follows:

The point **E** belongs to the parabola **AEK**, thus:
AG : EG = EG : BC. P.1

The point E belongs to the hyperbola **AEK**, thus:
EG : BC = BC : BG. P.2

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[24]. Khayyām also proves by reductio ad absurdum that the two conics actually intersect for any choice of a and h: cf. the footnote (25) below.

the one which depends on the intersection of two parabola[22]. This is actually equivalent to the construction of the side of a cube that is given in extension, i. e. to the solution of geometric equation-like problems of species 3: "A cube is equal to a number[23]". To prove that, Khayyām remarks that the equality

$$C(x) = P(c, u, u) \; ; \; [c = nu] \, , \tag{8}$$

—providing the condition for these problems—is a consequence of the continuous proportion

$$u : x = x : y = y : c, \tag{9}$$

since from such a proportion it follows the other proportion

$$Q(u) : Q(x) = x : c, \tag{10}$$

which states in its turn that the bases of *C(x)* and *P(c, u, u)* are inversely proportional to their altitudes (so that, according to the proposition VI.34 of Euclid's *Elements*, these solids are equal).

Once this is proved, one can use lemma 1 in order to replace, in the condition of any equation-like problem, a given parallelepiped with a given cube. This is actually what Khayyām does. His solution of problems "A cube plus some square are equal to a number" goes then as follows:

The segment a is given; take **AB** equal to it (fig. 2). Sol.1

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22. Cf. [4], III, 82-85 and [5], I, 251-255.

23. Cf. [12], 160-161. As a matter of fact, Khayyām's solutions of equation-like problems of the third class are ingenious generalization of such a solution.

his solution is not properly a synthesis, but rather a reduction obtained thanks to a trans-configurational analysis.

Once this reduction obtained, the construction of the sought segment—that is, the synthesis—is so easy that it is left to the reader.

The consideration of equation-like problems of the third class largely con- firms this interpretation. I shall show it by means of my second and third examples.

3. Second example: "A cube plus some square are equal to a number"

These[19] are problems of the third class, and thus Khayyām is not able to solve them under an arithmetical interpretation. Hence, he directly supposes that these problems are geometrically interpreted. According to the previous conventions, this means that the term "a cube" refers to a geometric cube constructed on the sought segment, that is the side[20], while the term "some square" refers to a parallelepiped whose base is the square constructed on this side and whose altitude is obtained by taking a certain number of times, say q, an unitary segment, and the term "a number" refers to a given parallelepiped, namely the one whose base is the square constructed on such an unitary segment and whose altitude is obtained by taking this last segment a certain number of times, say n. If we denote respectively by "C(α)" and "P(α, β, γ)" the cube constructed on the segment α, and the parallelepiped constructed on the segments α, β, and γ, and, as before, by "x" and "u" the side and the unitary segment, we thus have the condition:

$$C(x) + P(a, x, x) = P(c, u, u) \quad ; \quad \begin{cases} [a = pu] \\ [c = nu] \end{cases} . \qquad (7)$$

Khayyām's solution makes use of the first of three lemmas which he proves before passing from problems of the second class to problems of the third one. This[21] is lemma 1, and consists of the exposition of the second of Menaechmus' constructions of two mean proportionals,

[19]. Cf. [12], 170-175.

[20]. Cf. the note (8) above.

[21]. Cf. [12], 152-157.

new configuration corresponding to a new and essentially simpler problem: to construct a segment *z* satisfying the new condition.

$$Q(z) = Q(x) + R(a, x) + Q\left(\frac{a}{2}\right). \tag{3}$$

The synthesis is thus easy. One should first construct a rectangle equal to $R(b, u) + Q\left(\frac{a}{2}\right)$, that is, according to the propositions VI.16 and II.3 of Euclid's Elements, the rectangle $R\left(\frac{a}{2}, \frac{a}{2} + t\right)$ constructed on and on the segment *t* satisfying the condition.

$$\frac{a}{2} : b = u : t \tag{4}$$

(what is nothing but a fourth proportional). Then, according to the proposition VI.17 of Euclide's *Elements*, one should search for the segment *z* satisfying the condition.

$$\frac{a}{2} : z = z : \frac{a}{2} + t. \tag{5}$$

(what is nothing but a mean proportional). Because of the equality $x = z + \frac{a}{2}$. the sought root of the original problem will then be:

$$x = z - \frac{a}{2} . \tag{6}$$

This last argument relies on a number of Euclid's theorems that make useless the consideration of any diagram. These theorems work in it, so to say, as rules of constructive calculation. Notice however that Khayyām does not make the synthesis explicit. He concludes his solution by observing that since $Q\left(\frac{a}{2}\right)$ and *R(b, u)* are both known, *Q(z)* is known, and then *z* is known, so that also *x* is known[18]. Hence,

[18]. Here is Khayyām's whole arguments in his own words [cf. [12], 136-137]: "Let us suppose that the square AC plus ten of its roots is equal to thirty nine in number. Let us suppose, on the other hand, that ten of its roots are equal to the rectangle CE; the straight line DE is thus ten. Cut it at half in G. Since we have cut the straight line DE at half in G, and we have added AD on its prolongation, the product of EA and AD, that is equal to the rectangle BE, plus the square of DG, are equal to the square of GA; now the square of DG, that is the half of the number of roots, is known, and the rectangle BE, that is the given number, is known; the square of GA is thus known, and the straight line GA is known. If one takes away GD, it remains AD, known." Khayyām then presents another solution, corresponding to al-Khwārizmī's well-known geometric proof [cf. [6], 13-16].

As a matter of fact, Khayyām accompanies his argument with a diagram, but he does not base on it to conduct his analysis. This diagram (fig. 1)

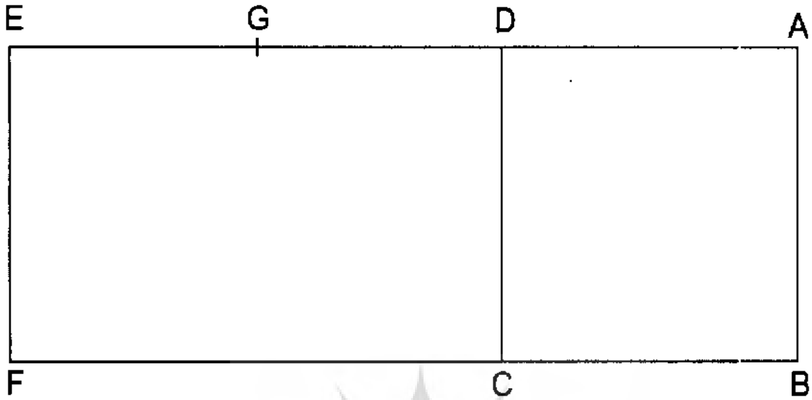

fig. 1

Simply represents (and thus individualizes and allow to gives a name to) the magnitudes involved in the problem: the segments **AD**, **DE** and **DG** represent respectively the root x and the given segments a and $\frac{a}{2}$, so that the square **ADCB** and the rectangle **DEFC** represent respectively the square Q(x) and the rectangle R(a, x), and the whole rectangle **AEFB** represents the sum Q(x) + R(a, x) that is, a rectangle equal to the other rectangle R(b, u). The square constructed on $x + \frac{a}{2}$ is not drawn; thus the equality

$Q(z) = Q(x) + R(a, x) + Q\left(\frac{a}{2}\right)$, which constitutes the essential step of the argument, is not deduced with base in the diagram, but rather using the proposition II.4 of Euclid's *Elements* as a rule of inference.

It should be clear, therefore, that Khayyām's argument does not aim to isolate a sub-configuration of given objects in the configuration of given and ungiven objects corresponding to the solution of the problem. Far from constituting an intra-configurational analysis, the two previous arguments form, thus, when they are taken together, a unique trans-configurational analysis able to transform the configuration corresponding to the conditions of the problem into a to a given rectangle, namely the one constructed on the unitary segment and another segment obtained by taking such an unitary segment thirty nine times. If we denote, as before, by "$Q(\alpha)$" and "$R(\alpha, \beta)$" the square constructed on the segment α, and the rectangle constructed on the segments α and β, and by "x" and "u" the root and the unitary segment, respectively, we have thus, in general, the condition:

$$Q(x) + R(a, x) = R(b, u) \quad ; \quad \begin{cases} [a = pu] \\ [b = nu] \end{cases} , \qquad (2)$$

where p and n are numbers.

Khayyām's solution actually consists of a two-fold analysis. The first part can be reconstructed as follows:

Suppose that x were given.

Then $Q(x)$ and $R(a, x)$ would be given, too. A.I.1

Thus $Q(x) + R(a, x)$ would also be given. A.I.2

But this is equal to $R(b, u)$ and thus this latter rectangle would also be given. A.I.3

This is a Pappusian analysis. It starts with (the supposition that) the required segment (is given) and proceeds to (the statement that) a rectangle that is actually given (is so). But Khayyām's argument does not stop here.

The second part of his argument can be reconstructed as follows:

a, b and u are given. A.II.1

Thus also $\frac{a}{2}$, $Q\left(\frac{a}{2}\right)$ and R(b, u) are given. A.II.2

But if $z = x + \frac{a}{2}$, them.

$Q(z) = Q(x) + R(a, x) + Q\left(\frac{a}{2}\right)$. A.II.3

Hence, for A.I.2, $Q(z) = Q(x) + R(b, u) + Q\left(\frac{a}{2}\right)$, A.II.4

so that Q(z) is given.

with. This is given by the choice of number ten as the numerical coecients for roots (that is, the replacement of the general term "some roots" with the particular one "ten (of its) roots") and of number thirty nine for the given quantity[14]. This is just al-Khwārizmī's example for the case where "roots and squares are equal to numbers" and just one square is considered[15], and Khayyām comes back to it to stay close to the tradition. His solution is nevertheless completely independent of the particular nature of such an example; that is, it does not depend on the choice of numbers ten and thirty nine. Hence, we might reformulate the problem, in our modern language, as that of solving the equation $x2 + px = c$, where p is a number and c a given quantity.
Khayyām begins by presenting al-Khwārizmī's solution[16] which can be expressed as follows:

$$x = \sqrt{\frac{P}{2}\frac{P}{2} + C} - \frac{P}{2} \qquad (1)$$

To him, this is the general solution of such a species of problems when they are interpreted as numerical ones, that is, when both the root and the given quantity are numbers.
When this is not so, i. e. the root is a segment, this solution does not apply. Hence a new and specifically geometric solution has to be provided[17]. According to the previous conventions, if the root is a segment, the term "a square" refers to a geometric square constructed on such a root, while the terms "ten (of its) roots" refers to a rectangle whose base is the root and whose altitude is obtained by taking ten times an unitary segment, and the term "thirty nine in number" refers

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14. Cf. note (9), above.

15. Cf. [6], 8.

16. To make clear that he is concerned with a species of equation-like problems and not just with a particular example of this species, Khayyām observes that such a solution is restricted by the condition that "the number is not greater than the square of the half of the number or roots". Otherwise, he adds, "the problem is impossible" [cf. [12] , 140-141].

17. Though he does not prove that al-Khwārizmī's solution is correct and he argues that a "numerical proof is conceived when a geometric proof is conceived" [cf. [12], 140-141], he maintains that the two solutions are distinct solutions of different problems.

problems. As long as the point is not only a terminological one, it pertains to Khayyām's justification of his solutions. And on this, he is quite clear. Here is how he betokens not to be able to solve equation-like problems of the third class when the root or side is a number[12]:

> But when the object of the problem is an absolute number, neither me nor any other man of this art [that is, algebra itself] has succeeded in the solution of these species but for the first three degrees, that are the number, the thing, and the square; perhaps someone else which will follow us will be able to do it. And I shall often point out the numerical proofs of what is possible to prove starting from Euclid's work. Mind that the geometric proof of these methods does not dispense you from the numerical proof if the object is a number and not a measurable magnitude.

In order to detail the previous preliminary characterization of Khayyām's algebra, one has to pass from the mere formulation and classification of equation-like problems to their solution. I shall limit myself to consider three examples, which illustrate quite well Khayyām's methods. They concern respectively: a species of problems of the second class whose condition is expressed by a three-term equality, corresponding to species [7] of the previous schema; a species of problems of the third class whose condition is expressed again by a three-term equality, corresponding to species [16]; and a species of problems of the third class whose condition is expressed by a four-term equality, corresponding to species [22]. By considering the second of these examples, I shall come *en passant* also to the problems of species 3, that are problems of the third class whose condition is expressed by a two-term equality.

2. First example: "A square plus ten of its roots are equal to thirty nine in number"

This[13] is one of the few cases where Khayyām considers a particular example of equation-like problems of the species he is concerned

12. Cf. [12], 124.
13. Cf. [12], 136-141.

arithmetical problems, since in this case Khayyām is obviously not able to provide a solution for the equation-like problems of the third class. Thus he restricts himself to present al-Khwārizmī's solution of the arithmetical equation-like problems of the first and second class, which are still general solutions: all of these problems, belonging to the same species, can be solved in the same way.

This general presentation of Khayyām's treatise makes possible to advance a preliminary characterization of what he understands it to be about. If, despite of the original meaning of the Arabic term "al-jabr" which enters its title, we admit that this treatise is about algebra, then it seems that for Khayyām algebra is a mathematical art aiming at: i) expressing the common form of equation-like problems both numerical and geometric; ii) classifying these problems; iii) showing how these problems can be systematically solved. It would thus be an art serving both to arithmetic and to geometry[11]. Still, it is certainly not a unitary theory whose objects are As an 7 quantities of any sort: although it employs a common language to speak both of numbers and geometric magnitudes, it is neither a general context where inferences concerning both numbers and geometric magnitudes can be warranted, nor a common domain where it is possible to assert something about these same objects. This common language is simply used to express the common forms of certain arithmetical and geometric problems and to classify them. But when these problems have to be solved, such a language has to be interpreted either arithmetically or geometrically. Hence, as long as these problems are solved, or simply are to be solved, they are either arithmetical or geometric problems and not both at the same time.

One could object that is not so, and argue that far from being concerned with the different solutions of arithmetical and geometric equation-like problems, Khayyām is concerned with arithmetical and geometric solutions of equation-like problems whose arithmetical or geometric nature is indefi- nite; that is, he is not actually solving separately arithmetical and geometric equation-like problems, but he is rather solving arithmetically or geometrically equation-like

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[11]. Cf. above, footnote (4). According to Khayyām, there is thus no opposition between algebra and geometry. This was not also the case for other Islamic mathematicians. An as example of an opposite conception, one could mention *Thâbit Ibn Qurra's Justification of algebraic questions through geometric proofs* [cf. [7]]. It seems however to me that early modern algebra is in this sense similar to Khayyām's one.

where x is the root or side, c is a given quantity, p and q are numbers, and the numbers in brackets indicate the species (in the order in which Khayyām considers them).

This notation is however essentially stranger to Khayyām's language and conventions. The language I have described above is precisely used to express the common forms respected by any problem of any species, independently of the fact that it is an arithmetical or a geometric problem. It is thus a common language for arithmetic and geometry. Take the species [19]. The formula used by Khayyām to characterize this species is: "a cube plus some squares, plus some sides are equal to a number". It refers both to an arithmetical equality where the third power of an unknown number x—that is, xxx—plus the second powers of x taken a certain number q of times—that is, qxx—plus x taken a certain numbers p of times—that is, px—are supposed to be equal to a given number—say c—, and to a geometric equality where the cube constructed on an unknown segment x—that is C (x) —plus the parallelepiped constructed on the square constructed on x and on a given segment, supposed to be unitary, taken a certain number q of times—that is, P(x, x, qu) = q [P(x, x, u)]—plus the parallelepiped constructed on x and on a rectangle constructed on the unitary segment and on this same unitary segment taken a certain number p of times—that is P(x, u, pu) = p [P(x, u, u)]—are supposed to be equal to a parallelepiped constructed on a given segment c and on the square constructed on the unitary segment—that is, P(u, u, c).

Though Khayyām has a common language for arithmetic and geometry that he uses to express the common forms of arithmetical and geometric equation-like problems, he does not have anything like a common method or a number of common methods to solve these problems independently of their nature as arithmetical or geometric problems. Thus, when he passes from the classification of his problems according to the form of the equality expressing their condition to the solution of them, he is forced to distinguish between the two possible interpretations of his language. Hence, the meaning of the term "systematically" I have used above depends not only on the previous threefold classification, but also on this distinction. When only geometric problems are concerned, it indicates that Khayyām associates to any species of his problems a method of solution that applies to any problem of this species. It is not the same for problems which neither involve a square or a cube, or that can be reduced to problems of this kind (that are, in modern language, the equation-like problems of the first degree or reducible to it); those which do not involve a cube, involving instead a square, or that can be reduced to problems of this kind (the equation-like problems of the second degree or reducible to it); and finally those that do involve a cube (the equation-like problems of the third degree).

The first class is composed by the problems belonging to three of the twenty five species distinguished before; the second one is composed by the problems belonging to other eight of these species; finally the third one is composed by the problems belonging to the other fourteen species.

This is a threefold classification which is only concerned with the form of the equality expressing the condition of the problem. If we use modern 5 notations to express this form, we can illustrate it by means of the following schema[10]:

	2 terms	3 terms	4 terms
1st class	[1] $x = c$ [4] $x^2 = px$ [6] $x^3 = qx^2$		
2nd class	[2] $x^2 = c$ [5] $x^3 = px$	[7] $x^2 + px = c$ [8] $x^2 + c = px$ [9] $x^2 = px + c$ [10] $x^3 + qx^2 = px$ [11] $x^3 + px = qx^2$ [12] $x^3 = qx^2 + px$	
3nd class	[3] $x^3 = c$	[13] $x^3 + px = c$ [14] $x^3 + c = px$ [15] $x^3 = px + c$ [16] $x^3 + px2 = c$ [17] $x^3 + c = qx2$ [18] $x^3 = qx2 + c$	[19] $x^3 + qx^2 + px = c$ [20] $x^3 + qx^2 + c = px$ [21] $x^3 + px + c = qx^2$ [22] $x^3 = qx^2 + px + c$ [23] $x^3 + qx^2 = px + c$ [24] $x^3 + px = qx^2 + c$ [25] $x^3 + c = px + qx^2$

10. My schema slightly divers from the one given by Rashed [cf. [12], 11], but it is still suggested by it.

problem involves a cube, then it never involve a square or a root or a side, but always "some squares" and/or "some roots" or "some squares". The term "some squares" is then used to denote a parallelepiped whose base is the square constructed on the root or side and whose altitude is obtained by taking a certain number of times a given segment supposed to be unitary, while the terms "some roots" or "some sides" are used to denote a parallelepiped whose altitude is the root or side and whose base is obtained by taking a certain number of times a square constructed on a given segment supposed to be unitary. If the problem does not involve a cube, but it involves a square, then it never involves a root or a side, but always "some roots" or "some sides". The terms "some roots" or "some sides" are then used to denote a rectangle whose base is the root or side and whose altitude is obtained by taking a certain number of times a given segment supposed to be unitary. Finally, in the only case where the problem does involve neither a cube, nor a square, it involves a root.

Such an apparently cumbersome language is in fact a quite sophisticated tool used in order to express the common forms of equation-like problems both arithmetical and geometric. As a matter of fact, Khayyām's treatise is about all the possible forms of numerical or geometric equation-like problems of the first three degrees. It aims to classify these different forms and to show how these problems can be systematically solved. The term "systematically" has to be explained. To do that, one has to expose Khayyām's classification of the equation-like problems he is concerned with. This is a multifarious one.

A criterion of it concerns the numbers of terms entering the equality which expresses the condition of the problem. According to such a criterion, these problems are separated in three groups, according to the fact that their condition is expressed by a two-term, a three-term or a four-term equality. After the general conditions characterizing Khayyām's equation-like problems, a two-term equality expressing these problems can take six different forms, a three-term equality can take twelve different forms, and finally a four-term equality can take seven different forms. Hence, one has twenty five different forms distinguishing twenty five different species of equation-like problems. Once this classification is given, Khayyām considers any species of his problems separately and distinguishes them in three classes: the

As we have just said, these problems are presented by stating a condition under the form of an equality to be satisfied and whose two arguments are composed by the adjunction of different terms. Each of these terms is provided either by a "number", or by one or some "sides" or "roots" (only in 3 one occasion Khayyām says "some things"[7]), or by one or some "squares", or finally by a "cube"[8]. Despite the term Khayyām uses to denote it, a number is a given quantity, that is, either a given number in the proper arithmetical sense of this term[9], or a given geometric magnitude, namely a segment, a rectangle or a parallelepiped. A side or root is an unknown quantity to be determined, that is either an unknown number or an unknown segment. A square is either the second power of the root or side (if this is a number) or a geometric square constructed on this root or side (if this is a segment). Analogously, a "cube" is either the third power of the root or side (if this is a number), or a geometric cube constructed on this root or side (if this is a segment). When the root or side is a number, then for "some roots", "some sides", or "some squares", Khayyām understands other numbers obtained by taking this number or its second power a certain number of times. When the root or side is a segment, the meaning of these expressions is variant. If the

←on the fact that these inferences concern "more abstract objects" [cf. [8], 248 and [9], 164]. It goes rather together with the understanding of these objects as arguments of relations that—at the light of appropriate theorems, specially from the books II, V and VI of the Elements—are quite independent of any positional features of them.

7. Cf. [12], 134-135 : "Some things are equal to a cube."

8. Khayyām speaks of one or more roots when the problem he considers does not involve a cube or can be reduced to a problem that does not involve a cube (it is an equation-like problem of the first or second degree, or it is reducible to such an equationlike problem), while he speaks of one or more sides when the problem he considers does concern a cube and cannot be reduced to a problem that does not concern a cube (it is a non reducible equation-like problem of the third degree). The reason of that is quite clear: he is able to solve the problems of this latter class only when they are interpreted geometrically.

9. From now on, I shall use the term "number" only in its proper arithmetical sense, unless it appears in a quotation from Khayyām. One can wonder whether for Khayyām a number in this sense is necessarily an integer positive or not. I shall not confront this question here, since from the point of view of my reconstruction it is hardly relevant. For simplicity, one could admit that this is just the case for the numbers entering the conditions of problems, and suppose that Khayyām was looking for numerical solutions of his problem under the form of a definite or indefinite rational approximation of an integer positive number.

What is searching for is of course (the determination of) the "root" or the "side" which satisfies these conditions, assuming that this is the side of the squares and cubes which occur in them. As long as the terms entering these problems are quite standard in Khayyām's language (I shall come back later on their respective meanings), we could take them as particular sorts 2 of notations and replace them with modern notations, so to get equations like:
$x = n$, $x^2 + n = mx$, $x^3 + lx^2 + mx = n$. However, the question is not simply that of transforming Khayyām's language in a more familiar one for us, but rather that of understanding how this language is associated with the methods that he uses for solving his problems. So, though one admitted that these translations are convenient and appropriate, one would not yet be authorized to attribute the use of modern algebraic formalism to Khayyām. Moreover, it seems to me that, according to him, to state a condition like "a square plus a number are equal to some roots" was properly to advance a problem, and not to present a mathematical object, as it is the case when a polynomial equation is presented in the context of algebraic formalism. Thus, in order to avoid any sort of possible misunderstanding, I shall refer to Khayyām's problems by the quite cumbersome expression "equation-like problems", rather than by the term "equation" alone, which is used in contrast in Rashed's translation[6].

[6]. The term "equation" is also used by Netz in his [8] and [9], where it is explicitly opposed to the term "problem". This is only an example of the differences between my reading of Khayyām's treatise and that of Netz. These differences certainly rely, for a great part, on the emphasis we bestow upon different aspects of such a treatise, but also concern the general interpretation of it. Netz seems to consider that it relies on equations (rather than problems) as long as it "downplays geometry" [cf. [9], 182] and that this goes together with its use of equalities (rather than proportions), its systematicity and its generality [cf. [8], 255-257 and [9], 182-186]. For him, Khayyām "opens up the possibility of considering his objects symbolically, as elements manipulated by the rules of calculation, yet essentially conceives of them as components in a geometric configuration" [cf. [8], 248 and [9], 163-164]. For me, Khayyām's objects are, without any ambiguity, numbers or geometric magnitudes, though he adopts a general language to speak of the common form of certain problems (the equation-like problems) that pertain to them. Khayyām's generality concerns thus for me more his language than his objects. Moreover, though I do not deny the systematicity of his approach, I do not think that this puts him away from geometry. When Khayyām's equation-like problems are interpreted on segments, they are genuine geometric problems. The systematic use of purely quantitative inferences—that is the aspect of Khayyām's treatise that I emphasize—neither brings nor depends →

different sort of problematic analysis. Though starting, as any problematic analysis, with the supposition that the object that is sought is given, it does not proceed to the statement that some objects that are actually given are so, but rather to a new configuration of the relations between the object that is sought and the given ones. It does not work within the configuration of given and ungiven objects corresponding the solution of the problem to isolate in it a sub-configuration of given objects that determines the entire configuration. It rather transforms such a configuration in a new one: it is not intra-configurational, but trans-configurational. In what follows, I shall present three examples of al-Khwārizmī's use of this kind of analysis: one concerning a problem that does not involve a "cube" (a second-degree problem) and two others concerning with problems that do involve a "cube" (two third-degree problems). Before that, some general considerations on Khayyām's treatise are in order.

1. The nature of Khayyām's problems

According to Khayyām, "the art of *algebra* and *al-muqābala*" is an art aiming to solve a certain class of problems, namely: to "determine unknown [quantities] both numerical and geometric[3]". These are particular sorts of problems which would be expressed today by means of polynomial equations of first, second or third degree[4]. The language used by Khayyām is codified enough to express these problems as different variants of a common form. But this form is not obviously the one of polynomial equations written in modern (that is Cartesian) notation. I present three examples, corresponding respectively to problems that we would express by means of an equation of first, second and third degree[5]:

A root is equal to a number.

A square plus a number are equal to some roots.

A cube plus some squares, plus some sides, are equal to a number.

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[3]. Cf. [12], 116-117. Translations from French are mine.

[4]. Khayyām seems to admit that these problems arise from different domains of mathematics, so that algebra is a sort of auxiliary art providing these domains with something like general lemmas. This is at least what is suggested in the introduction to his treatise: cf. [12],116-121.

[5]. Cf. [12], 128-129, 140-141 and 184-185.

On the use of analysis in Omar Khayyam's algebra

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Introduction

Khayyam's[1] *Treatise of Algebra and Al-muqābala*[2], is for many respects a systematic work. One of these respects depends on the uniformity of the method used to solve the different problems it is concerned with. *Stricto sensu*, these problems are not solved. They are rather reduced to other problems. In the great part of cases this reduction is proved to be correct *a posteriori*, but not justified *a priori*. For problems that do not involve a "cube", it was clearly suggested by al-Khwārizmī's solutions of these same problems. But how did al-Khwārizmī attain it in other cases? I guess that he did it by analysis. I moreover argue that this was not a Pappusian analysis, but a quite

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[1]. For its most part, the present paper is extracted by another larger one submitted to the *Revue d'histoire des mathématiques* ["On the Notion of Algebra in Early Modern Mathematics and its Relations with Analysis: Some Reflections on Bos' Definitions"]. I thank the editors of this review to have permitted such a partial anticipation of my text.
On the other hand, I thank the editors of *Farhang* specially Dr. Jafar Aghayani – Chavoshi to have accepted to publish it.

[2]. I refer here to Rashed's edition and French translation of such a treatise given in [12], 116-237. It is accompanied by an introduction, a rich and detailed commentary, and a reconstruction of the history of the text. All of them has been very useful for me. An older edition, also accompanied by a French translation, had been given by Woepcke in 1851: cf. [3].