here these fourteen cases, adding by way of information the nature of their solutions (real or complex, positive or negative) and the curves (circle, parabola, hyperbola) used by ‘Omar Khayyām to construct geometrically their positive solution.

1.	$x^3 = c$	$x_1 > 0; x_{2,3}\ \not{C}$	P, P
2.	$x^3 + bx = c$	$x_1 > 0; x_{2,3}\ \not{C}$	C, P
3.	$x^3 + c = bx$	$x_{1,2} > 0$ or $\not{C}; x_3 < 0$	P, H
4.	$x^3 = bx + c$	$x_1 > 0; x_{2,3} < 0$ or $\not{C}$	P, H
5.	$x^3 + ax^2 = c$	$x_1 > 0; x_{2,3} < 0$ or $\not{C}$	P, H
6.	$x^3 + c = ax^2$	$x_{1,2} > 0$ or $\not{C}; x_3 < 0$	P, H
7.	$x^3 = ax^2 + c$	$x_1 > 0; x_{2,3}\ \not{C}$	P, H
8.	$x^3 + ax^2 + bx = c$	$x_1 > 0; x_{2,3} < 0$ or $\not{C}$	C, H
9.	$x^3 + ax^2 + c = bx$	$x_{1,2} > 0$ or $\not{C}; x_3 < 0$	H, H
10.	$x^3 + bx + c = ax^2$	$x_{1,2} > 0$ or $\not{C}; x_3 < 0$	C, H
11.	$x^3 = ax^2 + bx + c$	$x_1 > 0; x_{2,3} < 0$ or $\not{C}$	H, H
12.	$x^3 + ax^2 = bx + c$	$x_1 > 0; x_{2,3} < 0$ or $\not{C}$	H, H
13.	$x^3 + bx = ax^2 + c$	$x_1 > 0; x_{2,3} > 0$ or $\not{C}$	C, H
14.	$x^3 + c = ax^2 + bx$	$x_{1,2} > 0$ or $\not{C}; x_3 < 0$	H, H

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powers (in our terms x^0, x^1, x^2) which are in *continued proportion*, that is, fulfil the proportion $x^2 : x = x : x^0$. Now this is no longer the case for third-degree equations, and therefore the former reasonings will not apply. Let us see what he has to say.

> Case of compound equations involving three elements not in continued proportion, or more, either in continued proportion or not. This is the case for the two possible categories involving three elements, namely, first, x^3, x^2 and number, and, second, x^3, x and number, which produce six compound equations; or the single category involving four elements, namely x^3, x^2, x and number, which produces seven compound equations; or others using higher powers. These do not admit of a treatment with our above algebraic resolutions, but only a geometrical one using conic sections.
>
> Case of the two trinomial categories mentioned above. The three kinds they each comprise do not belong to the domain of continued proportion. For the ratio of x^3 to x^2 is not equal to the ratio of x^2 to the number since there exists one power between x^2 and the number, namely that of x; neither is the ratio of x^3 to x equal to the ratio of x to the number since there exists one power between x^3 and x, namely that of x^2. Thus each of their six forms is beyond our above algebraic discourse. Indeed, the unknown which must be expressed and determined in each of these compound equations is the side of x^3, and the corresponding analysis leads to applying to a given straight line a given parallelepiped exceeding it, or falling short of it, by a cube. Now this can be constructed only by means of conic sections.
>
> Quadrinomial category, thus with an additional term. The situation of continued proportion is met, but the seven types are beyond the requirement of the general reasonings. For the unknown which must be determined is the side of the above mentioned x^3. But it cannot be expressed using algebraic reasonings but only, as already mentioned, by means of conic sections.

We have here quite clearly, around 1000, a statement that third-degree equations are not solvable by the usual algebraic reasonings and that their solution is possible by means of conic sections (if there is a positive solution). We have further a classification of all third-degree equations with positive terms and at least one positive solution, a classification ʻOmar Khayyām thought he was the first to establish: see the beginning of his *Algebra*, p. 3 in Woepcke's French translation, p. 160 in Mossaheb's Persian translation. (Our text only omits the first case, that of an equality between x^3 and a number, which amounts to no more than the extraction of a cube root). We shall conclude by enumerating

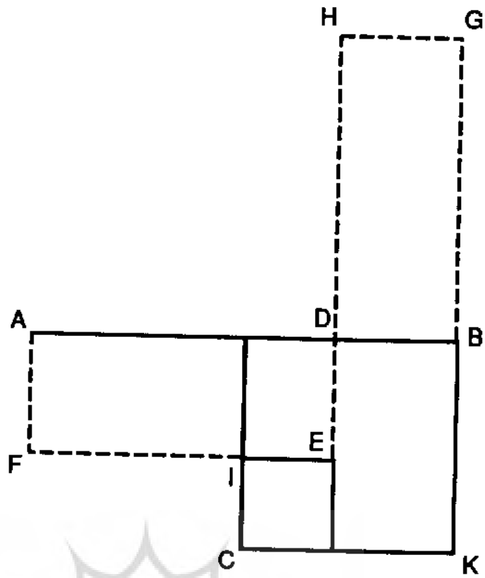

Fig. 6

3. Case of $x^2 = px + q$.

The construction is the same as in the first case, but this time the solution x is the segment of straight line AF. We have indeed AF BF=$x(x-p) = q$.

The text concludes these three constructions with the following remarks. (The "side of the unknown quantity" means x.):

> It has appeared clearly from the foregoing that the construction leading to the side of the unknown quantity in each of these three compound cases is the construction explained by Euclid towards the end of Book VI of his "Elements"; namely, the application to a given straight line of a parallelogram which exceeds this line, or falls short of it, by a square. Indeed, the side of the square in excess is the side of the unknown quantity in the first compound equation; in the second compound equation, the side of the unknown quantity is the side of the deficient square; in the third compound equation, it is the sum of the line to which the rectangle is applied and the side of the square in excess.

Our author then ends his treatise, the aim of which is to teach algebraic reasoning, thus *numerical* determination of the unknowns, with a note on higher-degree equations. He remarks first that the three compound equations seen previously admit of both an algebraic and a geometrical resolution (using Euclid's geometry) because they involve *three* considered in the form of products:

$$x^2 + px = q \qquad x(x+p) = q$$
$$x^2 + q = px \qquad x(p-x) = q$$
$$x^2 = px + q \qquad x(x-p) = q.$$

In our anonymous text these constructions are as follows.

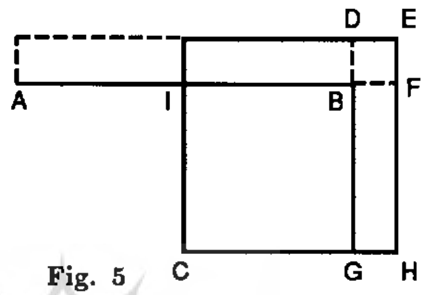

Fig. 5

1. Case of $x^2 + px = q$ (Fig. 5).

Let AB=p, AI=IB, and CB=$(\frac{p}{2})^2$. On the base CG of the square CB, describe the larger square CE=$(\frac{p}{2})^2 + q$, of which we know the side CH since we know how to construct the root of a given quantity. The required rectangle is then AE and the required solution BD=BF. Indeed, we see that the rectangle AE, being equal to the rectilineal figure ID+DF+FG, has the known area q and differs from AD, the rectangle on AB, by a square area.

2. Case of $x^2 + q = px$ (Fig. 6).

Let again AB=p and CB the square on its half. We construct the square CE=$(\frac{p}{2})^2 - q$ (with $(\frac{p}{2})^2 > q$), which is now smaller than the square CB. Their difference, q, is the sum of the two areas ID and DK, thus also equal to AI+ID=AE. In that case two rectangles fulfil the condition: AE, corresponding to the solution DE=DB=x, and DG, equal in size to the previous rectangle, corresponding to the solution AD=DH=x'.

IJ, we obtain $(x - \frac{p}{2})^2 = (\frac{p}{2})^2 + q$, from which the formula is deduced.

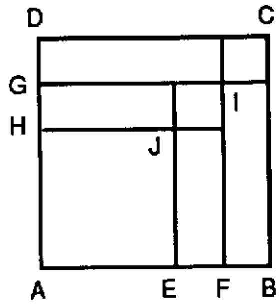

Fig. 3

All these figures illustrate the general formulae but do not represent graphically the solution to a specific equation since the length of x has been set to begin with. Euclid's *Elements* do, however, enable us to actually draw the solution and represent it as a segment of straight line. Our text explains how.

Three theorems from Euclid's *Elements* are used. The first, assumed to be known, is the construction of the root of a given quantity (that is, the root of a given segment of straight line). To do this, we add (Fig. 4) to the given quantity, say a, the unit segment and describe the circle with diameter $a+1$. The height at the extremity of a is then $\sqrt{a}$. This construction, an application of the theorem of height in a right-angled triangle, is *Elements* II.14.

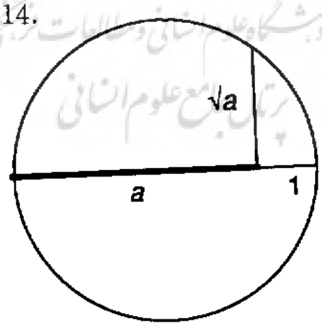

Fig. 4

The other two theorems are *Elements* VI.28-29, which teach how to construct on a known straight line a rectangle (generally, a parallelogram) equal to a given rectilineal figure and exceeding it or falling short of it by a square. For their application, the three equations are but also

AB AD=AE-CE-CD.

We find thus by addition that in both cases

2 AB AD=M+N+S+AE.

Now 2.AB.AD=px and AE=x^2, hence M+N+S=q. Thus, considering the equality of the two squares EC and each of the two possibilities for AD,

$$(\frac{p}{2})^2 - q = (\frac{p}{2} - x)^2 = (x - \frac{p}{2})^2$$

which illustrates the formula.

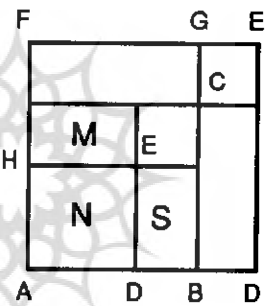

Fig. 2

(3) Case of $x^2 = px + q$ (Fig. 3).
The only positive solution is

$$x = \frac{p}{2} + \sqrt{(\frac{p}{2})^2 + q}.$$

Let ABCD represent x^2, EB be p, and let F be the middle of EB, so that EF=FB=$\frac{p}{2}$. Consider the completed figure, where GD=GH=$\frac{p}{2}$. Thus the rectangles GC and CF are each equal to $\frac{p}{2}x$, whence DI+IB+2 IC=px. Since IC=IJ, using the equation we find that GJ+AJ+JF=q. Adding now to each side of this equation the square

AC and CF and the rectangles CG and CE. (In Greek and Arabic texts, rectangular plane figures are usually designated by the letters at opposite angles). From the construction, we know that

$$\mathrm{CE} = \mathrm{CG} = \frac{p}{2}x.$$

Consider now the figure formed by CG, CA and CE. According to the equation, its area, which is $x^2 + px$, must equal q. Since CF=$(\frac{p}{2})^2$, the whole square AF is equal to $(\frac{p}{2})^2 + q$, but also, by construction, to $(x + \frac{p}{2})^2$. This illustrates the formula.

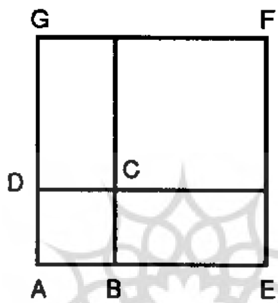

Fig. 1

(2) Case of $x^2 + q = px$ (Fig. 2).
The formula is

$$x = \frac{p}{2} \pm \sqrt{(\frac{p}{2})^2 - q},$$

thus with two positive solutions (provided that the discriminant is positive). In the anonymous treatise these two possibilities are represented in a single figure.

Let AB be $\frac{p}{2}$, thus AC=$(\frac{p}{2})^2$, and let AD represent the solution x, with either AD>AB or AD<AB, according to the two possible signs in the formula. We now complete the figure (keeping the same letters for the two solutions, as the manuscript does). Let us designate by N the smaller square AE and by M and S the (equal) rectangles adjacent to AE. In the case of the smaller solution,

AB AD=M+N

but also

AB AD=N+S=AE+S;

in the case of the larger solution, represented by the whole square AE,

AB AD=M+N+S+EC+CD

called "simple" (*mufrada*)

$$ax^2 = bx$$
$$ax^2 = c$$
$$bx = c$$

and the three equations called "compound" (*muqtarana*)

$$ax^2 + bx = c$$
$$ax^2 + c = bx.$$
$$ax^2 = bx + c.$$

The geometrical figures used to illustrate the formulae of the compound equations are different in nature in the treatises of Khwārizmī and Abū Kāmil. Khwārizmī's illustrations rely on an intuitive, visual geometry. Although the use of geometrical figures suggests a Greek influence, he does not mention the name of Euclid at all. Abū Kāmil has two kinds of illustration: one is similar to his predecessor's, but in the other Euclid is mentioned and reference is made to the two theorems *Elements* II. 5 and II. 6, of which this second kind of illustration is a direct application. That Euclid's name and theorems should appear in Abū Kāmil's *Algebra* but not in Khwārizmī's is, by the way, hardly surprising: Khwārizmī's treatise is elementary and does not suppose any prerequisites in (then) higher mathematics, whereas Abū Kāmil's *Algebra* is written specifically for mathematicians, that is to say, people trained in the study of Greek mathematics, chiefly Euclid's *Elements.*

Both kinds of illustration survived. Thus, we still find the visual kind in an anonymous treatise written in 1004/5 (395 of the hegira) which is said by its author to be a compilation from various sources (MS Mashhad 5325). The illustrations presented there for the compound equations are the following.

(1) Case of $x^2 + px = q$ (Fig. 1).
There is one positive solution, namely, as enunciated in the text,

$$x = \sqrt{(\frac{p}{2})^2 + q} - \frac{p}{2}.$$

Let the square ABCD represent x^2; let us extend AB by BE=$\frac{p}{2}$ and then complete the whole square AF, which then includes the squares

Towards 'Omar Khayyām

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Islamic algebra is said by Ibn Khaldūn in his *Muqaddima* to have begun with Khwārizmī (c. 820) and Abū Kāmil (c. 890). In Khwārizmī's largely accessible (and probably not very original) *Short account of algebra* are already found what were to be the three main characteristics of mediaeval algebra.

First, and unlike in the Greek algebra of Diophantus, there is a *complete absence of symbolism.* Everything, including numbers, is written in words. Only a few designations, such as those for the powers of the unknown, are specific to algebra: "thing" (*shay'*) is our x, "amount" (*māl*) is x^2, "cube" (*ka'b*) is x^3. The higher powers, found in later authors, are expressed, as were the Greek ones, by combining the words for x^2 and x^3.

A second characteristic of mediaeval algebra is the *recourse to geometrical figures* to illustrate the rules of algebraic reckoning or the resolution formulae for equations. In that sense, algebra can be said to have not yet fully gained autonomy; the geometrical proof was to remain, indeed for centuries, the criterion of mathematical truth.

A third characteristic, which was of ancient origin and, like the previous one, to last until late Renaissance times, is the *reduction of the* (then) *solvable algebraic equations to six specific types* with positive coefficients and at least one positive solution, namely the three equations

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