Appendix
TABLE 1

Agreement between the proposed geometric scheme based on Omar Khayyam's Triangle and the actual dimensions of the north dome

Dimensions of the North Dome	Components of Khayyam's triangle	Theoretical values (meters)	Actual measurements (meters)	Error (percent)
GM	CD	6.974	6.99	0.2
NO	AB	5.383	5.36	0.4
GO	GO	7.641	7.63	0.1
NR	CD	6.974	7.00	0.4
LR	GO	7.64I	7.63	0.1
KO	GO	7.641	7.64	0.0
GO	AC + ½AB	12.591	12.59	0.0
GK	½AC	4.950	4.95	0.0
KO	½AB	2.692	2.68	0.4
OO	½AC	4.950	4.96	0.2
KN	½BD	2.259	2.28	0.9
GJ	¼BD	1.129	1.14	0.9
JK	½GO	3.821	3.81	0.3
NP	½CD	3.487	3.47	0.5
PO	AB - ½CD	1.896	1.89	0.3
GN	½AC + ½BD	7.209	7.23	0.3
OS	√(AC x CD -	6.674	6.68	0.1
CS	BC	8.309	8.29	0.2
1M	CF	5.853	5.85	0.0

To end on a lighter note, let us see if we can detect any possible hint of the North Dome in Omar Khayyam's poetry. A rich accentuation of verticality is the first impression one receives of the interior space of the dome, known locally as Gunbad-i Khaki (the Dome of Earth). Can we trace the origin of this name to Omar Khayyam's metaphor in the following quatrain, in which he elegantly expresses the graceful and delicate beauty of an implied slender woman? This, perhaps, is the delightful package in which is wrapped the evidence we seek:

My beauty's rare, my body fair to see,
Tall as a cypress, blooming like the tulip;
And yet I don't know why the hand of Fate
Sent me to grace this pleasure-dome of Earth.[33]

---

[33]. Ali Dashti, *In Search of Omar Khayyam*. trans. L P. Elwell-Suton (London, 1971), 191

the ribs and the dome itself were constructed together, this unprecedented idea should have been conceptualized and worked out during the design process. Who but Omar Khayyam in Isfahan could have had the vision and expertise to conceive such a revolutionary design?

The geometric scheme outlined here is by itself inconclusive evidence that Omar Khayyam designed the North Dome, but with additional circumstantial evidence the idea becomes increasingly convincing. There remains the question why he, one of the great intellects the Islamic world produced, was never mentioned in contemporary sources as the designer. The answer may lie in the period of his life after the completion of the North Dome.

After Malikshah's death in 1092, the observatory was dosed and Omar Khayyam's reformed calendar abolished. Al-Zawzani writes in 1249 that the spirit of Omar Khayyam's poetry, in spite of its outer appearance, betrayed secret thoughts that incited religious hatred against the poet. He restrained his tongue and pen oftentimes to escape danger to his life. His pilgrimage to Mecca in his old age and his constant attendance in places of prayer and at religious services, said al-Qisti (1172-1239), were only for the purpose of concealing his ideas and thoughts from the public. He died at Nishapur some time around 1131.[31]

There is an inscription panel over the gate leading into the North Dome, placed there during the restoration of the mosque after a fire in 1121-22. It quotes the Koran 2:114: "And who is more unjust than he who forbids that in places for worship of God (*masājid*) God's name should be celebrated, and whose zeal is in fact to ruin them? It was not fitting that such (people) should themselves enter these (places of worship) except in fear. For them there is nothing but disgrace in this world and in the world to come an exceeding torment ."[32]

The verse is unusual for a building inscription and, as Grabar points out, clearly refers to some particularly revolting desecration that had befallen the mosque. Can we read in it a hint of Omar Khayyam's concessions out of fear? Could his design of the part of the mosque that gate leads to have been considered a revolting form of desecration? Perhaps encrypted in this verse is the explanation of why his name was never cited as the designer.

---

31. Daoud S. Kasir, *The Algebra of Omar Khayyam* (New York, 1931), 3-4.

32. Grabar (n. 2 above), 26.

knowledge: "The mason does not have the same significance as the architect and the architect not the same as the geometer. The geometer is Ptolemy, the architect is al-Battani, and my role is that of a mason."[29]

Although such a hierarchy is not likely for every construction, the ingenuity of the geometric scheme of the North Dome does indeed suggest the authorship of a mathematician and points specifically to Omar Khayyam. His prestige increased in the court shortly before construction began owing to the adoption of his calendar by order of the sultan. This could have persuaded the patroness, Terkan Khatun, to give him the commission for designing such a prestigious building.

Comparison of the North Dome and the South Dome (dating from 1086–87) provides further indication that a highly skilled mathematician-architect was in charge of the construction of the North Dome. Rassad's photogrammetric survey includes plans and sections of the South Dome.[30] The positions of its horizontal and vertical lines and particularly its vertical axis alignment lack the precision of the North Dome, to a degree that makes meaningful proportional analysis impossible. These buildings were completed so closely together that their construction must have over-lapped. It is unlikely that bricklayers working on the North Dome were better than their counterparts on the South. The disparity in precision of execution of the two schemes, therefore, can only be attributed to a difference of skill or expertise of those in charge. The architect of the South Dome was Abu'l Fath, son of Muhammad the Treasurer. His name implies that he had some political influence. Of his skills and expertise we know only what his work reflects. The contrasting exactitude of the North Dome suggests a meticulous person. Omar Khayyam's calendar testifies to his concern with accuracy, and his authority as an astronomer skilled in trigonometry makes him at the very least one of the most likely persons in Isfahan at the time to have executed the demanding alignment of the dome.

Another case in point is the pentagonal star formation of interlacing transverse ribs in the dome (fig. 1). It is the earliest example of decoration on the concavity of a domical surface. The geometric schema must have been planned on a two-dimensional surface, following which the straight lines were transformed into curves. As

29. Ibid.

30. Rassad Survey Company (n. 21 above), plates 8, 12.

Was Omar Khayyam the Designer?

Omar Khayyam was born at Nishapur about the year 1048.[27] After living in various cities in central Asia and Persia, in 1074 he was invited to take charge of the observatory in Isfahan by sultan Malikshah and vizier Nizam al-Mulk. Around this date he wrote the untitled treatise and his famous book on algebra, *Risāla fī'l-barāhīin 'alā masā'il al-Jabr wa'l-muqābala*. In 1077 he wrote his commentary on Euclid, Sharh ma ushkila min musadarat kitab Iklidis. In 1079 his reformed calendar, so precise that its error was only one day in a cycle of 5000 years, was adopted by royal decree. Construction of the North Dome started presumably in the early 1080s.

The likelihood that Omar Khayyam was aware of the musical proportion embodied in his triangle increases if my analysis of its significance to the geometric scheme of the North Dome is correct. In the short time between the discovery of the triangle and the construction of the North Dome, he seems the most likely person in Isfahan to have worked out the unpublished properties. Even if someone else did so, he would have been told about it as the discoverer of the triangle and the most prominent mathematician in Isfahan. This of course does not prove that Omar Khayyam was the designer of the North Dome; he might have shared a knowledge of the musical proportion of the triangle with an architect-artisan in a conversazione.

The mathematician al-Isfizari, contemporary of Omar Khayyam, tells us: "Geometry is the basis for architecture; that is why the geometer with his science constitutes the foundation. He is followed by the master builder who in turn is followed by the wage laborer (bricklayer). The geometer commands the second (i.e., master builder) and the master builder commands the wage laborer, while the wage laborer busies himself with water and clay."[28]

We can infer from these words, cited in the work of the Persian historian ai-Bayhaqi (1106-74), that the geometer in charge of a construction could appropriately be called the mathematician-architect. An entry about al-Qajini in al-Bayhaqi's work establishes a hierarchy in the building trade based on differing levels of geometric

---

27. For Omar Khayyam's life, see *Dictionary of Scientific Biography*. s.v. "Al-Khayyami (or Khayyam), Giyath al-Din Abu' l-Fath 'Umar ibn Ibrahim al-Nisaburi"

28. Necipoğlu (n. 6 above), 140, 177 n. 24.

perpendicular QS and side CS which is equal to BC.[23] QS corresponds to the height of the dome itself above the sixteen-sided belt. It was an important measure during construction, but after completion of the dome its phenomenal existence as an observable dimension ceased. What a person at ground level observes is the dome surface extending from the sixteen-sided belt to the apex. For proportional analysis the arc CS is the relevant component.[24] The ratio of CS to the span in fact equals the proportion that Omar Khayyam posed as the objective of his untitled treatise (CS:AC::BC:AC::BD:AB). This very ratio appears as a constant theme underlying all musical proportions of the chamber, filling the dome with echoes of the geometric mean between the span and the whole transition zone

(JM:GM::KL:PQ::QR:PQ::PQ:KN::KN:KO::NR:CS:: CS:AC).[25]

To conclude, the geometric scheme of the North Dome appears to be generated wholly from Omar Khayyam's triangle. The scheme, inevitably, is my own, but its very close agreement with actual dimensions (the average deviation is 0.012 m, of 0.2 percent) gives it credibility.[26] It can reasonably be assumed that the designer, probably Omar Khayyam, calculated the dimensions of the building with the aid of astronomical handbooks.

---

23. Theoretically $QS^2 = AC \times CD - \frac{1}{4}AC^2$. The side CS acts as the geometric mean between the side of the smallest triangle. CDS, and the base of the largest triangle, CSA. that is, CD:CS::CS:CA.

24. Arc CS is a segment of a circle of which CD is the radius. CS is a mathematical fiction, drawn to show proportions and the height of the dome. The real curve of the dome, VUS, begins at the upper edge of the vertical inscription band. The vertical band may obey constructional considerations: it allows a slightly increased thickness of the dome at its base. This condition required a four-centered profile in order to retain the predetermined height. The arcs VU and US appear to be determined by points D^x and C^x as centers, and D^xU and C^xS as radii.

25. JM corresponds to CF in Omar Khayyam's triangle. The height of the square zone, GM. and the horizontal distance from the far walls to the edge of the lower side arches, CT, acts as the two mean proportionals between JM and span (JM:GM::GM:CT::CT:AC). The ratio of height to span of the square zone, GM:AC, reappears as a secondary theme among subdivisions of the chamber ending at the base of the dome as the ratio of height to span of the transition zone, NR:AC(GMAC::GJ:KL::GJ:QR::JK:NQ:KOJK::NF.GK:KL:KN::KO:PQ::NR:AC). This ratio, which equals CRAC in Omar Khayyam's triangle, is the square of the primary theme (BC:AC::[CDAC]^2).

26. See n. 28 above. For all dimensions, the maximum deviation is about 0.02 m. The errors of KN and GJ appear larger than others because they are very short dimensions, determined by way of subtraction.

The heights of the square part, GM, and the transition zone, NO, correspond respectively to segment CD and side AB of the triangle.[21] Thus the square zone height is the harmonic mean between span and transition zone. The height GO of the central window sill in the upper main arch equals the arithmetic mean of span and transition zone. These constitute Iamblichus's musical proportion, which I qualify as "upward" (AC:GO::GM:NQ).

A horizontal band bearing inscriptions, QR, links the dome and the space below. Vertical in section, it visually extends the transition zone; circular in plan, it joins the dome. The height NR of the combined transition zone and inscription band equals CD. This seems intended as an upper harmonic mean between span and transition zone. LR, which equals GO, becomes the upper arithmetic mean between span and transition zone. Since QR and KL are equal, KQ also equals GO and seems intended as a lower arithmetic mean reflecting the double role of the inscription band. They form two downward musical proportions (AC:LR::NR:NQ::21C:KQ). Apparently the harmonic mean determined heights of the main zones, while arithmetic means were used to accent central features and link musical proportions together.

The height GQ below the dome equals the span plus half the transition zone height (AC + 1/2AB), and GK, KO, OQ, and KN correspond respectively to ½AC, ½AB, ½AC, and ½BD. In the lower square zone, the height GK determines both the springing level of the central arch and the frame of the lower side arches. The dado height GJ subdivides GK so that

GJ equals ¼ BD and JK is ½ GO.[22] The transition zone has a lower octagonal belt NP, and an upper sixteen-sided belt PQ. NP equals ½ CD and PQ equals AB - ½ CD. These divisions of the two zones produce two musical proportions of halves (GKJK::NP.KO::OQ:JK). The division of the transition zone repeats the ratio of the transition zone to the span (PQ:NP::NQ:AC). The span is the arithmetic mean between the height below the dome and the square zone height (AC = ½ [GQ + GNl). Omar Khayyam's triangle regulates the dome's steepness. Isosceles triangles CSA, TCS, and CDS all share a

[21]. Table 1, in the appendix to this article, assesses the agreement between the proposed geometric scheme based on Omar Khayyam's triangle and the actual dimensions of the North Dome.

[22]. Another property of Omar Khayyam's triangle is AC = GO +½BD. Accordingly ½AC-¼BD =½GO, which corresponds to JK.

something like "a set of equal ratios arranged so as to create an orderly, pleasant effect." I expect proportions of the North Dome Chamber to meet this definition.

Let us superimpose Omar Khayyam's triangle ABC over a drawing of the dome such that its hypotenuse AC matches the 9.90 m span at ground level (fig. 5).

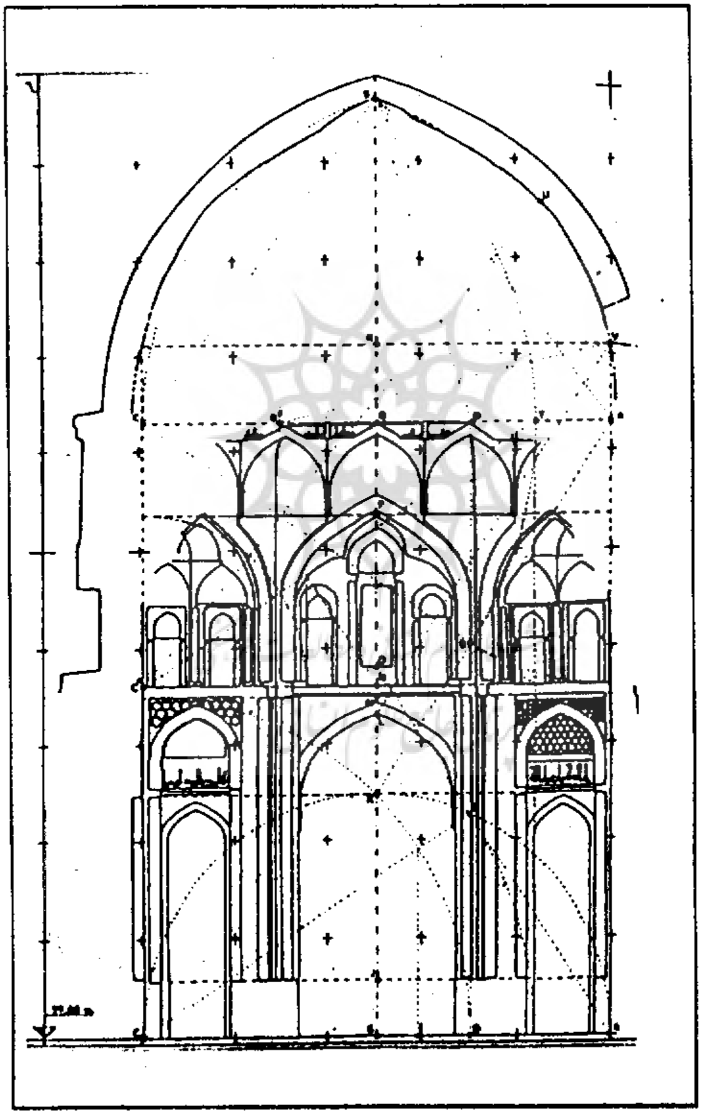

Fig. 5

This lower-part of the chamber is square. A narrow band separates it from a polygonal transition zone with muqarnas corbels (stalactites).

central arches and the dome is more significant. The axis crosses the apex of each arch and of the dome with an error of about 0.1 percent. To attain such precision aligning the dome axis with one of the arches would be a difficult task using modern technology, let alone that of the eleventh century. Such precision, judging by my experience surveying scores of medieval buildings in Anatolia, is very unusual for a Seljuqid building. It not only allows but motivates us to proceed with our analysis.

"Proportion" in common usage is defined today as "the correct relationship between the size, position, and shape of the different parts of a whole, especially as producing a beautiful effect?[19] The mathematical definition is specific: the equality of ratios. How would this term have been used in the Islamic world when it concerned architecture? Ibn Khaldun tells us: "In view of its origin, carpentry needs a good deal of geometry of all kinds. It requires either a general or specialized knowledge of proportion and measurement, in order to bring the forms (of things) from potentiality to actuality in the proper manner, and for the knowledge of proportions one must have recourse to the geometrician".[20] As carpentry was associated closely with architecture, we can infer from these words that mathematicians had a say in architectural proportions too. Their definition might be

---

← maximum deviation from vertical alignment of ±0.03 m. The average height of upper level of the dado, from which vertical mouldings start, is L15 m ±0.02 m. The frames of the lower side arches, corresponding to the central arch springing, are at 4.95 m. The apex of the lower central arch is at 6.60 m. The height of the lower edge of the narrow band separating the square and transition zones is 7.00 m ±0.02 m. The upper edge of this band, corresponding to the window sills (except for the central one), has an average height of 7.23 m ±0.02 m. The central window sill is at 7.63 m. The lower elements of the muqarnas begin at 10.70 m ±0.02 m. The upper level of the transition zone is at 12.59 m. The drawing shows only one window in the dome-, its sill corresponds to the upper edge of the inscription band, at 14.23 m. The apex of the dome is at 19.27 m. Deviations from average height in all horizontal lines of the building are minimal, yielding an almost negligible error of about 0.2 per-cent (in this analysis errors below 0.1 percent are considered negligible and reported as nil). The same is true for verticals: the span between the two walls, 9.90 m at ground level, increases by about 0.03 m in the middle portion and becomes 9.90 m again in the transition zone. The maximum deviation from true vertical alignment in the walls is less than 0.4 percent. The horizontal shift of vertical mouldings from the dado to the narrow band is no more than 0.02 m, yielding an error of 0.3 percent.

[19]. Longman Dictionary of Contemporary English (Bath, 1981), 881.

[20]. Ibn Khaldun, *The Muqaddimah*, trans. Franz Rosenthal (Princeton, 1967), 2:365.

reported accuracy can safely be used for our analysis.[15] Since the published survey includes only one section through the North Dome, my analysis should be regarded as a·preliminary evaluation until tested on a full survey.[16]

The coordinates of scores of points on the published drawing are deter-mined with the aid of a plotter and so provide an accurate numerical basis for analysis.[17] The construction of the building proves to be exceptionally precise: deviations of horizontal and vertical elements from true positions are within ± 0.02 m, yielding an average error of about 0.3 percent.[18] The position of the vertical axis of the

---

[15]. We were told at the symposium that much effort was expended so that publish. drawings would retain precisely the original scale. 1:50 (the general plan was reduced 1:400). The width of the inner frame of each sheet (shown here as Y-Z in fig. 5) w drawn exactly equal to 40 cm so that the scale of the drawing could be checked any tin in the future. I digitized the original published drawing using Summagraphics MG III 1000 lines per inch for accuracy and isolated the coordinates of relevant points AutoCAD version R12. Checking the distance Y-Z measured 40.00 cm.

[16]. A shortcoming of the section through the North Dome is that details of the upper parts are missing. Apparently they were not covered stereoscopically. Nevertheless. this does not affect our analysis seriously since most pertinent points are covered. The exception is the missing wall of the inscription band at the base of the dome. I recovered the missing information consulting other published drawings surveyed by direct method and pictures of the building: see Pope and Ackerman, *A Survey of Persian Art* (n. I above), 8:289-91. I established that the upper edge of the inscription band corresponds to the sills of the windows of the dome (one of which is shown on the photogrammetric drawing), and that its surface is not curved (as shown in broken lines in the photogrammetric drawing) but vertical (fig. 6). According to architectural photogrammetric conventions, when part of a building appears only in one of the pictures and is not covered stereoscopically, by checking the building itself, the plotter may complete the form in broken lines. The plotter here misinterpreted conditions and continued the curvature of the dome without the check

[17]. For instruments and methods employed see n. 22 above. The tip of the arrow indicating the 99.00 m altitude in the original drawing is adopted here as the datum, or reference point. Many points pertinent to our evaluation are shown: the first figure represents the horizontal coordinate, the second the vertical, rounded off to full centimeters (fig. 5).

[18]. The floor of the North Dome Chamber is precisely at the 99.00 m vertical datum of the whole building survey. As the mosque has been in continuous use since Seljuqid times, there is no reason to believe the original ground level has been changed. The floor is almost perfectly horizontal, deviating only -0.02 m at the middle. Coordinates of points measured at 2 m intervals along the walls give a →

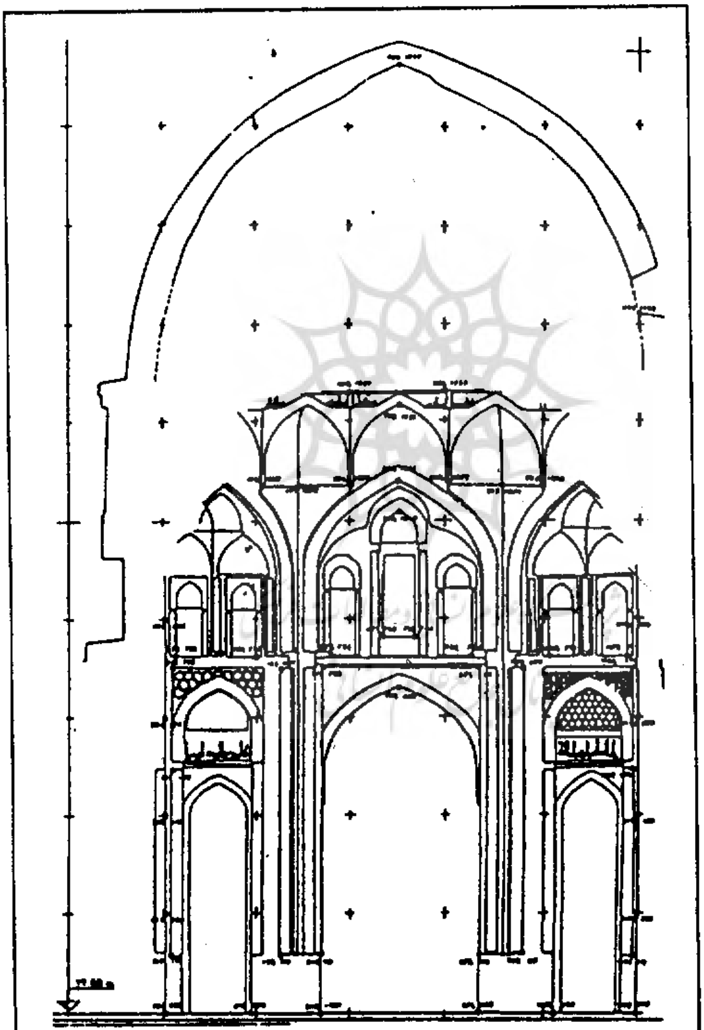

Fig. 4

The North Dome Chamber of the Masjid-i Jami` of Isfahan

The theory of proportion is not documented as solidly in architecture as in mathematics. An extensive literature on the subject exists, based generally on personal observations or convictions of modern scholars. Give weak concrete evidence of actual design principles, theories of proportion are developed by analyzing buildings that may have neither been execute with due precision nor surveyed with sufficient accuracy. Inevitably, theories contradict each other, even on the same building. The air of uncertainty prevailing in this field cautions us to be extremely careful in d analysis of the North Dome. Before analyzing its proportions, therefore, the accuracy of the available survey and the precision of construction of the building itself must be assessed.

The Rassad Survey Company, Tehran, has published a photogrammetric survey of the Masjid-i Jami` of Isfahan. Through the instruments ar methods employed, they attained an accuracy of better than 1:50,000 in the topographic survey and a "consistent high accuracy" in the photogrammetric one. One of the published drawings is a section through the Nor. Dome Chamber (fig. 4).[14]

The drawing was published at its original scale 1:50 and given its

---

[14]. Rassad Survey Company, "Masjed-e Jame' Esfahan" (paper presented to *International Committee for Architectural Photogrammetry at the Symposium on the Photogrammetric Survey of Ancient Monuments*, Athens, 1974), pl. 13. On the accuracy of the survey see pp. 1-4.

theory.[12]

The musical proportion of irrational magnitudes is a very interesting property of Omar Khayyam's triangle, but was he aware of it? An answer may lie in the third part of his commentary on Euclid. After discussing compounding ratios he argues that a geometrically obtained irrational ratio can be understood only by accepting irrationals as numbers - and in so doing he emerges as the first mathematician to admit irrationals to the status of numbers. He then addresses the use of ratios in music:

> The science of music is based upon combining of ratios. But the ratios used there are numerical, not geometrical. Decomposition of ratios in music is really a kind of combination. We leave it to your piercing intelligence to comprehend. We will mention a line about this idea on discussion of difficulties of books of music and science of numbers. [They say] "there is no need of geometry in it, and can be studied without geometry. This comes before geometry and there is no relation between them." But ... the science of numbers and geometry are two sciences neither of which comes before the other.[13]

His reason for discussing musical ratios apparently is to emphasize that irrational ratios have a role to play in musical science also. He appears to have a clear idea on this matter, which he leaves to his reader to comprehend. It may not be too far-fetched to suppose that his hint in the untitled treatise relates to musical proportion consisting of irrational magnitudes. At any rate, Omar Khayyam's triangle, with its richly interrelate proportions, presents itself provocatively as a tool suitable for architectural application.

---

12. Eilhard Wiedemann, "Über die Geometric and Arithmetik nach den Mafatih al-'Ulum," in *Aufsatze zur Arabischen Wissenschaftgeschichte* (Hildesheim and New York, 1970), 1:418.

13. Amir-Moéz, "Discussions of Difficulties" (n. 5 above), 301. To have an understanding of irrationals - which Greek mathematicians knew through geometry - Omar Khayyam defines them by infinite sequences. Archytas's definition of harmonic mean answers nicely to decomposition of ratios. Apart from that, however, I have failed so far to comprehend exactly what Omar Khayyam leaves to the piercing intelligence of I reader. A study of Islamic music theory may prove to be fruitful in this matter.

and BC become geometric means between the hypotenuse and the two segments, i.e., AC:AB::AB:AD and AC:BC::BC:CD.[8] Peculiar to Omar Khayyam's triangle, the ratio of the longer segment to the shorter side is the same as the ratio of the hypotenuse plus the shorter segment to the hypotenuse, i.e., CD:AB::(AC + AD):AC. Let us extend side A, until it meets at point W a perpendicular to the hypotenuse erected on point C. Let us also erect on the midpoint G of the hypotenuse the perpendicular GO. Then GO equals half of CW and is the arithmetic mean between the hypotenuse and the shorter side of the triangle, or GO = ½ CW = ½ (AB - AC).[9] The ratio of the hypotenuse to the arithmetic mean is the same as the ratio of the hypotenuse plus the shorter segment to the hypotenuse, i.e., AC:GO::(AC + AD):AC, hence AC:GO::CD-.AB. Now, the longer segment CD becomes by definition the harmonic mean between the hypotenuse and the shorter side, that is, (AC - CD):AC::(AD - AB)AB.[10] According to the theory of numbers, a proportion of four terms is formed by the arithmetic and harmonic means between two given numbers. To give a simple example, between 12 and 6 the arithmetic mean is 9 and the harmonic mean is 8, thus 12:9::8:6. In the case of Omar Khayyam's triangle, regardless of being irrational magnitudes, the segments GO and CD act respectively as arithmetic and harmonic means between the extreme sides, and, as we have seen above, they all are related to each other in the proportion AC:GO::CD:AB.

Iamblichus tells us that the proportion composed of arithmetic and harmonic means between two given numbers, called "musical proportion," was judged most perfect. According to tradition, Babylonians discovered it and Pythagoras introduced it in Greece.[11] After Islamic mathematics adopted the Greek theory of proportion and means, the harmonic mean played an important role in Islamic music

---

8. Archytas defines the geometric mean as "when of three terms, the first is to the second as the second is to the third" (ac = b2); see Thomas Heath, *A History of Greek Mathematics* (Oxford, 1965), 1:85.

9. Archytas defines the arithmetic mean as "when of three terms, the first exceeds the second by the same amount as the second exceeds the third" (a + c = 2b); see Heath. 85

10. Archytas defines the harmonic mean as "when the three terms are such that by whatever part of itself the first exceeds the second, the second exceeds the third by the same part of the third" ([a – b]):a::[b – c]:c): see Heath. 85.

11. Heath, 86.

him to write the treatise. That the triangle he discovered served later as the basis of an ornamental pattern suggests the meeting was a conversazione and that the people he addressed were artisans.[7] If the "similar problems" he referred to were ornamental patterns based on the same triangle, then the "other properties" he hints at must be ones irrelevant to the ornamental arts. To find the unmentioned properties, Omar Khayyam's triangle must be examined in light of the general topic of the treatise, the theory of proportion and means.

Analysis of the triangle using geometric, arithmetic, and harmonic means, the most widely used ratio systems in Greek and Islamic mathematics, reveals unexpected qualities. Let us construct Omar Khayyam's triangle ABC so that the perpendicular BD cuts the hypotenuse AC into two segments AD and CD (fig. 3).

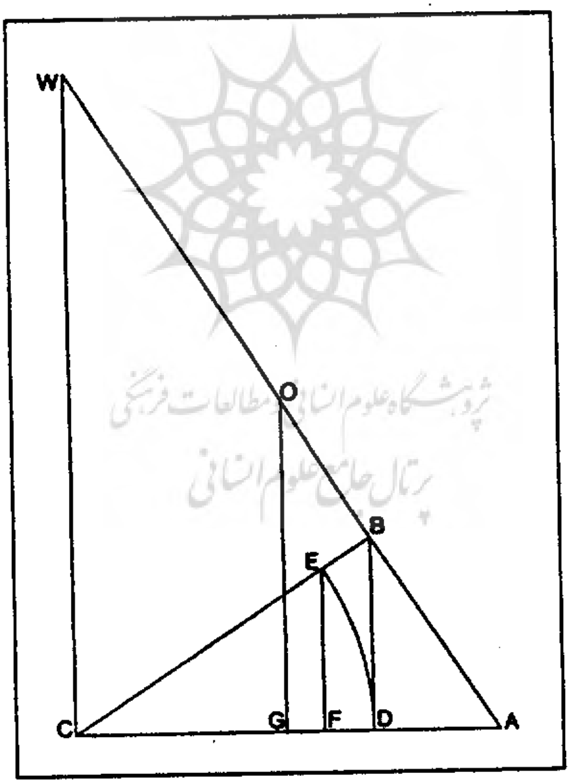

Fig. 3

According to general properties of a right triangle, the two sides AB

---

7. For a detailed discussion see Özdural, "Omar Khayyam, Mathematicians, and Conversazioni" (n. 6 above), 59-63.

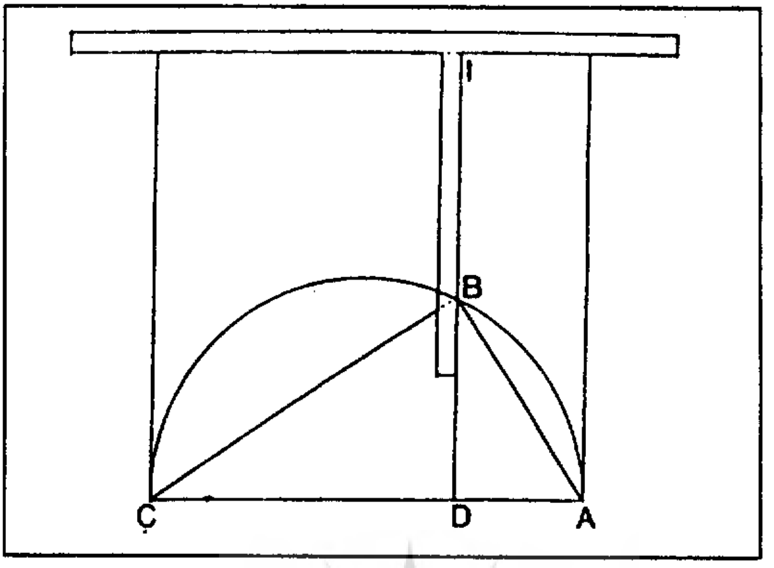

Fig. 2

On a given side AC draw rsquare and a semicircle. Slide a T-square along the side of the square opposite to AC, measuring the lengths of AB and BI at every position of poin B. When AB = BI the condition of Omar Khayyam's triangle, AB + BD = AC, is fulfilled.

This exemplifies a verging procedure, which can be define (as the mechanical equivalent of a cubic equation. In Greek and Islamic mathematics verging procedures aided by measuring instruments, some-times specialized ones, were used occasionally for practical purposes. This verging construction, aimed at producing a theoretically correct practical result, illustrates the role of mathematicians in Islamic art and architecture. The elegance of the procedure suggests its possible creation by Omar Khayyam himself.

Omar Khayyam's untitled treatise accommodates both theoretical and practical concerns to a degree unusual in his works, leaving the impression that he addressed it to practical people as well as mathematicians. It reads, for example: "This idea, that is, a triangle with mentioned properties is very useful in problems similar to this one. This triangle has other properties. We shall mention some of them so that whoever studies this paper can benefit from it in similar problems"[6] We can infer that the properties of the problem he does discuss are those relevant to a question raised at a meeting that in turn prompted

[6]. Amir-Moéz, "A Paper," 326.

He attempts first a solution based on conic sections but leaves that as an exercise for the student to complete. He then proposes an alternative method for readers not knowledgeable in conics. He defines the right triangle ABC as one whose hypotenuse equals the sum of the shorter side and the perpendicular to the hypotenuse (AC = AB + BD), and then proves that BC = AB + AD. He then assigns the arbitrary value 10 to AD and x to BD and reduces the problem to the cubic equation x3 + 200x = 20x2 + 2000.[2]

After solving the cubic equation by means of conic sections, Omar Khayyam offers another practical, approximate solution. Using astronomical handbooks and sexagesimal arithmetic he estimates the angle *BAC* as 57°, its sine as 0.833, its cosine as 0.544, and its versed sine (1 - cosine) as 0.455)0 Computing the values with modern tools after assigning to the hypotenuse AC a length 1, we obtain angle BAC = 57° 3' 53.565", BC = 0.8392868, AB = 0.5436890, and BD = 0.45631.[3] Omar Khayyam's approximations deviate 0.7 percent, 0.1 percent, and 0.2 percent, respectively, from our more accurate values.

Another solution of Omar Khayyam's triangle can be found in a later anonymous Persian work on ornamental geometry, On Interlocking *Similar or Corresponding Figures*. This appears to be a compilation of notes taken by a scribe of a series of *conversazioni*.[4] The work repeats the exact definition of Omar Khayyam's triangle but attributes it wrongly to Ibn al-Haytham.[5] The solution the scribe tries to explain, incompletely and confusedly, can be restored as follows (fig. 2).

---

2. Amir- Moéz "A Paper: 325-28, fig. 4. Omar Khayyam's equation is clumsy; defining x = 10y yields $y^3 + 2y = 2y^2 + 2$, which looks better. Other designations of the unknown and given magnitudes yield neater versions: $x^3 + x\text{-}^2 + x = 1$. or $x^3 + 2\text{-}x^2 = 2$.

3. Amir-Moéz, "A Paper." 336.

4. This assessment is based on Özdural, "On Interlocking Similar or Corresponding Figures," 191-211.

5. Özdural, "On Interlocking Similar or Corresponding Figures" (n. 6 above), 19: Ibn al-Haytham wrote over 180 treatises. For a list of their titles, see the *Dictionary Scientific Biography*, s.v. "Ibn al-Haytham," and Fuad Sezgin, *Geschichte der arabische Schrifttums*, 7 vols. (Leiden, 1967-1, 5:365-74, 6:254-61. Some seventy of his works a extant. None concern this particular problem. Necipoğlu believes that Ibn al-Haytha might have dealt with this problem not in a separate treatise but in the concluding chapters of the now lost *Kitāab al-abniya wa al-'ugtid* (Book of buildings and pattern Necipoğlu, Topkapi Scroll (n. 6 above), 178. Since On interlocking Similar or Corresponding Figures mentions specifically that this triangle was the subject of a treatise believe that the attribution to Ibn al-Haytham was wrong.

as referring to engineers, whereas the original sense of the word would be more accurate. I believe *muhandis* evolved to mean "engineer" as more mathematicians became involved with architecture through *conversazioni*.

Schroeder's attribution generally has been considered attractive but unverifiable. An untitled treatise by Omar Khayyam, first published in 1960, on a right triangle he discovered can shed new light on this issue. I shall readdress the question whether Omar Khayyam was the designer of the North Dome Chamber after analyzing the properties of his triangle and the proportions of the building.

Omar Khayyam's Triangle

In his untitled treatise, Omar Khayyam proposes this objective: divide a quarter-circle with its center at A by a point B so that if BD is drawn perpendicular to the radius AH, the ratio AH:BD equals AD:DH (fig. 1).[1]

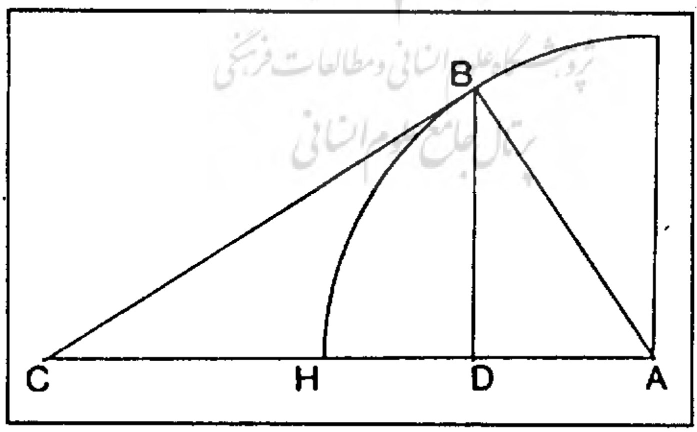

Fig. 1

[1]. Ali R. Amir-Moéz, "A Paper of Omar Khayyam," *Scripta Mathematica* 26 (1963): 323-37.

on inaccurate information: accurately surveyed dimensions of the North Dome show his pro-portions to be only rough approximations.4 Also, although Omar Khayyam understood the properties of the pentagon, he did not report them in his works.[5]

These defects do not mean Schroeder's hypothesis is misconceived. Many Islamic documentary sources refer to special meetings - which I call *consersazioni* for want of a better word - between artisans and mathematicians, suggesting their frequent collaboration.[6] Ornamental patterns based on cubic equations, or conic sections, bear further witness to their association since these powerful conceptual tools were available only to mathematicians at the time. Renata Holod has argued that a thorough awareness of engineering history may be necessary to explain aesthetic, structural, or spatial innovations in major Islamic centers.'[7] Her case hinges on a critical word, nuhandis, "engineer" in modern Arabic. Persian, and Turkish. Its original Arabic meaning was "geometer," or "mathematician" in a broader sense. Modern historians usually translate references to *muhandis* involved in building activities

---

4. According to Schroeder, 3:1008, fig. 347, the interior height is twice the base, but using actual dimensions. 19.27 m + 9.90 m ÷ 1.946 (fig. 5). He claims the line where the transition zone begins divides the total height into two parts whose ratio is the golden section (1.618...), but 19.27 m _ 12.04 m = 1.601. He adds that the ratio of heights of the upper to the lower main arches is also a golden section, whereas 10.81 m ÷ 6.60 m = 1.638.

5. On properties of the pentagon related to the golden section, Schroeder seems to allude to Omar Khayyam, *Sharh mā ushkila min musādarāt kitāb Iklīdis* (Commentary on the difficulties of the postulates of Euclid's book), written in 1077. In that work Omar Khayyam defines irrational numbers through infinite sequences, stating that there are magnitudes which cannot be divided into units of any kind. As examples he offers $\sqrt{5}$ and $\sqrt{10}$; beyond that there is no specific reference to the pentagon or to extreme and mean ratio. See All R. Amir-Moéz, "Discussions of Difficulties in Euclid." *Scripta Mathematica* 24 (1959): 275-303.

6. For collaborations between mathematicians and artisans see Alpay Özdural, "Omar Khayyam, Mathematicians, and Conversazioni with Artisans; *Journal of the Society of Architectural Historians* 54 (1995): 54-71, and "On Interlocking Similar or Corresponding Figures and Ornamental Patterns of Cubic Equations," *Muqarnas* 13 (1996): 191-211. For a broad discussion of the link between arts and mathematics see Gülru Necipoğlu, The Topkapi Scroll: *Geometry and Ornament in Islamic Architecture*. pt. 4. "Geometry and the Contribution of Mathematical Sciences" (Santa Monica, Calif., 1995), 129-81.

7. Renata Holod, "Text, Plan and Building: On the Transmission of Architectural Knowledge," in *Theories and Principles of Design in the Architecture of Islamic Societies*, ed. M. Ševčenko (Cambridge, Mass., 1988), 1-2, 11 n. 4.

Omar Khayyam and the Friday Mosque of Isfahan

Alpay Özdural

Introduction

On the north end of the main axis of the Masjid-i Jami′ (Friday Mosque) of Isfahan stands the elegant North Dome Chamber of Terkan Khatun, built in 1088-89. It corresponds to the majestic, slightly earlier dome of Nizam al-Mulk on the south. Although the dome chamber is relatively modest in size and austere in materials, the art historian Eric Schroeder has judged it to marry genius and tradition more elegantly than many other famous domed structures.[1] The identity of the designer of this remarkable building is a mystery that has intrigued many architectural historians.

The perfection of the North Dome led Oleg Grabar to assume the presence in Isfahan of a particularly inventive designer.[2] Grabar reports Schroeder's argument that since its proportions were derived from the golden section and the poet and mathematician Omar Khayyam at that very time had identified properties of the pentagon, he might have been the designer.[3] But Schroeder's syllogism is based

---

[1]. Eric Schroeder, "Seljuq Architecture". in *A Survéy of Persian Art*, ed. Arthur Upham Pope and Phyllis Ackerman, 2nd ed. (Tehran. 1977), 3:1005.

[2]. Oleg Grabar, *The Great Mosque of Isfahan* (New York and London, 1990), 64-65.

[3]. Schroeder never published this argument. See Grabar, 35, n. 5. The golden section can be defined as dividing a straight line into two segments such that the ratio of the whole line to the greater segment is the same as the ratio of the greater segment to the lesser, that is, $\frac{1}{2}(\sqrt{5}+1):1$. it was called "extreme and mean ratio" in Greek and Islamic mathematics and was used in defining the pentagon and the decagon. It came to be called the golden section in the West after Luca Pacioli's Divina proportione (1509).