bibliographical and historiographical connections. When working on a mathematical issue, the goal is to bring it to its proper place inside Mathematics in a capital M – the Mathematics which is an ideal book-type object, containing all the cases and all the details in perfect order. So much, deuteronomic culture. It may sometimes, indeed, be merely pedantic or scholastic; add in genius, however, and you may get the likes of Khayyām.

To sum up, my hypothesis concerning Khayyām is as follows. The cultures of the codex gave rise to a new kind of dominant writing – the deuteronomic. This naturally allows for a treatise motivated by the impulse to provide exhaustive lists; and in the context of such a treatise, the very same geometric problems studied by the Greeks suddenly obtain a new, algebraic meaning. Finally, then, it is this deuteronomic aspect of medieval culture that accounts for the transformation of Greek geometry into algebra.

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clusions – which are not *in* mathematics, but *about* mathematics: about the classification of problems, whether historically, bibliographically, or more meta-mathematically in a more technical sense. This finally is true of Khayyām's treatise as a whole – as already argued in section 3.1. above. The treatise is an unending introduction; it does contain, to be sure, many problems set out and solved – but it is considered throughout not *through* problems, but *about* problems.

In other words, the main difference between Archimedes and Khayyām is that, whereas Archimedes separates clearly his introductions from his main text – and uses them, so to speak, merely as introductions – Khayyām does not separate his general claims at all from his actual mathematics, and allows the general claims, instead, to govern the particular claims.

Now, to produce, for the fist time, an exhaustive list of equations up to the third degree, and to solve them all, and to achieve all this with great elegance and precision, is a task calling for genius. Thus it is not as if the basic stylistic difference between Archimedes and Khayyām *explains* Khayyām's treatise. Not anyone letting his introductions run wild would write Khayyām's Algebra.

But while this stylistic difference does not provide sufficient conditions for the writing of the Algebra, it does provide, I argue, a central necessary condition. For the Algebra to be written, one needed first of all to have a culture where writing *about* mathematics was part and parcel of the writing *of* mathematics.

Now, in a previous article (R. Netz 1998: 261-288) I have argued that the medieval cultures of the codex provided just that. These cultures were marked by the dominance of deuteronomic texts. (I.e. the dominance of texts, such as commentary, that essentially depend upon previous texts). Thus, these are, in general, cultures where writing about writing was part and parcel of writing itself. Little wonder, then, that writing *about* mathematics was part and parcel of the writing *of* mathematics. I shall not repeat the arguments of R. Netz (1998) – they are of course reinforced by the discussion offered here – but shall instead sum up my main conclusion. I have argued that this dominance of the deuteronomic accounts for features of medieval science for which it is often criticized: its "scholasticism" or "pedantry". The interest of "scholastic", "pedantic" authors, is to put works in textual order: to add in all the details, to exhaust the field, to make all the relevant lem of finding lines satisfying a certain ratio is not related to other problems of lines satisfying certain ratios, but is related to a different 'kind' of problem, that of cutting a sphere. The Khayyāmite context is, as it were, horizontal: the problem of finding lines satisfying a certain equality is not related to other problems from which it may arise, but is instead related other problems of finding lines satisfying other equalities. This difference in context fully determines the mathematical difference between Archimedes and Khayyām. Khayyām differs from Archimedes in his foregrounding of study of cases, and of equalities, both deriving from his different type of context. Thus, merely by being set in different types of context – with no deep difference in admissible mathematical operations – the very nature of the proposition has been transformed, and a geometrical problem has become a cubic equation.

The question arises, why does Khayyām's context differ so markedly from that of Archimedes. And, in a sense, we already have been given a possible answer to this question. When surveying the overall structure of Khayyām's treatise, we saw that the impulse to provide exhaustive lists is closely related to a basic feature of the work, namely the continuity it displays between introduction and discussion. General, meta-mathematical claims, are interspersed with more specific mathematical claims at the object level, and the claims at the object level gain their significance from the claims at the meta-mathematical level.

In technical terms, we can see this phenomenon, of foregrounding of the general, in Khayyām's treatment of the cubic equation. I have noted how the specific geometric properties serve to show claims about possibility or impossiblility of problems. In other words, the goal is not at the object-level, to obtain geometric properties, but is, instead, meta-mathematical – to show the possibility of impossibility of tasks under varying conditions. In stylistic terms, the foregrounding of the general is seen in the two conclusions Khayyām reaches in his treatment: "This tangency or intersection was not grasped by Abu al-Jud, the eminent geometer, so that he reached the conclusion that if BC is bigger than AB, the problem would be impossible; and he was wrong in this claim. And this kind is the one that baffled Māhānī (among the six kinds). So that you shall know. ... So it has been proved that this case has different cases, some may include impossibilities, and it has been solved by the properties of two sections, both a parabola and a hyperbola". The mathematical discussion is governed by those two coning to given ratios is often an interesting task, *just because* proportions are more complicated. In his same treatise, the *Second Book on Sphere and cylinder*, Archimedes does mention, of course, tasks involving simple equalities. There is the task to find a plane equal to the surface of a given sphere; or to find a sphere equal to a given cone (or cylinder). But those problems are absolutely trivial. The first does not even get a diagram, and is effectively dismissed as obvious from the facts known from known results; the second gets a brief treatment in the first proposition of the book, where the proof, once again, is a mere quick unpacking of well-known results.[49] The remainder of the treatise is then dedicated to real problems, which are all defined by proportion or by the (equivalent) relation of similarity.

Archimedes' problem arises, as it were, in 'real-life geometry', and its shape is determined by the demands of this 'real-life geometry'. Khayyām's problem arises from its position in a list of problems – the list deriving not from an external, geometrical investigation, but from its own independent listing principle.

This comparison is crucial. There are of course classical Greek mathematical texts that list results: these are known as Elementary results. But the essence of Greek elementary results – the very way in which Greeks understood what "elementary" means[50] – is that those results serve in some other, advanced situations. Thus, *Book II* of the *Elements* – once considered as an example of "geometrical algebra" – is in fact motivated by an interest in specific geometrical configurations, arising in specific advanced problems. K. Saito has shown that the results of *Book II* are arranged not according to some principle internal to the work itself, but by geometrical motivations that are external to it (K. Saito 1985: 31-60). Thus, it is only natural that no attempt is made, in *Book II*, to obtain anything like an exhaustive list. The list is not interesting for its properties as a list, but is a mere 'repository' of results, useful vase by case.

The very same problem, we see, may be set in very different types of context. The Archimedean context is, as it were, 'vertical': the prob-

(49 Archimedes does refer to a separate proof concerning the impossibility. This proof is studied in R. Netz (1999), where I claim that, crucially, Archimedes avoids, in that proof as well, any mention of cases. The proof simply unfolds for a single case. Instead of making his arguments through cases, Archimedes develops two separate, case-free lines of argument: one on the solution of the problem, the other on the conditions for solubility of the problem.

(50 For these two problems, see Heiberg (1910-1915): pp. 170-174.

4.2. Conclusion:

How does a Geometrical Problem become a Cubic Equation?

Going back, then, to the structural observation made in Section 3.1. above concerning Khayyām's treatise, we can see how those structures are reflected in the mathematical detail of the work. Most obviously, the foregrounding of the study of cases is a feature of the work at all levels – the overall treatise as well as the individual proof. Throughout, Khayyām is motivated by the impulse to provide exhaustive lists. And it is because this proof serves as a "case", that it is analyzed according to tits cases. Khayyām, as it were, never really set out to solve a problem – this was not the issue. The issue, for him, was to catalogue a certain problem according to the properties of its solution.

So much for the foregrounding of the study of cases over geometrical properties. Inside geometrical properties, once again though in a less obvious way, the foregrounding of equalities over proportions is determined by the overall impulse to provide exhaustive lists. Equalities lend themselves to an exhaustive survey; proportions do not. Equalities have the simplest possible surface structure: a pair of symmetrical positions. Proportions have four positions, symmetrical in some ways and asymmetrical in others. Also, subtraction can always be eliminated from equalities (instead of A=B-C, you can have A+C=B), but not from proportions, lending a further dimension of complexity to proportions. Those brute facts alone make it almost inevitable that, when motivated by a desire to provide exhaustive lists of mathematical relations, equalities will be foregrounded over proportions.

On the other hand, Archimedes is throughout motivated by immediate geometrical tasks – in this case, to divide a sphere according to a given ratio. This ultimate goal determines the nature of Archimedes' treatment, just as Khayyām's exhaustive list determines his own treatment. Archimedes foregrounds the solution with its specific geometrical property, because this geometrical property is the external function of the proof; and he foregrounds proportions, because this external function is ultimately determined by a ratio. Truly, proportions are not easy to catalogue, but Archimedes was never interested in cataloging his problem. In his treatment, Archimedes' problem seems to be a one-off, totally unrelated to any other problem. Archimedes is simply interested in obtaining interesting geometrical tasks, and obtaining results accordproblem is set out by Archimedes as that of finding a proportion, it is to a proportion that his argument would lead; while Khayyām starts from an equality and must return to it. As it were, in the different melodies of their mathematical arguments, Archimedes has "proportion" as the tonic – the note from which he started and to which his readers expect him to return; while Khayyām has "equality" has the tonic.

In short, then, "proportion" gets foregrounded by Archimedes, "equality" by Khayyām. It is for this reason that Archimedes' lines are so clearly felt as "lines": a ratio involving four lines and areas, and ultimately dependent upon some geometrical similarity, is not easy to read off as a quantitative statement, but makes more sense as qualitative statement about a geometrical object. This is true even of algebraically-seeming statements, e.g. what we might express by a:b=ak:bk. Consider: "(18) and as the <line> ΓZ to the <line> ZN, (taking ZH as a common height) so is the <rectangle contained> by ΓZH to the <rectangle contained> by NZH". Inside a complex grid of lines, and inside a complex four-term expression, this claim becomes easier to read as a statement about figures in space, and not just about manipulated quantities. Khayyām's simpler equalities, on the other hand, are very easy to interpret as simple results of calculation, so that, even though his conception must have been thoroughly spatial, it becomes much more natural to read those equalities in abstraction from space – as it were, the equalities tend to become "equations".

We may sum up the comparison like this, then. Archimedes foregrounds geometrical properties, backgrounds study of cases; Khayyām foregrounds study of cases, backgrounds geometrical properties. Within geometrical properties, Archimedes foregrounds porportions, backgrounds equalities; Khayyām foregrounds equalities, backgrounds proportions. Put schematically:

Archimedes: (Proportions>Equalities)>Cases,
Khayyām: Cases>(Equalities>Proportions).

It is this inverse ordering of foreground and background which makes the proofs so different, which finally makes us feel that Khayyām's proof "just couldn't be Greek" – that it is, indeed, already *Algebra*. The mathematical materials are all the same, but they are arranged in a completely new structure. It is at this structural level, then, that Khayyām's originality has to be understood.

compare with 4 proportions.

Once again, the distinction between "foreground" and "background" is more qualitative than quantitative. As noted above, Archimedes has many geometrical constructions whose main function is to yield proportions – in particular, the grid of parallel lines, with its ensuing similar triangles. Khayyām has no need for such auxiliary structures and derives his relations in a much more direct way, from the equalities inherent in the conic sections; hence his much simpler figures.

Another example helps to bring forwards the sense of "foreground". I mentioned above the "9 or 11 equalities" Archimedes has: this is because some of his equalities are, as it were, self-effacing. Consider: "(9) it is: as the <line> EA to the <line> AΓ, so the <area> Δ to some <area> smaller than the <square> on BE, (10) that is, <smaller> than the <square> on HK". Now, the mathematical content of Step 10 is

(sq. BE)=(sq. HK),

But this is expressed through the "that is" operator, an after-thought to Step 9, so that, syntactically, we are invited to read Step 10 as a truncated way of stating

(EA:AΓ)::(<area> Δ :<area> smaller than sq. HK).

Thus, the equality is truly a background to the main statements, which are all about proportion. Put simply: for Archimedes, ecualities are ways of getting at proportions while, for Khayyām, proportions are ways of getting at equalities. As in the issue of cases versus geometrical properties, the main question is which serves which. We may compare, for instance, the ways through which the two geometrical proofs reach their goals:

Archimedes: "(36) while the <rectangle contained> by ΣZN is equal to the <square> on ΣΞ, (37) that is to the <square> on BO, (38) through the parabola. (39) Therefore as the <line> OA to the <line> AΓ, so the area Δ to the <square> on BO".

Khayyām: "(28) So the ratio of the square of GC, the first, to the square of BC, the second, as the ratio of BC, the second, to GA, the fourth. (29) So the cube of BC – which is equal to the given number – is equal to the solid whose base is the square of GC, and its height GA".

Archimedes develops some equalities – only to translate them into proportions; Khayyām develops some proportions – only to translate them ito equalities. The reason for this is, in fact, obvious: the way in which the goal is obtained is determined by the goal itself. Since the

In his article "Steps towards the Idea of Function: a Comparison between Eastern and Western Science of the Middle Ages", M. Schramm commented on Khayyām's failure to study the point at which the parabola and the hyperbola are tangents. This point is exactly one third the way above the given line – thus, an interesting property, which we would expect Khayyām to notice. Archimedes devotes his entire study of limits of solubility to this property (R. Nets 1999: 1-47). As M. Schramm put it:

> It is strange to find that 'Umar al-Khayyāmī does not mention this condition, already known to Archimedes. He likes to leave something for his readers to do. (M. Schramm ?: ?)

In fact Khayyām's silence on this point – as well as Archimedes' eloquence – are easy to explain. Since Khayyām's study of cases is logically prior to his study of geometrical properties, he is not interested in the geometrical properties of the point that define cases, as long as the points can be stated in terms of his exhaustive lists. For Archimedes' on the other hand, the cases are reached through an investigation of the geometrical properties of the configuration, hence he very naturally states the conditions for the tangencies of the sections. The different priorities determine, quite naturally, which questions you pursue and which questions you choose to leave aside.

This then is one major difference between the two proofs, having to with their overall aims and interests. Another major difference has to do with the technical tools used to achieve those aims, especially ratios and proportions.

Once again, this difference may be expressed in simple quantitative terms: Archimedes' solution has many more proportion statements than Khayyām's. Of Archimedes' 40 Steps, 16 assert proportions (40%); of Khayyām's 35 Steps, only 4 assert proportions (11%). Instead of *proportions*, Khayyām more often asserts *equalities*, and he asserts 8 equalities in his argument. Of course, equalities are much less central to Khayyām's overall argument than proportions are to Archimedes, but this is because many of Khayyām's claims have to do directly with possibility or impossibility under various inequalities. Both proportions and equalities are backgrounded in Khayyām's treatment – relative to the study of cases – while they are both foregrounded in Archimedes' treatment. What we now see is that, *among* the two, Archimedes foregrounds proportions, while Khayyām foregrounds equalities. Archimedes' 16 Proportions compare with 9 or 11 equalities: Khayyām's 8 equalities as well – now well into the middle of the proof – is not the solution itself, but its division into cases. This is the heart of Khayyām's proposition – the moment where he stops to make historical and bibliographical statements, comparing his achievement with previous achievements. It is precisely such division into cases of which he prides himeself: "This tangency or intersection was not grasped by Abu al-Jud, the eminent geometer, so that he reached the conclusion that if BC is bigger than AB, the problem would be impossible; and he was wrong in this claim". Now, the next Step in Khayyām's proof, 21, is another brief claim concerning cases; and then Steps 21-32 provide the geometrical argument concerning the solution, which is now seen as dependent upon the main claims. What Khayyām's solution at Steps 21-32 does, given its context, is not so much to solve a problem, but to show that *a solution is possible given a certain condition.* Finally, Steps 33-35 wrap up the argument by suggesting how the same solubility may be seen for the other configurations.

Khayyām's proof, then, is not so much a solution to a problem, as a study of the cases arising out of the problem, arranged according to two exhaustive lists of equalities or inequalities:

$$\text{(Content of Step 1):} \quad H \leq, \geq AC$$
$$\text{(Content of Step 11):} \quad BC \leq, \geq AB.$$

The first part of the proof, Steps 1-10, studies the cases of possibility and impossibility arising from the first exhaustive list. The second part of the proof, Steps 11-35, studies the cases of possibility and impossibility arising from the second exhaustive list. The main geometrical property – Steps 21-32 – serves, in context, merely as an element inside this second study.

Khayyām looks at the problem, distinguishes its cases and studies them as items in an exhaustive list of equalities and inequalities; geometrical comments being made to the extent that they contribute to this study. Archimedes looks at the problem and develops its geometrical properties, realizing that these may also fall into different cases. This difference is one of the major reason why Khayyām's problem feels more "algebraical" – why his lines tend to appear like sheer quantities. Since he plunges directly into cases and develops them before developing his geometrical study, he is bound to single out simple equalities or inequalities, which do not call for any geometrical imagination – the simple exhaustive lists of steps 1 and 11.

begin with, it if is greater,[47] the problem may not be constructed, as has been proved in the analysis; (3) and if it is equal, the point E produces the problem. (4) For, the solids being equal, (5) the bases are reciprocal to the heights, (6) and it is: as the <line> EA to the <line> AΓ, so the <area> Δ to the <square> on BE". For Archimedes, we see, the study of cases is simply a way of getting the main solution off the ground. In one case, the problem is insoluble, so this can be put aside, no further comment being made;[48] in another case, the solution is effected in a simple, direct way; so, having said that, the proof can unfold, without any further mention of cases being made.

Khayyām's solution is of course totally different. In mere quantitative terms, Khayyām's preliminary study of cases has 10 steps out a total of 35 Steps of the proof (29%), as against Archimedes' 6 out of 40 (15%). Indeed, the qualitative gap is wider, since steps 3-6 in Archimedes' proof are not primarily a study of cases, but simply part of the solution: the division into cases serves not as an end, in this case, but as means for the solution. Thus we are left with step 1 alone, which is a mere claim, not an argument, so that, in short, *Archimedes offers no argument whose end is the study of cases.* Khayyām, on the other hand, not only dedicates ten Steps for this preliminary investigation: he goes on showing the same approach in the solution itself. We immediately notice that he offers not one, but three separate diagrams, corresponding to three possible geometrical configuration. And once again, these are not mere tools for obtaining the solution. Having made the necessary constructions and preliminary statements, Khayyām reveals the main interest of this study by division: "(18) So if the two sections meet, by a tangency at another point or by an intersection, then the perpendicular drawn from this <point of meeting> will have to fall between the points A, B; (19) and the problem is possible; (20) otherwise it is impossible". In other words, the configurations are simply another way of yielding cases of possibility and impossibility, so that the goal of this discussion

(47 Any comparison of Khayyām's language with that of Greek geometry opens up an important question: which form of Greek geometry was Khayyām acquainted with? What did he think was Greek geometry "written like"? It may well be that the translations from Greek mathematics with which Khayyām was acquianted already contained some original linguistic features. Once again, I ignore this historical question, and concentrate on the direct comparison between the two languages – Archimedes' and Khayyām's.

(48 I.e. (<area> Δ, on the <line> AΓΔ) > (<square> on BE, on the <line> EA).

nate lines in the section AIL; (25) so its <=IG> square shall be equal to the product of AG by BC". Thus, the difference between "product" and "rectangle" is in a sense no more than that of notation: in terms of admissible operations, Khayyām's terminology carries no consequences. Most tellingly, at the moment where Khayyām's treatment is most reminiscent of *Al-Jabr wa l-Muqābala* – when a quantity is added to two sides of an equation – there is nothing algebraic to his argument. "(29) So the cube of BC – which is equal to the given number – is equal to the solid whose base is the square of GC, and its height GA. (q) And we add the cube of GC as common; (30) so the cube of GC with the given number is equal to the solid whose base is the square of GC, and its height AC". The operation through which we obtain the equality

(solid whose base is the square of GC, and its height GA)+(cube of GC)=
(solid whose base is the square of GC, and its height AC).

Has nothing algebraic about it, and is instead classical Greek cut-and-paste derivation, strongly based on unpacking information from the diagram. Apart from their strange initial formulation, then, Khayyām's proofs could be read, without perplexity, by any Greek mathematician.

But could they have been *written* by any Greek mathematician? While Khayyām uses the idiom of Greek mathematics, he also uses it in his own way, meaningfully different from, say, Archimedes'. At a mathematical, technical level, Khayyām's proof is clearly distinct from that of Archimedes. Let us try to analyze this sense of difference.

Once again, to have a sense of the difference, we should also notice the similarities. Both proofs, after all, are based upon an intersection of a parabola and a hyperbola, and both offer a study of cases, connecting it with the conditions of solubility. To some extent, such similarities may have historical explanations. (Khayyām is likely to have been familiar with some version of Archimedes' solution). There are also technical mathematical facts that account for the similarity: the problem is after all the same; cubic equations are indeed equivalent to proportions involving lines and squares, and there are only so many curves that satisfy such proportions. Thus, in a sense, history and mathematics both determine a certain convergence between Archimedes and Khayyām.

Which makes their divergence all the more apparent. This divergence has two aspects: the different roles the study of cases, and the different roles of ratios and equations.

For the study of cases, consider Archimedes' discussion: "(2) To number of cases, names are allowed to switch: "(25) so its <=IG> square shall be equal to the product of AG by BC. (26) So the ratio of BC to IG is equal to the ratio of IG to GA. (27) So the four lines are proportional: the ratio of GC to CB as the ratio of CB to IG, and as the ratio of IG to GA. (28) So the ratio of the square of GC, the first, to the square of BC, the second, as the ratio of BC, the second, to GA, the fourth". In the course of these four steps – the key to the main geometrical property – AG has switched into GA, while BC has switched into CB and back again into BC. Thus, the reference of those two-lettered objects can not be purely symbolic – it is precisely their identity as symbols which such a permutation destroys. The identity of these objects is clearly given by the diagram where, indeed, it makes no difference whether you read them, as it were, from left to right or from right to left.

In short, we see that Khayyām opens the possibility of considering his objects symbolically, as elements manipulated by the rules of calculation; yet essentially conceives of them as components in a geometrical configuration. This is seen at the most elementary level – the use of letters; but, as always, we encounter the same structural forces at all levels of analysis. For, after all, the entire treatise is determined by Khayyām's open-ended list of degrees – on into square-squares, square-cubes, and beyond; and his explicit decision, to limit himself to the four basic degrees alone. Most importantly, the same duality, with a preference to the geometrical is shown in the kinds of mathematical statements and operations allowed. In this problem, we see Khayyām making a few claims whose geometrical significance is not apparent: "(a) We suppose AC as the quantity of the squares; (b) we construct a cube equal to the given number". What is the meaning of a line being "supposed as a quantity"? Or of a cube "being equal to a number"? Thus, an equivalence between geometrical and more abstract objects is being suggested. However, those kinds of non-geometrical claims are limited to the stage of setting-out, where the general problem is set in geometrical terms. Following this setting-out, the argument proceeds strictly according to geometrical manipulations. None of the derivations made by Khayyām would have been inadmissible for Archimedes. Truly, Khayyām speaks of "product" where Archimedes speaks of "rectangle". However, Khayyām obtains his products through precisely the same geometrical techniques Archimedes could use for obtaining his rectangles: "(24) And IG is among the ordithe expression often used by Khayyām, "the square of AB", is truly indeterminate: it can refer both to the square (in terms of calculation) of the magnitude AB, or to the square (in the geometrical sense) produced from the line AB. It is indeed interesting to note that when Khayyām wishes to refer in non-ambiguous terms to a geometrical square, he does so by a different mode of naming of squares: "(f) and a square DC shall be completed in the three diagrams". By referring to the square through two opposite vertices, the reference can no longer be to 'square' in the terms of calculation, and must be to 'square' in terms of geometry. On the other hand, in some other expressions, the language of calculation seems dominant, as in, e.g. "(17) ... the product of AB by BC". Archimedes would probably have "the <rectangle contained> by the <lines> AB, BC", but the absence of the formula "the <line> AB" makes it much more natural to refer to the product not as a geometrical, two dimensional object, but as a result of calculating with two symbolic objects, AB, BC.

And yet, while opening up these radically new ways of reading his text, it remains clear that Khayyām himself does not intend his text to be read in this way. There are many indications Khayyām conceives of his lines as geometrical configurations, and not as more generalized magnitudes represented symbolically.

Most simply, he operates upon them, even at the symbolic level, according to their geometric configuration. In keeping with Greek practices, Khayyām allows lines to be represented by the diagram, in whatever is the most natural way. Consider the line H: since it does not form part of the continuous geometrical configuration, it does not intersect with any other line and is thus not distinguished by any of its points. Thus, it becomes natural to refer to it as a single unit (and not, as is done for other lines, through the points at its two limits). The result is that most, but not all lines in Khayyām's solution are two-lettered. This heterogeneous way of naming lines makes it somewhat less natural to see the expressions "H", "AC" as mere symbols. As mere symbols, they are homogenous; their heterogeneity is a function of the geometrical configuration.

The same grounding of the symbol in the diagram is seen in another phenomenon of Khayyām's treatise: the permutability of names. Again, as is also true of Greek mathematical practices, once a name is attached to an object it is generally kept the way it is. However, in a significant points on the line. The Greek words *eutheia gramme*, "straight line", are dropped. (In my translation, I insert back the word "line", alone, inside pointed brackets). However, these words are understood: the expression is merely a way of referring to specific lines in a specific figure. While my translation is no doubt irritating in its plethora of pointed brackets, those pointed brackets do serve a function in reminding us how much the Greek reader fills in, and how much it is felt that the text refers throughout to geometrical objects.

Khayyām's text is different, and this particular formulaic form is dropped altogether. This is indeed natural in a translated context: the easiest way to render the Greek *he AB* in another language is simply by AB, if only because the expression *he AB* contains nothing to translate besides definite article and Greek letters. Further, Arabic does not possess a declension of the definite article: but without the feature 'feminine' spelled out on the definite article, it loses even the minimal meaning it had in the Greek. Finally, even phonologically, the expression "the AB" is problematic in Arabic, in which the definite article joins with the noun it governs to form a single word. The hypothetical expression *al-AB would be particularly strange, as the definite article would have to combine with a peculiar, extra-linguistic object – the letters of the diagram. Such linguistic speculations aside, it is clear that Khayyām's text differs from Archimedes in its avoidance of this particular formula – with which go many other, more complex formulae. To put it simply, my translation of Khayyām contains far fewer pointed brackets than my translation of Archimedes.

This could have a consequence for the way in which geometric objects seem to be understood in Khayyām's text. Of course Khayyām occasionally does refer explicitly to lines as "lines" – as indeed Greek mathematicians also do. But Khayyām would drop this explicit reference in the contexts where a Greek mathematician would use only the abbreviated form *he AB*. Thus, the text would seem to speak not about lines as such but rather about objects represented by diagrammatic letters. In an expression such as, e.g., "(9) Then, if H is bigger than AC...", the "bigger" relationship holds, as far as the text is concerned, not between lines as such, but between such objects as are designated by diagrammatic letters. Since Khayyām does belong to a world where such letters can be used in calculation, and not only in geometry, his expressions now allow for a systematic ambiguity. Thus, for instance,

<=AC> is equal to the given quantity (of squares).[44] (32) And this is the goal.[45]

(33) Analogously with the two remaining cases, (34) except that the third has to give rise to two cubes, (35) since each perpendicular cuts from CA the side of the cube, (36) as has been proved.

So it has been proved that this case has different cases, some may include impossibilities, and it has been solved by the properties of two sections, both a parabola and a hyperbola.

4. The Originality of the *Algebra*

4.1. Reading Khayyām in light of Archimedes

We have now seen the two separate treatments, by Archimedes and by Khayyām, and it is time to consider Khayyām's treatment in the light of Archimedes'. One is at times struck by the degree of continuity between the two treatments, at times struck by Khayyām's originality. By delineating the lines of difference and similarity, then, we may obtain a finer understanding of the sense in which Khayyām's work was an 'Algebra'.[46]

I start with a detail of Khayyām's exposition that is very typical of this duality – opening up non-geometrical possibilities, while manifesting a sustained geometrical conception of the problem. I refer to Khayyām's system for naming lines.

To begin with, notice how my translation of Archimedes is peppered by the phrase "the <line> AB". This is a very minimal Greek expression, in transliteration:

he AB

The Greek definite article, in its singular feminine form, followed by two Greek letters (or, less frequently, three letters), for letters standing at

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height GC. Add it to the solid whose base is the square of GC, and its height GA, and you have a new solid, whose base is the square of GC, and its height (GC+GA). GC+GA is the same as AC, hence "the solid whose base is the square of GC, and its height AC", mentioned by Khayyām.

(44 AC was set down as the quantity of squares, in the very first Step a.

(45 We have produced a line – GC – whose property is that: its cube, together with a given number, equals a given number of its squares.

(46 As always, my comments about Khayyām sometimes apply to other Arabic mathematicians as well, and sometimes apply to Khayyām alone, and I do not try to distinguish between the two.

at another point or by an intersection, then the perpendicular drawn from this <point of meeting> will have to fall between the points A, B; (19) and the problem is possible; (20) otherwise it is impossible.

This tangency or intersection was not grasped by Abu al-Jud, the eminent geometer, so that he reached the conclusion that if BC is bigger than AB, the problem would be impossible; and he was wrong in this claim.

And this kind is the one that baffled Māhānī (among the six kinds).[39] So that you shall know.

(21) And in the third diagram, the point D shall be interior to the parabola,[40] so the sections cut each other at two points.[41]

(n) And, in all, we draw, from the point of meeting, a perpendicular on AB, (o) and let it be, in the second diagram, IG; (p) similarly, <we draw> from it <=D> another perpendicular, on CE, namely IK. (22) So the rectangle IC is equal to the rectangle DC (*Conics II* 2000: 3), (23) so the ratio of GC to BC shall be as the ratio of BC to IG (*Les Éléments VI*: 16). (24) And IG is among the ordinate lines in the section AIL;[42] (25) so its <=IG> square shall be equal to the product of AG by BC. (26) So the ratio of BC to IG is equal to the ratio of IG (*Les Éléments VI*: 16). (27) So the four lines are proportional: the ratio of GC to CB as the ratio of CB to IG, and as the ratio of IG to GA. (28) So the ratio of the square of GC, the first, to the square of BC, the second, as the ratio of BC, the second, to GA, the fourth. (29) So the cube of BC – which is equal to the given number – is equal to the solid whose base is the square of GC, and its height GA. (q) And we add the cube of GC as common: (30) so the cube of GC with the given number is equal to the solid whose base is the square of GC, and its height AC,[43] (31) which

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that the parabola passes: a converse of *Conics I* (p. 13).

(39 It is not altogether clear which "six kinds" are referred to: they could be either the six kinds to which this kind belongs in Khayyām's treatise, or some six kinds Māhānī was baffled by. The reference to Māhānī, at any rate, is a follow-up of the brief mention in the introduction, where it was also mentioned that Māhānī studied the Archimedean Problem; this is as much as Khayyām says explicitly to connect this Problem with Archimedes.

(40 The same reasoning as used in Step 17.

(41 The hyperbola now needs to "escape" from inside the parabola, in 'both' directions.

(42 I.e., it is one of the lines defined in such a way that the square on them is equal to the rectangle contained by: (1) the line they cut from the axis, and (2) the *orthia* (*Conics I* 2000: 13).

(43 The cube of GC is, in fact, the solid whose base is the square of GC, and its

(b) So we cut BC, equal to H, from AC. (11) So the line BC shall be either equal to AB, or bigger than it, or smaller. (c) So let it be, in the first diagram, equal to it; (d) and in the second, bigger than it; (e) and in the third, smaller than it. (f) And a square DC shall be completed in the three diagrams, (g) and we produce, at the point D, a hyperbola, asymptotic to AC, CE, (h) which is DG in the first diagram, (i) DI in the second and the third. (j) and we produce a parabola, whose vertex is the point A, and whose axis is AC, and its *orthia* is BC;[34] (k) which <parabola> is AI in the first diagram, (l) and AL in the second, (m) and AK in the third. (12) And the sections shall be known in position.[35] (12) So in the first <diagram>, the parabola passes at the point D, (13) since the square of DB is equal to the product of AB by BC:[36] (14) so D shall be on the perimeter of the parabola; (15) and it <=the parabola> will meet <the hyperbola> at another point – which you can grasp, with the least thought.[37] (16) And in the second, the point D shall be outside the perimeter of the parabola, (17) since the square of DB is bigger than the product of AB by BC.[38] (18) So if the two sections meet, by a tangency

(34 *Orthia* is a formulaic Greek expression, literally meaning something like "the rightish <line>", transformed in Arabic into the equally formulaic expression "the right side", and which I finally transliterate back into the original Greek. It refers to the line, defining a parabola so that – applying modern terms to, e.g., diagram 1 of this proposition – every perpendicular from the parabola on the axis, such as DB, satisfies $DB^2=(Orthia)^*(BA)$ or – as this construction stipulates – $DB^2=(BC)^*(BA)$.

(35 The claim is that a hyperbola is determined by a point through which is passes, together with its two asymptotes (*Conics II* 2000: 4), while a parabola is determined by its vertex, axis and *orthia* (*Conics I* 2000: 52).

(36 The square of DB is the square EDBC and, by the definition of diagram 1, AB=BC and so $AB^*BC=BD^2$; by a converse of *Conics I* (p. 11), the parabola must therefore pass at the point D.

(37 Once again, Khayyām addresses the reader with an "exercise", this time curiously explicit. The truth of the claim is visually compelling, but ancient and medieval readers would probably prefer not to rely on the diagram for exploring the relations of conic sections, as these were drawn (intentionally) falsely, by arcs of circles. I have given the matter a least thought, and then some more thought, and finally I think as follows: if the two sections cut each other at D, the claim is indeed obvious (for the hyperbola will have to 'escape' from inside the parabola, so as to avoid cutting the asymptote). The two sections cannot be tangent at D, since this would imply that, with the tangent produced, it should be cut into equal segments at the touching point (*Conics II* 2000: 3), which iu turn would imply that CB, that is DB, is equal to the segment from B to the cutting-point of the tangent and of the line BA produced; but DB is already equal to BA and an impossibility arises.

(38 And it is at the point on the line DB, where the square is equal to the product,

fig. 5. From R. Rashed & B. Vahabzadeh 1999: 176-178.

(a) We suppose AC as the quantity of the squares; (b) we construct a cube equal to the given number, and let its side be H.[28] (1) And the side H will have to be either equal to the line AC, or greater, than it, or smaller. (2) So, if it is equal to it, the problem is impossible, (3) since the side of the required cube will have to be equal to H, or smaller, or greater. (4) So, if it is its equal, the product of AC by its <=the required cube's side> square is equal to the cube of H; (5) and the number shall be equal to a quantity of squares, and there will be no need to add the cube.[29] (6) And if the required side is smaller than it <= than H>, the product of AC by its <=the required cube's side> square is smaller than the given number, (7) so the quantity of squares will be smaller than the given number, even without the addition.[30] (8) And if the side is greater than H, its cube is greater than the product of AC by its <=the required cube's side> square, even without the addition, to it, of the number.[31] [32] (9) Then, if H is bigger than AC, the impossibility in the three cases shall be even greater.[33] (10) So it shall be necessary that H will be smaller than AC, and otherwise the problem is impossible.

(28 This point is rather confusing: the problem sets out a cube that, together with a number, equals a (multiple of a) square – the cube and the square being related in that they share the same side. Now, Khayyām immediately moves on to construct a further, auxiliary cube – not to be confused with the one set out by the problem itself – which is equal to the given *number*. Its side is H so that one may say that the given number equals (in modern symbolism) H^3.

(29 I.e., the original equality is "cube with number equals quantity of squares", but we have "number equals number of squares", i.e., in effect, no cube – so obviously the problem is impossible (in our terms, it may be said that Khayyām does not consider zero to be a solution to the problem).

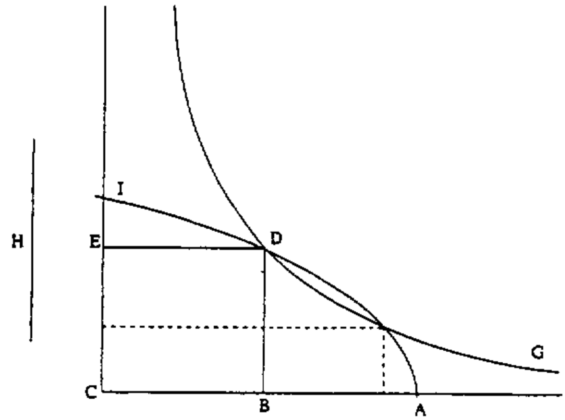
(30 I.e., the original equality is "the cube with numbers equals quantity of squares", but we already have "number greater than quantity of squares", and adding in a cube to the number will not make it any smaller! (In our terms, it may be said that Khayyām does not consider negative numbers as solutions to the problem).

(31 I.e., the original equality is "the cube with number equals quantity of squares", but we already have "the cube is greater than quantity of squares", and the addition of a number can only make this worse. (In our terms, it may be said that Khayyām does not consider negative numbers as possible parameters).

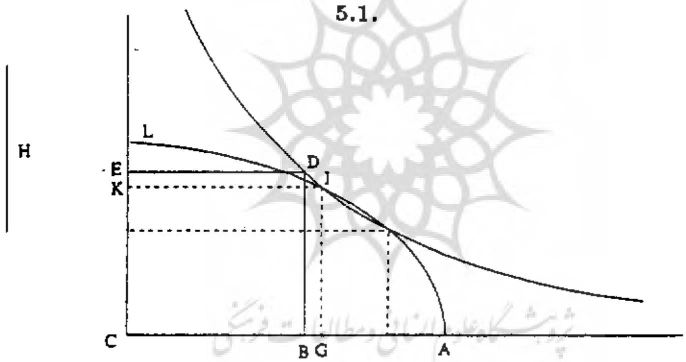
(32 Steps 3-8 are all governed by step 2, and together show the impossibility of the case $H=AC$.

(33 Khayyām intends that we verify by going through the previous three cases, which the reader may now do. This Step 9 shows the impossibility of the case $H>AC$ so, together with Steps 2-8, the ensemble of Steps 2-9 shows that the only case which may at all be possible is $H<AC$, as asserted in the following step.

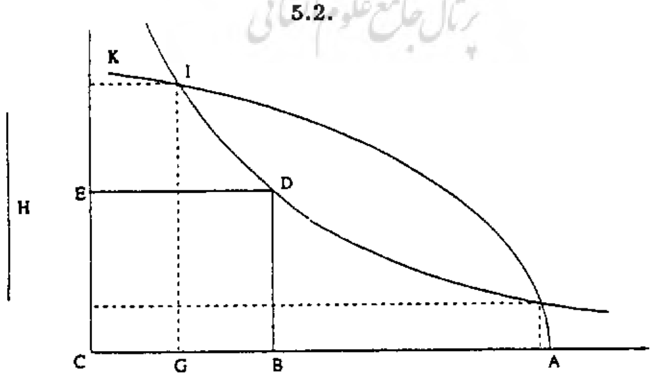
termed kinds': A cube and a number equal a square (fig. 5).[27]



5.1.



5.2.



5.3.

[27 Sc. *a certain quantity* of squares.

والمفرداتُ ستة أصناف:
آ: عدد يعدل جذراً   بَ: عدد يعدل مالاً   جَ: عدد يعدل كعباً
دَ: جذور تعدل مالاً   هَ: أموال تعدل كعباً   وَ: جذور تعدل كعباً

وأما المقترنات فمنها ثلاثية ومنها رباعية:
أما الثلاثية فاثنا عشر صنفاً، فالثلاثة الأولى منها:
آ: مال وجذر يعدل عدداً   بَ: مال وعدد يعدل جذراً
جَ: جذر وعدد يعدل مالاً
والثلاثةُ الثانيةُ منها:
آ: كعب ومال يعدل جذراً   بَ: كعب وجذر يعدل مالاً
جَ: كعب يعدل جذراً ومالاً،

. والستةُ الباقية من الأصناف الاثني
عشر:
آ - كعب وجذر يعدل عدداً   بَ - كعب وعدد يعدل جذراً
جَ - عدد وجذر يعدل كعباً   دَ - كعب ومال يعدل عدداً
هَ - كعب وعدد يعدل مالاً   وَ - عدد ومال يعدل كعباً

...، وهو أربعة أصناف:
آ - كعب ومال وجذر يعدل عدداً   بَ - كعب ومال وعدد يعدل جذراً
جَ - كعب وجذر وعدد يعدل مالاً   دَ - كعب يعدل جذراً ومالاً وعدداً

وهو ثلاثة
أصناف.
آ - كعب ومال يعدل جذراً وعدداً   بَ - كعب وجذر يعدل مالاً وعدداً
جَ - كعب وعدد يعدل جذراً ومالاً

fig. 4. From R. Rashed & B. Vahabzadeh 1999: 125-129.

The translation has no claims for style or precision. It is brought here so that we can discuss the text and, to make the comparison with Archimedes easier to follow, I adopt the same conventions adopted in my translation of Archimedes. The fifth kind of the 'six remaining three-overall argument, surveying the domain of algebra.

To sum up, then, we saw three structural features of Khayyām's *Algebra.* The first was an inter-penetration of the introduction, and the treatise proper: the treatise was a direct continuation of the introduction, since the treatise was simultaneously, *in* algebra, and *about* algebra. The second was the strongly articulate, systematic nature of the treatise: it constantly arranged itself in various divisions and lists. Finally, we saw how the two features are connected through the principle of exhaustive lists. The interests of the treatise is in arranging claims – and objects – into systematic orders, so those separate claims become, simultaneously, components in a large-scale claim about the entire domain of algebra.

Having made those general observations on the treatise, it is time to see its part devoted to the Archimedean problem.

3.2. The Archimedean Problem Solved by Khayyām

I offer here a translation, based on R. Rashed's and B. Vahabzadeh's important new edition and translation of Khayyām's work, of a problem in Khayyām's algebra that has a certain affinity with the Archimedean problem. It is, as its title mentions, the fifth problem in a group of six problems of three terms; modern editors sometimes number the problems in this treatise, and then it becomes "problem 17". There are altogether 25 problems, so this problem occupies an advanced position in the book (fig. 4).

In an interesting complication, this example has a two-tiered exhaustive classification (within a certain possibility, further sub-possibilities are surveyed). Exhaustive lists, that is, can become complex, many-dimensional systems.

It should be noted that this interest in argument through exhaustive lists is remarkable, given the subject matter taken by Khayyām. In the terms of Greek mathematics, Khayyām deals almost exclusively with problems: that is, he defines situations, and sets himself the task of finding lines satisfying the definitions. Now, argument through exhaustive lists is often used in Greek mathematics – but mainly in two contexts. One is that of *Reductio* arguments, which work through the exhaustive principle that P or not-P, showing that P is impossible and thus deriving not-P. Another – essentially a development of Reductio arguments – is what is called (for other reasons) 'the Method of exhaustion'. There, it is argued that a certain object is either greater, smaller or equal to another one; the "greater" and "smaller" options are ruled out and the "equal" is thus proved. Both Reductio arguments, and the Method of exhaustion, are useful, for obvius reasons, not for *problems*, that achieve a task, but for *theorems*, that state a truth. Finally, a very special work within the corpus of ancient Greek mathematics (but one in which Arab commentators had a special interest) does work through the principle of classification: this is Elements Book *X*.[26] This book classifies the kinds of relations of incommensurability. Once again, however, classification is used in the context of theorems. (Furthermore, the classificatory object of the work remains mostly implicit). It is a peculiarity of Khayyām's argumentative style, then, to rely so heavily on exhaustive lists in a treatise dedicated to *problems*.

But then again, exhaustive lists is what this treatise is about: Khayyām's main claim is not that he proved this or that result, solved this or that problem, but that he had encompassed an entire domain. The goal of the treatise is totality: thus to claim that an object has a certain position in the system is not some tool used for listing objects, a mere signpost. The signaling of positions in a system is a tool used in the exhaustive survey of the *entire* system. Each separate part of the treatise – each case within a problem, each kind of equality, each group of kinds – participates simultaneously at two levels. At one levels, it makes a specific claim, separate to it; at another level, it functions in an

(26 For its reliance on the principle of classification, see B. Vitrac (1998): pp. 51-63.

So the work is characterized throughout by an impulse to divide, to articulate, to put into systematic structure. To complete our observations, a final feature of the treatise must be added: the impulse is, often, not merely to articulate domains, but fully to exhaust such domains. Once again, it is instructive to take a non-mathematical example, namely the historical excursus. In surveying the domain of previous works in his field, Khayyām proceeds by an exhaustive division into "ancients" and "moderns", and then reasons as follows for the ancients:

> We have no treatises from them concerning it: perhaps, after having studied and looking for it, they failed to grasp it; or their theories did not lead them to study it; or their treatises were not translated into our language. (R. Rashed 1999: 117)

What we see here is Khayyām's urge to obtain truth by encompassing a domain of possibilities. This immediately becomes a defining feature of the work. The subject-matter, algebra, is exhaustively defined, in many aspects. The kinds of quantities, as we have noted, are enumerated, in an exhaustive list which – purely for exhaustion's sake – includes time in addition to the other mathematical quantities. (It is in this context that reference is made to the *Categories*, a work that Khayyām must have understood as an exercise in exhaustive systematization). Then the various degrees are spelled out, from the root upwards (and, much later in the work, form the root *downward*, dealing with 'parts'). Then Khayyām stops short the infinite expansion of degrees (to square-square, square-cube, cube-cube and beyond) by insisting on the geometrical meaningfulness of quantities: "since there is no other dimension [beyond cubes], the square-square and what comes beyond it are not among the magnitudes". Thus, the exhaustive list of kinds of magnitudes helps to delimit an exhaustive set of kinds of degrees (number, root, square, cube), and this immediately leads on to the heart of the treatise, which is the exhaustive list of kinds of equations defined by those four degrees.

Thus at the most global level the treatise operates through exhaustive listing. But the same principle is operative in many individual proofs. This is the essence of Khayyām's interest in "cases" in proofs, which derive from some exhaustive list of a set of possibilities: "And these two <conic> sections will either meet or not meet"(R. Rashed 1999: 167). Having made such an assertion, Khayyām then moves on to study each of the possibilities. Many proofs of the treatise are structured by such exhaustive lists, and we shall see an example in the following subsection.

of the treatise – how its different parts relate to each other – is always an interest of Khayyām. Thus different problems are related, in what may be considered, anachronistically, a "reduction": one problem is shown to be equivalent to another. Thus Khayyām states explicitly that a certain species of problems is all equivalent to another, and then proves this equivalence, each time using particular examples, sometimes to substitute the general argument, sometimes to corroborate it (R. Rashed 1999: 147-153).

The word "example" is one kind of local signpost used to articulate the work; other words are used as well, such as "by numbers" and "by geometry" which we have seen already, as well as, simply, "proof": that is, here and there, following a general statement, Khayyām would introduce his mathematical argument by the single word "proof". (By my counting, this minimal title occurs 10 times in the treatise, though I may have missed some occurrences). This is correlated with several expressions similar in meaning to QED: "and that's the goal", "and that's what we wanted to prove", etc., I count 24 occurrences of this expression.

By far the most important signpost is, of course, the word "kind", followed by an ordinal, and often introduced by a connector. So, for instance, a problem is introduced by the words "And the second kind of this" (R. Rashed 1999: 141).[24] This constant repetition of the word "kind" is the main structural feature of the work, and may well have been so even at the visual level. While of course no autograph survives of the work, at least one manuscript (BN Arabe 2458) systematically sets out the expressions containing the word "kind" in bigger characters: this has a marked visual impact.[25] (Note that this kind of visual articulation is common in many Arabic scientific manuscripts, though sometimes using colour instead of size). Finally, in some parts of the work, a similar effect of articulation is obtained by the figures, which are (as is the standard elsewhere) positioned near or at the end of their respective problems, thus enhancing visually the verbal articulation of the work.

(24 Notice how the system works as a whole: the word "kind" is the local signpost, signaling the start of a new problem; the word "and" positions the kind inside a sequential system; the word "second" provides the place in the sequential system; the word "of this" hints at the system being referred to. The whole expression, finally, is an unpacking of an entry from the original set of tables.

(25 This manuscript apparently is, according to R. Rashed (1999): pp. 109-113, the copy closest to the autograph, though of course this does not guarantee that this particular feature is authorial.

The impulse to divide and to list goes, however, well beyond those basic grids. The work is articulated, throughout, by comparisons and parallel parts. In some simple cases, having offered one proof, Khayyām often moves on to offer another one, alternative to it. Once again, this is often explicitly marked accorrding to a preconceived grid. The sixth kind, for instance, is defined, and then immediately we have the words “proof by numbers”, followed by a very brief proof; and then “by geometry”, and another brief proof follows (R. Rashed 1999: 135). In some other, more complex cases, the nature of the problem makes it natural to distinguish, not kinds of proofs, but kinds of situations arising within a single proof. In the next subsection, for instance, we shall see a case where Khayyām distinguishes three possible configurations that may arise from a single geometrical situation. Typically, the distinction is made explicit, and is even marked out in the layout of the work, as the three figures are labeled “first”, “second” and “third”.

Thus different proofs, and different cases within proofs, are put side by side. Further, Khayyām puts side by side proofs, and examples. Consider again fig. 2, with the table setting out ‘degrees’ and their corresponding ‘parts’. We may now notice that, besides the named degrees and parts, the table also lists numbers: those numbers are examples of such degrees and parts (taking 2 as the basic root). Such articulations of general proof, and of particular numerical examples, are often repeated through the work. In some cases, Khayyām uses a particular example instead of a general proof. For example, instead of solving generally the case of “a square equal a number of roots”, Khayyām simply offers a special case, “a square equals five times its root”, allowing the general solution (the root is equal to the number of roots mentioned in the problem) to be apparent from the particular case (R. Rashed 1999: 133). In some other problems, general statement exists alongside a particular example, as in the immediately following problem, “<a number of> things equal a cube”. Khayyām explains explicitly that this general problem is essentially like the problem “a number equals a square”. This is explained as follows: “example: four roots are equal to a cube; it is like has been said: four, a number, is equal to a square”. (R. Rashed 1999: 135).

Notice how, in the text quoted above, the word “example” is used explicitly – a sort of local signpost. The articulation of the work is never implicit. Indeed, as the same example also shows, the structural features

جزء الكعب		جزء المال		جزء الجذر
١		١		١
٨		۴		٢
الواحد	الجذر		المال	الكعب
١	٢		۴	٨

fig. 3. From R. Rashed & B. Vahabzadeh 1999: 219.

There are many further divisions and lists made throughout the book, in the course of the mathematical argument itself. Several distinctions occupy Khayyām explicitly. Most important is the distinction between problems that do not require conic sections from those that do. This, indeed, is the main division of the book (R. Rashed 1999: 153): following a list of problems and solutions which do not require conic sections, Khayyām makes a break in the argument. 'After introducing these kinds could be proved from the properties of the circle, that is from the book of Euclid, let us discuss now the kinds that can not be proved except with the properties of the <conic> sections'. (Note, incidentally, how mathematical and bibliographical distinctions coincide). The break is very noticeable in the overall structure of the book as, for once, Khayyām deviates from the structure set out by the division of equalities, and introduces further auxiliary lemmas on solid figures (R. Rashed 1999: 155-161). Another crucial distinction for Khayyām is that between problems that are always soluble, and those that are not: those distinctions do not divide the book neatly, as the circle/conic sections division does. Thus, Khayyām makes those distinctions case by case: following each kind of problems, he notes whether or not they are always soluble. Thus, e.g., at the end of 'the fifth kind of the remaining six kinds of three terms' Khayyām notes that 'this kind has different cases, some of which may be impossible', while at the end of the next kind he notes that 'this kind has no different cases, and none of its problem is impossible' (R. Rashed 1999: 181-183). In other words, the treatise sets out to impose three separate grids on the universe of algebraic problems: the grid defined by number of terms and their relations (the one set out at the original table); the grid defined by the mathematical/bibliographic distinction of circle from conic sections; and the grid defined by the presence or absence of impossible cases. We see that one of the explicit interests of Khayyām is to investigate the pattern of this triple superposition.

in systems of all kind. Khayyām is constantly interested in articulating domains: dividing them, and organizing them according to some overarching principles. This indeed is the very start of the work, with its species-genus arrangement: (A) wisdom, in it (B) 'mathematical', in it (C) '*Al-Jabr wa l-Muqābala*', in it (D) 'kinds' (of a more difficult nature). This Prophyry's tree is but the first of many lists and divisions made in the treatise. In history, people are either 'ancients' or 'later' (R. Rashed 1999: 117). In the metaphysics of algebra, its objects are 'the line, the plane, the solid, and time' (R. Rashed 1999: 121) – tellingly, Khayyām immediately refers to Aristotle's *Categories* (as well as to a *Categories* – based comment in the *Physics*). Khayyām lists the 'degrees' (R. Rashed 1999: 121): thing, square, cube, square-square, etc.; towards the end of the treatise, he reverts to the same list, now to list it together with its correlate, list of 'parts' ('part of a square' is what we would call 'lover square': if the square is 4, part of the square is $\frac{1}{4}$). The one-dimensional list of degrees thus becomes a two-dimensional grid and, in acknowledgement of that, Khayyām explains that he decided, for clarity's sake, to set out the 'parts', together with the original degrees, in a table (R. Rashed 1999: 219) (fig. 2). Now, similar tables form what may be considered the heart of the treatise. Near the beginning of the work, following other divisions concerning Algebra, Khayyām sets out the various kinds of equalities (R. Rashed 1999: 125-129) (fig. 3). These form, once again, a many-tiered genus-species structure: equalities are either 'simple' (binomials) or 'complex' (polynomial). 'Complex' equalities are either with three, or with four terms (note that Khayyām does not deal with degrees beyond the cube: this results from his deeply geometrical conception of mathematics, to which we shall return in the next section). For several of the species obtained in this manner, Khayyām distinguished further species (e.g. between equations that were treated by earlier mathematicians, and those that were not), so that finally each *infima species* contains no more than a few equalities (six at most). The bulk of the treatise is an unpacking of this preliminary list: a set of solutions of those equalities, always following this genus-species structure. Overarching division is thus, quite simply, what the book is about. Tellingly, even the *names* of the species and genera derive from the list, as they are called, e.g., '<the kind of> the six kinds' etc.

without a previous mastery of this background.[23]

Khayyām's introduction does not stop there, and now he goes on to discuss the nature of Algebraical equations, from metaphysical and mathematical points of view, and this survey leads on, very naturally, to a survey of the types of equations studied in this field. This survey of types of equations, finally, constitutes what may be considered the treatise proper. The language gradually becomes now that of Greek-Arabic geometry and algebra, with figures lettered by the Arabic alphabet, and the language of theorems and proofs. Notice, however, that the early types of equations dealt with are very simple, they do not call for detailed mathematical discussion. Thus, the continuity between "introduction" and "treatise" is further stressed: the text, even in its more mathematical part, starts out as relatively "discursive", ordinary scientific Arabic, and only gradually it becomes more specifically mathematical. Finally, even the later part of the work – which contains many complex mathematical propositions, naturally in the mathematical mode of exposition – more general, discursive remarks are frequently made. For example, Khayyām systematically describes the type of equations dealt with, in more general terms (e.g. whether or not it has cases). Also, when such comments suggest themselves, he notes the relation between his works and earlier works. We shall see discursive remarks of this kind in the text quoted in the following subsection; though it should be noted that this text is one of the least discursive ones in the treatise.

Briefly, then, Khayyām's treatise is characterized by a seamless transition from general, contextual comments, to the mathematical results themselves. Indeed, the context – setting out the results as belonging to a certain system – is not some marginal comment, but is the key to the work, which is all about setting out cases. Thus the introduction, in a real sense, never ends. It is typical that the word "introduction" is supplied by the modern editor (R. Rashed 1999: 117.4): it is not in the original, because the original is not neatly divided between "introduction" and "text". The work, as it were, is not just Algebra, but also "An Introduction to Algebra".

The central role of the introductory material is related, as we see, to another important feature of this treatise, namely its strong interest

(23 The bibliographic coordinates of the work keep being provided later on in the treatise: quite frequently, Khayyām refers explicitly to propositions from the three books mentioned, naming book and proposition as the authority for a certain claim: I count at least 19 such references in the work.

So: (A) wisdom, in it (B) 'mathematical', in it (C) '*Al-Jabr wa l-Muqābala*', in it (D) 'kinds' which are especially difficult: it is this, fourth layer of systematic analysis to which Khayyām's treatise is dedicated. As can be seen, the systematic position immediately gives rise to a mathematical, or even bibliographic position (the kinds 'require preliminary propositions') as well as a historical position (the solution was 'inaccessible to most researchers'). It is to this historical context that Khayyām now proceeds, nothing first the absence of ancient (i.e. Greek) extant works, then the limited success of later (i.e. Arabic) works. This historical notice is of special interest, as Khayyām mentions explicitly the Archimedean Problem: Māhānī tried to solve it without success, Khāzin then solved it. This is about as much as the moderns achieved, according to Khayyām, until his own time. Thus, the historical context leads smoothly to the autobiographical context: Khayyām tells us about his lifelong desire to study this field, the obstacles put on his way – not least by some obnoxious people. Finally, he tells us of his studies with Abu Ṭāhir 'Abd or-Raḥmān ibn 'Alak (for whom he has very warm words) and of his ultimate success in producing this work: the historical route, from Archimedes, through Abu 'Abdollāh Māhānī, Abu Ja'afar Khāzin and Abu Ṭāhir, ends with Khayyām himself.[22]

With this personal note, it would seem that the introductory material was over; but Khayyām presses on with a more detailed mathematical-philosophical positioning of the field. *Al-Jabr wa l-Muqābala* is defined; the quantities it deals with are enumerated and analyzed, from both metaphysical and mathematical points of view (typically, two previous authors are mentioned in this context: Aristotle and Euclid). Khayyām then specifies further the scope of the specific field he deals with: as we have been led to expect from the very start, this is done by reference to the preliminary propositions required, i.e. here arrives the bibliographic context. This is a set of three works: Euclid's *Elements* and *Data*, as well as Apollonius' *Conics*. Readers are warned not to attempt the treatise

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Rashed's, his is the more precise rendering of the Arabic.

(22 As will be noted below, introductory material keeps being provided later in the work, including historical context: this is done in particnlar in an excursus added at the end of the work, R. Rashed (1999): 227 ff. Further information, particularly on Ibn Haitham, is mentioned towards the end of the work proper, R. Rashed (1999): pp. 223-225; while many other references to "previous", unnamed mathematicians are made throughout the work, e.g. R. Rashed (1999): p. 197. Finally, in an interesting complication, Khayyām refers to a treatise by himself, in R. Rashed (1999): p. 129.

according to a given ratio. The problem is then transformed, and then solved, always following the principle of transforming geometrical ratios, until simple ratios between lines are obtained. At the moment where the ratios, while linear, become too complex to handle, Archimedes moves into a higher plane of generality, ignoring some specific properties of the problem at hand: but the purpose of this transition into generality is merely to arrive at ratios that are more simply defined. To sum up: both the motivation for the problem and the tools used for its solution, are geometrical. Let us now compare this with Khayyām.

3. Khayyām's Problem

3.1. Some Structural Observations on Khayyām's *Algebra*

In what follows I offer a number of observations on Khayyām's treatise. I explain immediately: those observations are not intended to exhaust the structure of the treatise, but merely to point to certain features of it I find relevant for the following discussion. Further, I do not claim those features are specific to Khayyām himself. Some are typical of Arabic and Medieval mathematics in general, some are more specific to Khayyām, and I do not try to distinguish between the two.

A basic feature of the treatise is the central role played in it by introductory statements. Reflections upon the treatise, and the treatise itself, form a continuous whole. Khayyām's *Algebra* is marked by a strong, explicit setting in a historical, bibliographic, philosophical, indeed even an autobiographical context. This setting is not a marginal, "colouring" addition to the work, but a fundamental constituent, and indeed "setting" and "work" are hard to tell apart.

Let us look at the introduction, then.[20] Khayyām begins his treatise by putting his subject-matter in its philosophical, 'systematic' position.

> One of the scientific principles required in that part of wisdom known as 'mathematical' is the art of *Al-Jabr wa l-Muqābala* ... and in it, there are kinds in which one requires kinds of preliminary propositions which are very hard, and whose solution inaccessible to most researchers.[21]

(20 For the following analysis of the introduction, cp. R. Rashed (1999): pp. 117-125.

(21 All the translations offered here are based on a combination of H.J.J. Winter and W. Arafat (1950), with R. Rashed (1999). I sometimes deviate from both, mainly to accommodate the text to my terminology used in the translations from the Greek. Obviously, readers should assume that, whenever my translation conflicts with R.

on his formulation of the problem, then, it could have produced the following three-dimensional equality:

(parallelepiped cont. by sq. ΔX, line XZ) equals
(parallelepiped cont. by sq. BΔ, line ZΘ).

Now we can see that the bottom side is known – both square and line. Thus we are asked simply to cut a line so that the square on one segment, together with the other segment, produce a parallelepiped of a given volume. The seemingly intractable ratios of spheres, their segments and their cones, have been reduced to a truly elegant task.

Let us now translate the problem even further, now into modern terms, so as to have some neutral point from which, finally, to compare Archimedes with Khayyām. So, the problem is that of cutting a line (call it a) so that the square on one of its segments (call this x^2) "multiplied" by the other segment $(a - x)$ equals a certain given magnitude (call it b):

$$x^2(a - x) = b \quad \text{or} \quad x^2 a - x^3 = b \quad \text{or} \quad x^3 + b = x^2 a.$$

This final re-formulation of the problem, as we shall see below, is highly reminiscent of a problem that arises, and is solved, in Khayyām's *Algebra*. We shall immediately move on to make some observations on that treatise: but not before noting how radically different this algebraic formulation has become from Archimedes' statement of the problem.

The tools used by Archimedes in his synthesis are primarily parallel lines – with their implied similarities and equalities, based on elementary properties of plane geometry. Those similarities allow, essentially, transformations upon ratios. The same is true for conic sections. This may be less obvious in this synthetic presentation, but – concentrating on the proof for the limits of solubility to the problem – I have argued in a precious article (R. Nets 1999) that, for Archimedes, a conic section essentially defines a ratio involving both areas and lines. It thus may sometimes allow the transformation of ratios between squares into ratios between lines. So we see a need, to manipulate and simplify geometrical ratios, a need which determines the use of parallel lines and of conic sections. Finally, the same impulse is shown in the very approach to the problem of cutting the sphere: the sphere is transformed into cones, so that their heights will form the lines along which ratios are measured.

So the Archimedean Problem arises directly from a well-defined geometrical task, of an immediate, "tangible" interest – to cut the sphere pass through O.[16] (m) Let it pass, and let it be as the <line> ΓΟΣ. (31) Now, since it is: as the <line> OA to the <line> AΓ, so the <line> OB to the <line> BΣ (*Les Éléments I* 1998: 29, 32 & *VI*: 4), (32) that is the <line> ΓZ to the <line> ZΣ (*Les Éléments VI* 1998: 2), (33) and as the <line> ΓZ to the <line> ZΣ (taking ZN as a common height) the <rectangle contained> by ΓZN to the <rectangle contained> by ΣZN (*Les Éléments VI* 1998: 1), (34) therefore as the <line> OA to the <line> AΓ, too, so the <rectangle contained> by ΓZN to the <rectangle contained> by ΣZN. (35) And the <rectangle contained> by ΓZN is equal to the area Δ,[17] (36) while the <rectangle contained> by ΣZN is equal to the <square> on ΣΞ, (37) that is to the <square> on BO,[18] (38) through the parabola.[19] (39) Therefore as the <line> OA to the <line> AΓ, so the area Δ to the <square> on BO. (40) Therefore the point O has been taken, producing the problem.

2.3. A wider Perspective on Archimedes' Problem

It can be seen that in his solution Archimedes sometimes uses a language different from that of ratios alone: this could have been used, in principle, to simplify the problem further. The following then is no longer Archimedes' own formulation of the problem, but still does represent his mathematical tools. This simplification would be important when comparing Archimedes to Khayyām.

Recall the ratio obtained by Archimedes – the starting-point for the problem:

$$(\text{sq. on} B\Delta) : (\text{sq. on} \quad \Delta X) :: XZ : Z\Theta$$

Now, there being four lines in proportion, A:B::C:D, we deduce an equality between two rectangles:

(rectangle contained by A, D) equals (rectangle contained by B, C). While the extension of this result to parallelepipeds has a less compelling intuitive character – and is not proved in the *Elements.* We just saw Archimedes taking it for granted in some moves of his solution. If applied

(16 Step 30 is better put as: "The diagonal of the parallelogram ΠΣΖΓ passes through O", which can then be proved as a converse of *Éléments I* (p. 43).

(17 Step h. The original Greek is literally: "To the <rectangle contained> by ΓZN is equal the area Δ" (with the same syntactic structure, inverted by my translation, in the next Step).

(18 18. Steps a, e, k, l, *Les Éléments I* (p. 34).

(19 A reference to *Conics I* (p. 11) – the "symptom" of the parabola.

to the <square> on HM; (16) and alternately, as the <square> on ΓZ to the <rectangle contained> by ΓZN, so the <rectangle contained> by ΓZH to the <square> on HM (*Les Éléments V* 1998: 16). (17) But as the <square> on ΓZ to the <rectangle contained> by ΓZN, the <line> ΓZ to the <line> ZN (*Les Éléments VI* 1998: 1), (18) and as the <line> ΓZ to the <line> ZN, (taking ZH as a common height) so is the <rectangle contained> by ΓZH to the <rectangle contained> by NZH (*Les Éléments VI* 1998: 1); (19) therefore also, as the <rectangle contained> by ΓZH to the <rectangle contained> by NZH, so the <rectangle contained> by ΓZH to the <square> on HM; (20) therefore the <square> on HM is equal to the <rectangle contained> by HZN (*Les Éléments V* 1998: 7). (21) Therefore if we draw, through Z, a parabola around the axis ZH, so that the lines drawn down <to the axis> are, in square, the <rectangle applied> along the <line> ZN – it will pass through M.[9] (i) Let it be drawn, and let it be the <parabola> MΞZ. (22) And since the <area> ΘΛ is equal to the <area> AZ,[10] (23) that is the <rectangle contained> by ΘΚΛ to the <rectangle contained> by ABZ,[11] (24) if we draw, through B, an hyperbola around the asymptotes ΘΓ, ΓZ, it will pass through K[12] (through the converse of the 8th theorem of <the second book of> Apollonius' *Conic Elements*).[13] (j) Let it be drawn, and let it be as the <hyperbola> BK, cutting the parabola at Ξ, (k) and let a perpendicular be drawn from Ξ on AB, <namely> ΞΟΠ, (l) and let the <line> ΡΞΣ be drawn through Ξ parallel to the <line> AB. (25) Now, since BΞK is an hyperbola (26) and ΘΓ, ΓZ are asymptotes,[14] (27) and the <lines> ΡΞΠ[15] are drawn parallel to the <lines> ABZ, (28) the <rectangle contained> by ΡΞΠ is equal to the <rectangle contained> by ABZ (*Conics II* 2000: 12); (29) so that the <area> ΡΟ, too, <is equal> to the <area> ΟZ. (30) Therefore if a line is joined from Γ to Σ, it will

(9 The converse of *Conics I* (p. 11).

(10 Based on *Éléments I* (p. 43).

(11 As a result of Step a (the angle at A right), all the parallelograms are in fact rectangles.

(12 Converse of *Conics II* (p. 12).

(13 This note was not put in by Archimedes, but by the later commentator Eutocius; interestingly – and typically – Eutocius' reference assumes a text of the *Conics* different from ours. For Eutocius' practices, particularly in regard to the *Conics*, see M. Decorps – Foulquier (2000).

(14 Steps 25-26: based on Step j.

(15 An interesting way of saying "the <lines> ΡΞ, ΞΠ".

(a) Let the <line> AE be taken, a third part of the <line> AB; (1) therefore the <area> Δ, on the <line> AΓ[3] is either greater than the <square> on BE, on the <line> EA, or equal, or smaller.

(2) To begin with, it if is greater, the problem may not be constructed, as has been proved in the analysis;[4] (3) and if it is equal, the point E produces the problem. (4) For, the solids being equal, (5) the bases are reciprocal to the heights (*Les Éléments XI* 1998: 34), (6) and it is: as the <line> EA to the <line> AΓ, so the <area> Δ to the <square> on BE.

(7) And if the <area> Δ, on the <line> AΓ is smaller than the <square> on BE, on the <line> EA, it shall be constructed like this: (a) Let the <line> AΓ be set out in right <angles> to the <line> AB, (b) and let the <line> ΓZ be drawn through Γ parallel to the <line> AB, (c) and let the <line> BZ be drawn through B parallel to the <line> AΓ, (d) and let it meet the <line> ΓE (<itself> being produced) at H, (e) and let the parallelogram ZΘ be completed, (f) and let the <line> KEΛ be drawn through E parallel to the <line> ZH. (8) Now, since the <area> Δ, on the <line> AΓ is smaller than the <square> on BE, on the <line> EA, (9) it is: as the <line> EA to the <line> AΓ, so the <area> Δ to some <area> smaller than the <square> on BE,[5] (10) that is, <smaller> than the <square> on HK.[6] (g) So let it be: as the <line> EA to the <line> AΓ, so the <area> Δ to the <square> on HM, (h) and let the <rectangle contained> by ΓZN be equal to the <area> Δ.[7] (11) Now since it is: as the <line> EA to the <line> AΓ, so the <area> Δ, that is the <rectangle contained> by ΓZN, (12) to the <square> on HM, (13) but as the <line> EA to the <line> AΓ, so the <line> ΓZ to the <line> ZH,[8] (14) and as the <line> ΓZ to the <line> ZH, so the <square> on ΓZ to the <rectangle contained> by ΓZH (*Les Éléments VI* 1998: 1), (15) therefore also as the <square> on ΓZ to the <rectangle contained> by ΓZH, so the <rectangle contained> by ΓZN

(3 The expression "area, on line" means "the parallelepiped with the area as base, and the line as height". See discussion of this expression in R. Netz (1999).

(4 The reference is to a later part of the same argument, showing the limits of solubility of the problem – translated and discussed in R. Netz (1999).

(5 The closest foundation in Euclid is *Les Éléments VI* (p. 16), proving that if $a^*b = c^*d$, then $a : d :: c : b$ (for a, b, c and d being lines).

(6 Steps b, e, f, *Les Éléments I* (p. 34).

(7 Steps g and h define the points M, N respectively, by defining areas which depend upon those points.

(8 Steps b, e, f, *Les Éléments I* (p.p. 29, 32) & *VI* (p. 4).

As it were, the point O serves two masters: once, it defines AO, thus serving the ratio AO:AΓ; once again it defines OB, serving the ratio (area Δ):(square on OB). Can one be the servant of two masters? Yes, if the service is identical: the two ratios must be the same. It is as such – as a complex ratio – that Archimedes understands and solves the problem.

The following section 2.2. is a translation of the synthetic part of Archimedes' solution. Following the translation, I offer a few remarks in section 2.3., before moving on to Khayyām.

2.2. The Archimedean Problem Solved by Archimedes[2]

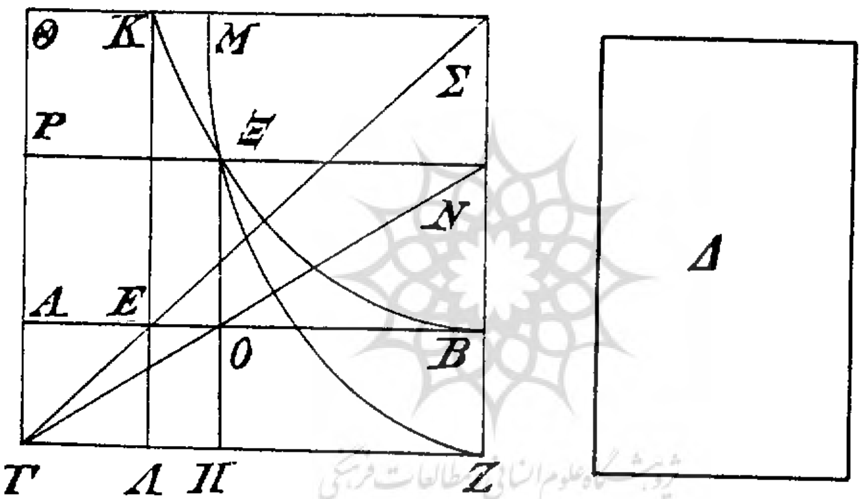

fig. 2. From J.L. Heiberg 1910-1915: vol. III, 139.

And it will be constructed like this: let the given line be AB, and some other given <line> AΓ, and the given area Δ, and let it be required to cut the <line> AB, so that it is: as one segment to the given <line> AB, so the given <area> Δ to the <square> on the remaining segment.

(2 The following is a translation of J.L. Heiberg (1910-1915): pp. 136, 14-140, 20. The argument that this text is indeed by Archimedes is not straightforward. It derives, in fact, from Eutocius' commentary to Archimedes second *Book on Sphere and Cylinder* (6^{th} century AD). Eutocius thought this text was by Archimedes, as it was written in Doric, in archaic terminology: probably he was right. For an exposition of the special textual difficulties surrounding this text, as well as an explanation of the conventions of translation adopted here, see R. Netz (1999) with the exception that (for reasons which will become apparent in section 4., below) I do not abbreviate the Greek expression "the <line> AB" into "AB", as I did in that translation.

be given by the terms of the problem: it is the ratio of the sum of the cones (i.e. the sum of the segments of sphere, i.e. simply the sphere) to the smaller cone, i.e. the smaller segment: so if the problem is to cut the sphere in the ratio 2:1, the ratio PΛ : ΛX is 3:1. BZ, again, is simply the radius, so the point Θ is fully defined by the terms of the problem.

What happens now to the cutting-point itself, X? Our goal now is to manipulate our ratios so that we define the point X with the various lines we have defined by the terms of the problem. Archimedes reaches such a ratio:

$$(\text{sq. on}B\Delta) : (\text{sq. on}\Delta X) :: XZ : Z\Theta.$$

In other words, the terms of the problem define a line ΔZ, and our task is to find a cutting-point on it, X. This cutting-point has a complex defining property.

The cutting-point cuts the line into two smaller lines, ΔX, XZ. Now, we have The Defining Square – the one on BΔ; and The Defining Line – ZΘ; both are fully determined by the terms of the problem. The Defining Property is this: The Defining Square has to the square on one of the smaller lines (ΔX) the same ratio which the other smaller line (XZ) has to The Defining Line.

The problem becomes truly irritating in its details if we continue to think about the specific characteristics of The Defining Area and The Defining Line, in terms of the problem. For instance, The Defining Area happens to be the square on two-thirds the given line ΔZ; while the definition of ZΘ is truly complex. It is much easier, then, simply to leave those details aside and to look at the problem afresh, without the specific characteristics: we can always get them in later when we wish to. So the problem can be re-stated as follows:

Let us assume we are given a line and an area – any line, any square. Let us re-name them, now, as the line AB and the area Δ. Now the problem is, given another line, which we call AΓ, to find a point on AB – say O – that defines two segments of AB, namely AO, OB. Those two segments should now satisfy:

$$AO : A\Gamma :: (\text{area}\Delta) : (\text{square on}OB).$$

This is the Archimedean Problem.

Is the problem as stated now soluble? This is not yet evident.

the sphere share a common base – the plane at which they are divided – and certain solid and curvilinear figures are relatively easy to handle once their base is made equal: these are cones. The ratio of cones of equal base is the same as the ratio of their height – in other words is it a simple linear ratio. Therefore, we shall try to convert the segments of sphere into cones. This is relatively easy to obtain, following results Archimedes had proved in his first *Book on Sphere and Cylinder.* Hence the figure of this Proposition 4 (fig. 1): ΑΒΓ, ΑΔΓ are the two segments of sphere; ΑΡΓ, ΑΛΓ are the cones equal to them, respectively. The question "where to cut the sphere" is the question of the ratio between the diameter (ΒΔ) and one of the cut lines (e.g. ΔΧ). In the simplest case of equality, this ratio is 2:1, but in all other cases it still eludes us; but, with the cones, we have a way forwards.

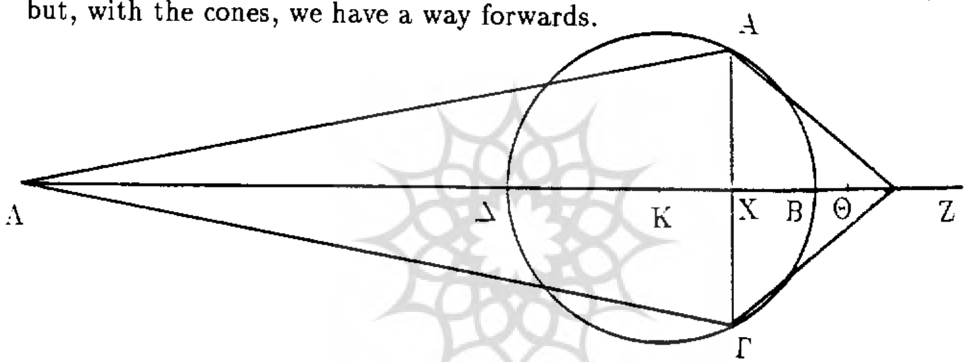

fig. 1. From J.L. Heiberg 1910-1915: vol. I, 188.

Now, to get the cones, a relatively complex ratio defines the lines ΧΡ, ΧΛ in terms of the position of the point X. For instance, the length ΡΧ is defined by (transforming into a modern notation).

$$(\mathrm{K}\Delta + \Delta\mathrm{X}) :: \mathrm{PX} : \mathrm{XB}.$$

Clearly, all the lines except for ΡΧ are given by the point X itself, so that, in general, the cones are well-defined and with them the ratio of the two segments of sphere. Thus a single manipulation by ratios, albeit a complex one, transforms a ratio defined by solid, curvilinear figures, into a ratio defined by lines alone.

Archimedes introduces now two auxiliary lines (that ultimately simplify the ratios). The line ΒΖ is defined in a simple way, ΚΒ=ΒΖ. As for the line ΖΘ, it is defined in a more complex way: ΡΛ : ΛΧ :: ΒΖ : ΖΘ. Notice however that while this ratio is somewhat complex, it is still "manageable", since the ratio ΡΛ : ΛΧ is essentially the ratio we would problem, as I shall show below, is essentially geometrical. In Khayyām's *Algebra*, however, it becomes much more algebraic: in fact, it can now be validly seen as "a cubic equation". The question then is that of the title: how does a geometrical problem become a cubic equation? It is with this narrow question that this article deals.[1]

I sketch here a possible approach to this question. In section 2., I describe the starting-point in the Archimedean Problem, showing how it arises, and offering a translation of the synthetic part of its solution. In section 3., I make some general observations on structural features of Khayyām's treatise, and then concentrate on the Archimedean Problem as it is formulated and solved by Khayyām. In section 4., I look at Khayyām's treatment in light of Archimedes', and suggest a possible account for the difference between Archimedes and Khayyām.

2. The Archimedean Problem

2.1. The Archimedean Problem Obtained

In his second *Book on Sphere and Cylinder*, Archimedes offers a series of problems concerning spheres. The goal is to produce spheres, or segments of spheres, defined by given geometrical equalities or ratios. In Proposition 4, the problem is that of cutting a sphere so that its segments stand to each other in a given ratio. For instance, we know that to divide a sphere into two equal parts, the solution is to divide it along the middle, or, in other words, at the middle of the diameter. But what if want to have, say, one segment twice the other? Cutting it at two-thirds the diameter is clearly not the answer, and the question is seen to be non-trivial, for two separate reasons: it involves solid figures, and it involves curvilinear figures – both difficult to handle by simple manipulations of lines.

However, a dirction forwards suggests itself. The two segments of

(1 Notice that this article is restricted to a *comparative* methodology. I put side by side two solutions, by Archimedes and Khayyām, and make comparative remarks. I ignore the *historical* question of the way in which the Archimedean Problem was transmitted and gradually transformed, in late antiquity and in the middle ages. For all those questions, see R. Netz (1999), and, for some further bibliographic references, R. Rashed (1999). To clarify, however: Khayyām had many Arabic antecedents, some of whom he even mentions explicitly. Indeed, among the historical issues ignored here is a brief mention by Khayyām himself to the same problem in the earlier treatise *On the Division of the Quadrant*. This article merely sets two marks on the road from problems to equations – a road I hope to map it in detail in the future.

‘Omar Khayyām and Archimedes:

How does a Geometrical Problem become a Cubic Equation?*

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1. Introduction

Works such as Archimedes' *Sphere and Cylinder* and ‘Omar Khayyām's *Algebra* are among the greatest achievements of humankind. Arguably, they belong to the sphere of "genius", not so much to the sphere of history: the overpowering individualities of Archimedes and of Khayyām seem to defy any historical labeling. Still, it is by mighty logs, not by specks of dust, that we learn of the flow of rivers. And the direction of the river of mathematics, from Archimedes to Khayyām and onwards, seems to be well known: starting already in antiquity itself and reaching early modern Europe, Hellenistic Greek geometric science is transformed into what we call algebra.

What was this process like? Using our two "logs", this question can be put in more precies terms. Khayyām's *Algebra* can be said to have its origins in a certain problem put forward and solved by Archimedes, in an appendix to his second *Book on the Sphere and the Cylinder*. This

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