considers is a divisible unit;[23] and that it is by assuming that a number such as '2' is composed of divisible units that one will then be able to speak of an irrational number such as '√2' - unlike the Greeks, for whom a notion such as the irrational number, '√2' did not seem to have any meaning: they would only speak in this case of the ratio of two incommensurable straights lines, namely the ratio of the diagonal of a square to its side.

This is how Omar Khayyâm- by examining in detail the connection between the concept of ratio and the concept of number, and by raising *explicitly* the theoretical problems related thereto - made a decisive contribution both to the theoretical study of the concept of irrational number, and to the understanding of its status as a mathematical entity in its own right[24].

[23]. The unit considered by Khayyâm being a magnitude, it is in fact divisible ad infinitum - and this is something which is required for finding approximations to an irrational magnitude.

[24]. As regards the part played by Arabic algebraists - especially al-Karajî - in the genesis of the concept of irrational number, we refer the reader to the research of R. Rashed in *Entre arithmétique et algèbre: recherches sur l'histoire des mathématiques arabes* (Paris, 1984), notably pp. 34-36, 48-49, 192, and 310-311.

consider the quantity of the ratio of two magnitudes, one will usually end up with what is called an *irrational* number - of which there seems to be no trace in Greek mathematics.[22]

Let us return to Omar Khayyâm. It is in Book 3 of his commentary that he sets out to prove the above-mentionned proposition in the general case where one considers any three magnitudes. And it is precisely in this context that he engages in a detailed study of the quantitative nature of ratio and the concept of irrational number.

For Khayyâm, every ratio expresses a measure ; this is to say that a certain magnitude is assumed as unit, and the other magnitudes of the same kind are related to it. For example, the meaning of 'the ratio of three to five' is 'three-fifths of a unit'. And in case the ratio considered is between two magnitudes A and B, that is in case the ratio of A to B is not necessarily equal to the ratio of two (whole) numbers, he then considers a magnitude G such that its ratio to the unit is equal to the ratio of A to B. It is this magnitude G which will express the measure of the ratio of A to B.

Khayyâm explains that the study of the connection between the concept of ratio and the concept of number is a philosophical study to which the geometrician must by no means devote himself; that it will be sufficient for him to convince himself that this connection between ratio and number indeed exists; and that it is in this manner that the compounding of ratios should be conceived.

While proving the proposition in question, Khayyâm goes back over these notions. He explains that the magnitude G - i.e. the magnitude whose ratio to the unit is equal to the ratio of A to B - should not be regarded as being a line, or a surface, or a solid, but that it should on the contrary be abstracted from these concrete things and be considered as a number, *whether the ratio of A to B be rational or irrational*. In this manner, Khayyâm will then be able to reduce the *compounding* of ratios to the *multiplication* of the numbers which express their respective measure. He also explains that the unit he

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22. Take for instance $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, &c. These were not considered by Greek mathematicians as irrational numbers, but rather as straight lines - i.e. as geometrical magnitudes - which were incommensurable with the unit of length. See notably T. L. Heath, *A History of Greek Mathematics* (Dover Publications, New York, 1981), vol. I, pp. 90-91, and *Plato's Theaetetus*, 147D - 148B.

If an intermediate term be taken between two numbers or magnitudes, the ratio of the numbers taken first will then be compounded of the ratio which the first has to the intermediate number and of the ratio which the intermediate number has to the third. It must first of all be remembered how a ratio is said to be compounded of ratios: it is, as in the *Elements*, when the quantity of the ratios multiplied together produce some quantity; obviously meaning by *quantity* the number whereby the given ratio is denominated, as it has been said by other authors, such as Nicomachus in his first book on music, and Heronas in his commentary on the *Introductio Arithmetica*; which amounts to saying that the quantity is the number which, when multiplied by the consequent term of the ratio, produces the antecedent.[19]

Eutocius then proceeds with the proof of the proposition in question. But his proof only takes into account ratios between numbers, and therefore does not apply to ratios between incommensurable magnitudes.

If we have mentioned this passage of Eutocius' comments, it is mainly because it allows to illustrate the fundamental problem which arises in this context. And this is that according to Greek views, a number is, in the strict sense of the word, a collection of indivisible units.[20] Consequently, the «quantity» of the ratio of two numbers cannot, in the strict sense of the word, be considered as a number unless the antecedent of the ratio be a multiple of the consequent. If, therefore, one wants to consider as a number the ratio of any two numbers (the first not being necessarily a multiple of the second), it will then be necessary to suppose that the unit is in fact divisible.[21] In the first case, one will end up with what is called a *whole* number, and in the second case with a *fractional* number. And in case one wants to

[19]. Translated from *Les oeuvres complètes d'Archimède, suivies des commentaires d'Eutocius d'Ascalon*, traduites du grec en français avec une introduction et des notes par P. Ver Eecke, 2ème éd. (Liège, 1960), vol. II, pp. 628-9.

[20]. This is implied by Definitions VII. 1 & VII. 2, namely: "1. An unit is that by virtue of which each of the things that exist is called one. 2. A number is a multitude composed of units." See for instance Euclide d'Alexandrie, *Les Éléments*, vol. II, pp. 248-249.

[21]. And this is what Eutocius does suppose. He indeed explains that the quantity of say the ratio of 12 to 6 is 2, and that of the ratio of 9 to 12 three-fourths, that is three-fourths of a divisible unit (see *Les oeuvres complètes d'Archimède*, vol. II, pp. 629 & 631).

two magnitudes are equal; or the less is a part or parts of the greater; or this relation is characterized by means of the anthyphairetic process. Whence it follows that two ratios will necessarily be the same if they yield the same series of numbers. As to greater ratio, Khayyâm also defines it through anthyphairesis.

He then proves that the anthyphairetic definitions of equal and unequal ratios are equivalent to the corresponding definitions of the Elements, namely Definitions V. 5 & V. 7. Therefore, all the propositions which had already been established within the frame work of the "Euclidean theory of proportion" will remain valid within the framework of the "anthyphairetic theory of proportion": so that these propositions will not need to be proved once more.

The compounding of ratios

We find in the beginning of the sixth Book of Euclid's *Elements* a definition which states that "A ratio is said to be compounded of ratios when the sizes of the ratios multiplied together make some (? ratio, or size)."[16] But Euclid nowhere explains what is meant by the «size» of a ratio; to say nothing of the «multiplication» of these sizes. However, he does make use of the compounding of ratios, notably in Proposition VI. 23[17] *and in Proposition* VIII. 5[18]. In both cases, he admits without proof, that *if there be given any three magnitudes - or any three numbers - A. B and C, then the ratio of A to C will be compounded of the ratio of A to B and of the ratio of B to C.* It is this last statement - the one people use in practice when they want to compound ratios - which has often been considered as a proposition which should be proved.

This is in fact what Eutocius (5-6th century A.D.) applies himself to when commenting on the fourth proposition of the second Book of the treatise On the Sphere and Cylinder, in which Archimedes had used the preceding statement on the compounding of ratios. Says Eutocius:

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16. That is, Definition VI. 5. This definition is now considered to be an interpolation.

17. That is: "Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides."

18. That is: "Plane numbers have to one another the ratio compounded of the ratios of their sides."

regard to the second case, Ibn al-Haytham uses three premises: 1. *If four magnitudes be proportional, the first will then be to a part*[13] *of the second as the third to a part of the fourth. 2. Magnitudes can be halved indefinitely. 3. Given two unequal magnitudes, if the lesser magnitude be multiplied continuously, it will finally be greater than the greater magnitude.* By means of these three premises, he is able to reduce the non-numerical case to the numerical case.[14] However, it does not appear that Ibn al-Haytham has studied the connection between Definition V. 5 and the anthyphairetic definition of equal ratios - nor that he had even defined proportionality through anthyphairesis.

The commentary of Omar Khayyam

In his *Commentary on the Difficulties of Certain Postulates of Euclid's Work*, completed in December 1077 A.D., Omar Khayyâm intends to amend what he considers to be the more important difficulties found in Euclid's Elements. The first Book of Khayyam's commentary deals with the Theory of Parallels[15], the second with the concepts of ratio and proportionality, and the third with the compounding of ratios.

The concepts of ratio and proportionality

In Book 2 of his commentary, Khayyâm intends to deal thoroughly with the concepts of ratio and proportionality between magnitudes. He indeed considers that the question had never been treated in a satisfactory and philosophical way, and therefore proposes to remedy this situation.

For Khayyâm, two things enter into the concept of ratio between two magnitudes: the *relation* between these magnitudes as to equality and inequality: and the *quantity* - or the size - of this ratio, which he considers as a number. As to his interpretation of the relation between two magnitudes, it is essentially the same as that of some of his predecessors (notably al-Mâhânî and al-Nayrîzî), namely: either the

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13. This is not an aliquot part.

14. B. H. Slide, op. cit., pp. 188-204.

15. We will not study this Book herein.

of one of the two magnitudes with regard to the other. He characterizes this measure by means of the series of numbers obtained through the anthyphairetic process. Two ratios will then be the same if they are both characterized by the same series of numbers. He then states a definition of greater ratio which also involves the anthyphairetic process. The aim of al-Mâhânî will then be to prove that his definitions of equal and unequal ratios are in fact equivalent to the corresponding definitions found in the *Elements*.[9]

In his commentary on Euclid's Elements, al-Nayrizî (ca. 922) has also interpreted Definitions V. 3 & V. 5 in terms of anthyphairesis. But unlike al-Mâhânî, he considers that it is unnecessary to prove Definition V. 5, since in his opinion it is in reality something which belongs to the principles of Book V. Neither does he study the connection between the anthyphairetic definition of equal ratios and Definition V. 5.[10]

Ibn al-Haytham (965-1039) also deals with this question in his *Commentary on the Premises of Euclid's Elements*[11]. Ibn al-Haytham is of the opinion that Definition V. 5 is true, but that it is not obvious.[12] Therefore he considers this Definition - as well as Definition V. 7 - as a convertible proposition which should be proved. In order to do that, Ibn al-Haytham distinguishes two different cases: the case where the ratios are numerical, and the case where the ratios are not numerical. He proves that in case the ratios are numerical, Definition V. 5 will then be equivalent to Definition VII. 20. With

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9. Let us recall Euclid's definition of greater ratio, i.e. Definition V. 7: "When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to have a greater ratio to the second than the third has to the fourth."

10. In reality the Arabic text of the commentary on Definition V. 5 is not so clear. Al-Nayrîzî only says that it is not necessary to demonstrate "this," without our being able to know for certain what exactly he is referring to. See R. O. Besthorn, J. L. Heiberg, G. Junge, J. Raeder, W. Thompson, Codex leidensis 399, 1. *Euclidis Elementa ex interpretatione al-Hadschdschadschii cum Comentariis al-Narizii*, III, 2 (Hauniae, 1932), p. 16.

11. Books I to VI of this work have been edited and translated into English by B. H. Sude in her Ph.D *Dissertation Ibn al-Ha ytham's Commentary on the Premises of Euclid's Elements*, Princeton University, September 1974.

12. B. H. Sude, op. cit., pp. 181-182.

meaning. Thirdly, this Definition only considered the *similarity* of two ratios; so that it did not allow for a meaning to be given to the concept of ratio *in itself*

We do not know of any Greek commentary on Definition V. 5. But the situation is completely different with regard to Arabic mathematics. This definition has indeed given rise to many commentaries among Arabic mathematicians, which were aimed at either vindicating it by means of a proof, or replacing it by another definition known as the «anthyphairetic definition of equal ratios». The latter definition consists in applying to two homogeneous magnitudes a certain process known as «anthyphairesis».[5] That is, given two homogeneous magnitudes[6], one should consider the number of times the first magnitude measures the second, leaving a remainder less than the first; then the number of times this remainder measures the first magnitude, leaving a remainder less than the first remainder; then the number of times the second remainder measures the first remainder, leaving a third remainder less than the second remainder: and so on *ad infinitum*. The series of (whole) numbers thus obtained will then be «characteristic» of the ratio of the first magnitude to the second. Now if the ratio of two other homogeneous magnitudes is also characterized by the same series of numbers, the four magnitudes will then be said to have the same ratio.[7]

The first mathematical writing we know of in which the anthyphairetic definition is explicitly mentioned is the *Treatise on the Difficulty Relative to Ratio* by al-Mâhânî (ca. 880).[8] In this writing, al-Mâhânî defines the ratio of two homogeneous magnitudes as expressing the measure

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[5]. From a Greek word meaning alternate subtraction. This process is also known as «Euclid's algorithm».

[6]. We suppose here that the first magnitude is less than the second.

[7]. We think that the main idea underlying this definition is that it allows the ratio between the first and the second magnitude to be characterized independently of the ratio between the third and the fourth; in other words, it allows to give a definite meaning to the concept of ratio in itself - which is a necessary condition in order to consider a ratio between magnitudes as a number. For more details on this point, we refer the reader to our Paper "Al-Khayyâm's conception of ratio and proportionality," *Arabic Sciences and Philosophy*, 7, 2 (1997): 247-263, see on pp. 253-7.

[8]. Regarding the possible existence of a Greek definition of equal ratios based on the anthyphairetic process and the problems involved therein, we refer the reader to the comments of B. Vitrac in Euclide d'Alexandrie, *Les Éléments*, vol. II, pp. 515-523.

A brief survey of the fifth Book of Euclid's *Elements* and of Omar Khayyam's predecessors[2]

The theory of proportion as expounded by Euclid in Book V of the *Elements* was based on the definitions found in the beginning of the Book, of which three deserve our undivided attention:
3. A *ratio* is a sort of relation in respect of size between two magnitudes of the same kind. 5. Magnitudes are said to *be in the same ratio*, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order. 6. Let magnitudes which have the same ratio be called *proportional*.
As regards Definition V. 3[3], it must be noted that the meaning of the words "a sort of relation in respect of size" is nowhere explained in the Elements, which therefore poses a problem as to how these words should be interpreted. As for Definition V. 5, it played a major part in the theory in that it applied to magnitudes in general, that is whether the magnitudes in question were commensurable or incommensurable- unlike the other definition of proportionality found in Book VII of the *Elements*, which only applied to numbers and to commensurable magnitudes.[4]
However, Definition V. 5 posed a certain number of difficulties. First of all, comparing multiples of the given magnitudes did not seem to have any determinate and manifest relation to the notion of proportionality. Secondly, Euclid did not give any explanation on how this Definition had been conceived or established; which would have probably helped mathematicians to clarify and understand its

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2. Cf. E. B. Plooij, *Euclid's Conception of Ratio and his Definition of Proportional Magnitudes as Criticized by Arabian Commentators* (Rotterdam, 1950), chapters III & IV; A. P. Youschkevitch, *Les mathématiques arabes* (Paris, 1976), pp. 80-90; and Euclide d'Alexandrie, *Les Éléments*, traduction et commentaires par Bernard Vitrac, vol. II (Paris, 1994), pp. 539-548.
3. This notation means the third definition of the fifth Book of Euclid's Elements.
4. That is, Definition VII. 20: "Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth."

what one may call a «modern» conception, in which proportions between magnitudes are replaced by equations between numbers.

In order to illustrate this important point, let us consider the measurement of the surface of a circle. In Greek mathematics, this measure was expressed by saying, that "circles are to one another as the squares on their radii"[1]; in other words, any two circles being given, the ratio of these circles will be the same as the ratio of the squares on their radii. Whereas one would nowadays express this theorem by saying, that the surface S of a circle of radius r is given by the formula $S = \pi^2$, where $\pi = 3.14159..2$" We see, therefore, that we are in the presence of two different ways of measuring magnitudes: in the first case, we only have a similarity between ratios; whereas in the second case, we have a relationship, or to be more precise an equation, between numbers.

What should be noted in this context, is that expressing the measure of any magnitude by means of an equation presupposes that it is an established fact that the ratio of any two magnitudes can indeed be considered as a number. Thus, in the formula $S = \pi r^2$, S is in fact a number equal to the ratio of the given circle to the unit of surface, π a number equal to the ratio of the circumference of a circle to its diameter, and r a number equal to the ratio of the radius of the given circle to the unit of length.

We will, in this Paper, confine ourselves to but one of the numerous steps which led mathematicians to consider a ratio between magnitudes as a number, namely the commentary of Omar Khayyâm on Euclid's *Elements*. But we will in the first place recapitulate certain basic definitions as well as the works of some of Omar Khayyam's predecessors.

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[1]. The exact Greek statement is found in the second proposition of the twelfth Book of Euclid's Elements, that is: "Circles are to one another as the squares on the diameters." All quotations of Euclid are taken from Heath's translation of the Elements (T. L. Heath, *The Thirteen Books of Euclid's Elements*, second edition, Dover Publications, New York, 1956).

Omar Khayyam and the Concept of Irrational Number*

Bijan Vahabzadeh

Introduction

One of the main features of Islamic mathematics is the numerous commentaries on the fifth Book of the *Elements,* in which the celebrated Greek mathematician Euclid *(ca.* 300 B.C.) had expounded the notions of ratio and proportionality between magnitudes, that is what is usually known as the "theory of proportion". This theory was the one which was used by Greek mathematicians for measuring magnitudes: and it is this very same theory which was still in use during the 17th century, until it was progressively abandoned and replaced by another conception of measurement based on the notion of number. Thus one will notice, when comparing the way the Ancients measured magnitudes with ours, that there is on the one hand an Euclidean conception, in which only proportions between magnitudes are considered; and on the other hand

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