which it is the origional relation in the arithmetical triangle.
Nasir al-Din has never told about the discoveror of the Triangle and the approximate method for the extraction of roots and presumably he has borrowed it from the prior mathematicians.
Since this is the same method that Khayyam has claimed to discover it, we can conclude that Nasir-al-Din has borrowed it from Khayyam's method.[7]
Also, considering that Khayyam was completely familiar with the "arithmetical triagle" and used it for the extraction of roots-the same method which became common in Europe after Khayyam-so it is reasonable to call this triangle "al-Kharaji-Khayyam's triangle".

(7 Youschkevitch, having studied Nasir al-Din's book had come to the same conclusion; he affirms that, Nasir-al-Din has probably borrowed the arithmetical triangle from Khayyam and says that:
"Al-Tusi ne pretend en aucune manière avoire decouvert lui-meme toutes ces formules. Comme il lui est arrive de developper dans certains cas des idees d'al-Khayyam, on peut supposer qu'il s'est inspire ici de l'ouvrage de ce dernier...".
(A. P. Youschkevitch, A. P. 1976. *Les Mathematiques arabes* (VIII-VX siecles). Paris. 1976 p. 80)

be; and no one has done this before"[6]
The book, in which Khayyam pointed out, has not been obtained yet, but from Nasir-al-Din's treatise on arithmetics, we can easily understand what Khayyam means.
It is nessesary to say that Nasir-al-Din was completely familiar with Khayyam's books and he has often described or developed his viewpoints.
Therefor, presumably he has used Khayyam's missing treatise in his own thesis about calculus.
In his treatise, ***Javame'al-Hisab bi'l Takht va'l Turab,*** in order to find the root of $\sqrt{N}$-where N is an integer number-he takes:

$$N = a^n + r$$

where n has no integer root and r shoud satisfy the following inequality:

$$r < (a+1)^n - a^n$$

Therefore, the approximate root of $\sqrt[n]{N}$ is:

$$\sqrt[n]{N} \approx a + \frac{r}{(a+1)^n - a^n}$$

Then Nasir-al-Din for calculatig $(a+1)^n - a^n$ provided a table of binomial coefficients up to $n = 12$ which is the same 'arithmetical triangle".
He describes this figure with its all detailes and even talkes about the relation between numbers in it.

Using modern notation, it is shown:

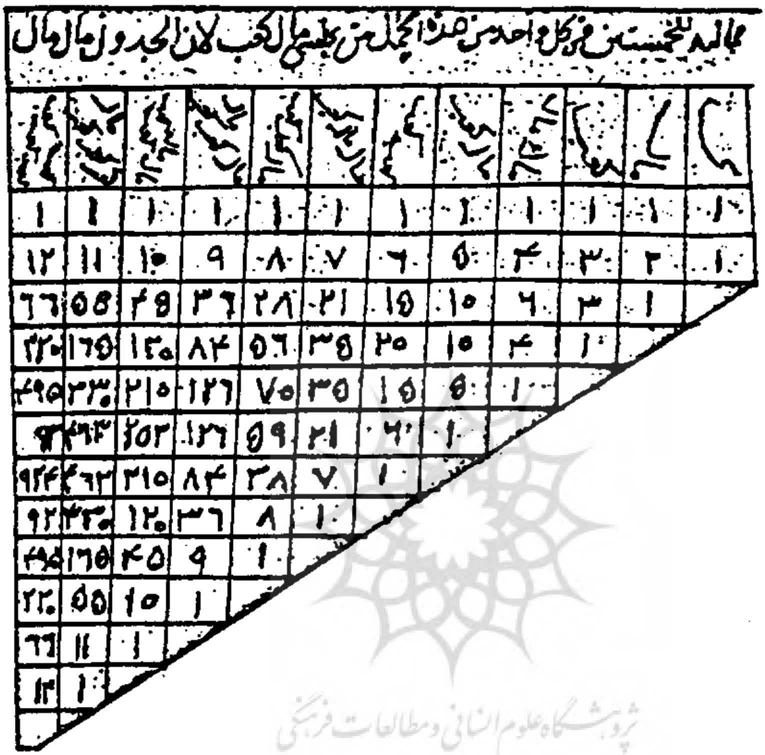
$$\binom{m}{n} = \binom{m}{n-1} + \binom{m-1}{n-1}$$

(6 H. J. J. Winter & w. Arafat, "The Algebra of 'Umar Khayyam". *Journal of the Royal Asiatic of Bengal Science.* Vol. XVI, No. 1 (1950). p. 34
The text of Khayyam is:

«...وللهند طرق فی‌استخراج اضلاع المربعات وللکمبات مبنیةٌ علی استقراء قلیل، و هو معرفه مربعات الصورالتسعة، اعنی مربع‌الواحدوالاثنین والثلاثه، و کذلک مضروب بعضها فی بعض، اعنی مضروب‌الاثنین فی‌الثلاثه و نحوها.ولنا کتاب فی‌البرهان علی صحة تلک‌الطرق و تادیتها الی‌المطلوبات. و قد غزرنا انواعها، اعنی من‌استخراج اضلاع مال‌الکعب و کعب‌الکعب، بالغا ما بلغ، و لم نسبق‌الیه...»

(Rashed, R. et Vahabzadeh, B. *Al-Khayyam Mathematicien.* Paris: Blanchard, 1999 p. 131.)

used in as-Samawal's treatise.



This is not the end of the history of the arithmetical triangle, because Omar Khayyam, living a century after al-Karaji, claims also the discovery of a method for extraction of roots with positive integer exponents and in his treatise on algebra, says that:

"The Hindus have their own methods for extracting the sides of squares and cubes based on a little calculation, which is the knowlage of the squares of 9 integers, i. e., the squares of 1, 2, and 3, etc., and of their products into each other, i. e., the product of 2 with 3, etc. I have written a book to prove the validity of those methods and to show that they lead to the required solutions, and I have supplemented it with their kinds, i. e., finding the sides of the square of the square, and the square of the cube, and the cnbe of the cube, however great they may

Using the combinatorial analysis, he found:

$$\binom{n}{x} = \binom{n-1}{x-1} + \binom{n-2}{x-1} + \cdots + \binom{x-1}{x-1}$$

and as mentioned before, this is one of the most basic relations in the arithmetical triangle.
Substiuting different numbers in it, first it was Ibn-Munim who discovered the arithmetical triangle using the combinatorial analysis.[4]
(2)-**Mathematics:**
Second group of Islamic scientists-mainly al-Karaji-were mathematicians who obtained the arithmetical triangle by the calculation of the coefficients of binomials.
Unfortunatly we have not found any treatise of al-Karaji about the description of his method, but presumably it is the same treatise, *Fi Hisab-al-Hind*, which he has pointed out, in his book, *al-Badi fi'l Hisab.*
Fortunatly Moroccan mathematician as-Samaw'al (D. 1149A. C.)-in, *al-Bahir fi'l ilm-al-Hisab*, expresses the construction method of the arithmetical triangle.
Saying nothing about al-Karaji's treatise, he uses it to determine the coefficients in the expansion of $(a+b)^n$.
It is nesessary to say that Karaji's instruction for the expansion of $(a+b)^n$ is:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{2} a^{n-r} b^r$$

which is based on relation:

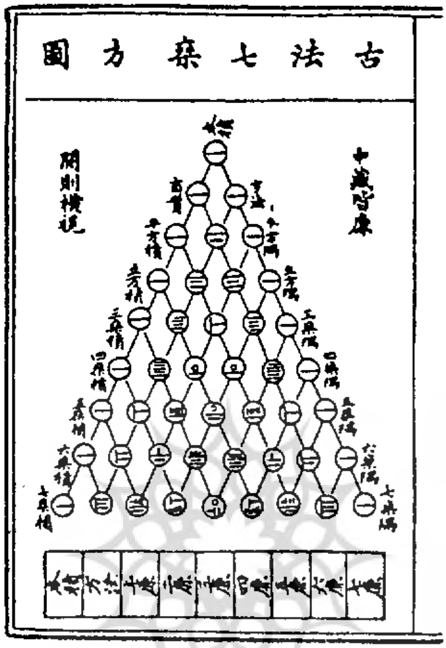
$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

and as seen, this is one of the most important relations in the arithmetical triangle.[5] The arithmetical triangle, shown in the following figure,

(4 A. Djebbar, L'analyse combinatoire au Maghreb entre le XII et le XIV siecle", *Cahiers d'histoire et de philosophie des sciences* [Nouvelle serie] n 20 1987 p. 232.

(5 R. Rashed, ≪L'induction mathematique: al-Karaji, as-Samaw'al≫, *Archive for History of Exact Sciences*, vol. 9, n° 1, pp. 5-7

There is another figure of the Triangle found in *Valuable Mirror of Four Elments* written by Chu Shi-Kie. -Chiness mathematician in 14th century-.



In this chapter, let us describe that how Islamic scientists discovered the "arithmetical triangle".

This figure appeared in Moslem World by two different traditions, those are: Linguistics and Mathematics.

(1)-**Linguistics:**

The discoveror of this method is a famous linguist, Khalil-ibn-Ahmad (D. 749 A. C.).

He tried a lot to find the laws of meter and rhyme in Arabic poem. Describing his results in, *Kitab al-aiyn,* he did some calculations, which we can call them, the combinatorial analysis in modern terminology. However Khalil did not discover the arithmetical triangle, but opened the way for further research in subject.[3]

For instance, we can name the Morocan mathematician, Ibn-Munim (D. ?), who continued Khalil's works in his book, *Feq-al-Hisāb.*

(3 R. Rashed. "Algèbre et linquistique: l'analyse combinatoire dans la science Arabe", *Philosophical Fondation of Science,* Hollande. 1974 pp. 396-397

thematical books is known as "Pascal's triagle".
Because "Delegue" submitting his thesis, *Proof of Newton's binomial by Pascal,* in the University of Sorbonne in 1869 A. C., discussed about Pascal's treatise which was about the arithmetical triangle and used it in order to determine the coefficients of the binomial expansion.[1]
Since that time this triangle is called Pascal's triangle.
But soon later historians of sciences found that the Triangle was discussed not only in noneuropeans' mathematical works but also in Europeans' living before Pascal. -Such as Mechaeel Stifel (D. 1567 A. C.)
Stifel, in his book, *Arithmetica Integra,* pointed out to the Triangle and used it in the arithmetical calculations.
According to Bosmans' researches, Pascal was familiar with Stifel's book and used it in his treatise which was about the arithmetical triangle.
In fact the only novel work, which he did, was the use of the triangle in the calculation of probablities.[2]
The arithmetical triangle shown in the following figure used in Pascal's thesis.

	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	
3	1	3	6	10	15	21	28	36		
4	1	4	10	20	35	56	84			
5	1	5	15	35	70	126				
6	1	6	21	56	126					
7	1	7	28	84						
8	1	8	36							
9	1	9								
0	1									

(1 Delègue. *Démonstration de la formula du binom de Newton d'après Pascal.* Paris 1869.

(2 Bosmans, H. "Note historique sur triangle arithmétique dit de Pascal", *Annales de la Société scientifique de Bruxells,* vol. 31 (1906), p. 72.

hand, taking in account the relation:

$$\binom{n}{1} = \underbrace{1 + 1 + \cdots + 1}_{\text{sum of } n \text{ terms}} = n$$

then we can write:

$$\binom{n}{2} = \underbrace{(n-1) + (n-2) + \cdots + 1}_{\text{sum of (n-1) terms}} = \frac{n(n-1)}{2} \qquad (5)$$

considering $\binom{n}{1}$ is equal to $n/1$ and knowing tht $n/1$ multiplied by (n − 1)/2 yields $\binom{n}{2}$. That is:

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

Similarly for $\binom{n}{3}, \cdots$ by mathematical induction, we have:

$$\binom{n}{3} = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$$

$$\binom{n}{x} = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \cdots \times \frac{n-x+1}{x}$$

or:

$$\binom{n}{x} = \frac{n(n-1)\cdots(n-x+1)}{x!}$$

and if we multiply both numerator and denominator of the right side of this equality by (n − x)! gives:

$$\binom{n}{x} = \frac{n!}{x!(n-x)!} \qquad (6)$$

(b) **History of the Arithmetical Triangle:**
As mentioned before the "arithmetical tiangle" among Europeans' ma-

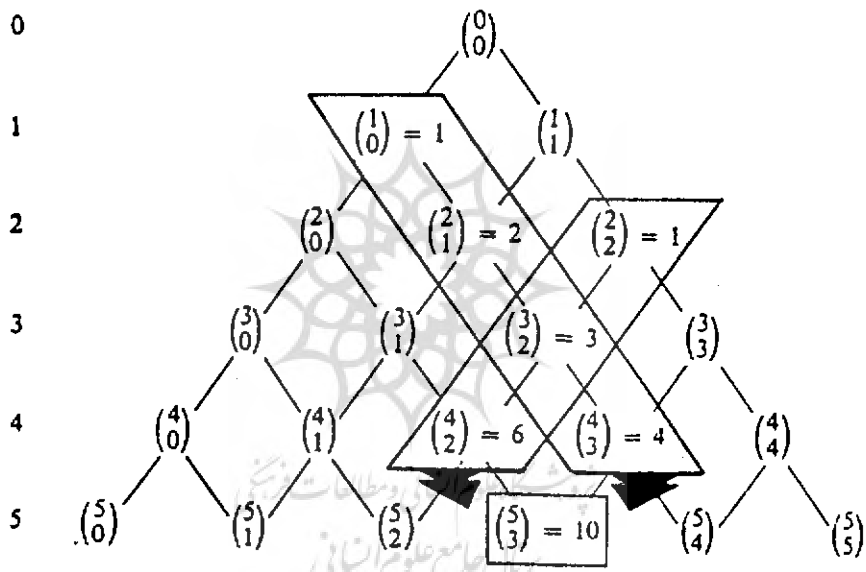
which states that each term (number) in the arithmetical triangle can be obtainded by two methods.

-Example:

If n = 5, x = 4, using relation (4) for (5, 3), we have:

$$\binom{4}{3}+\binom{3}{2}+\binom{2}{1}+\binom{1}{0}=\binom{4}{2}+\binom{3}{2}+\binom{2}{2}$$

The plot of the first six terms of the arithmetical triangle explains the relation (4) very clearly.



Multiplications of Numbers in the Arithmetical Triangle:

When n = k, the relation (3) takes the form:

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-2}{k-1}+\cdots+\binom{0}{0}$$

If k = 2, we have:

$$\binom{n}{2}=\binom{n-1}{1}+\binom{n-2}{1}+\cdots+\binom{1}{1}$$

Note that $\binom{1}{1}$ and $\binom{m}{1}$ are equal to 1 and m respectively, on the other we have:

$$\binom{n}{x} = \binom{n}{n-x}$$

(3): With regard the formation law of the arithmetical triangle, each term (number) is the sum of its two adjacent upper numbers, that is:

$$\binom{n}{x} = \binom{n-1}{x} + \binom{n-1}{x-1} \tag{1}$$

In the same way, for the other terms (numbers) of triangle, we have:

$$\binom{n-1}{x-1} = \binom{n-2}{x-1} + \binom{n-2}{x-2}$$

$$\binom{n-2}{x-2} = \binom{n-3}{x-2} + \binom{n-3}{x-3}$$

Now, owing to the properties of combinatorial analysis, two sides of these equalities can be added to yield:

$$\binom{n}{x} = \binom{n-1}{x} + \binom{n-2}{x-1} + \binom{n-3}{x-2} + \cdots + \binom{n-x+1}{0} \tag{2}$$

Considering relation (1), it is clear that we can define relations as followes:

$$\binom{n-1}{x} = \binom{n-2}{x} + \binom{n-2}{x-1}$$

$$\binom{n-2}{x} = \binom{n-3}{x} + \binom{n-3}{x-1}$$

Using the combinatorial analysis, we have:

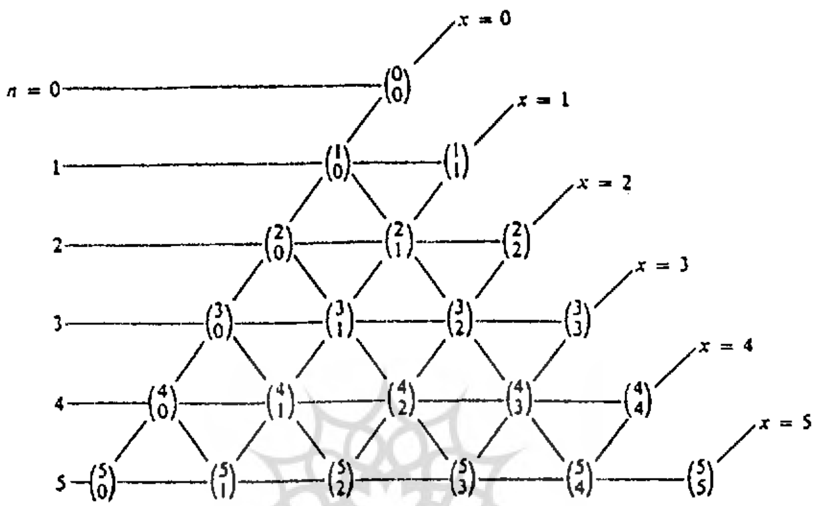
$$\binom{n}{x} = \binom{n-1}{x-1} + \binom{n-2}{x-1} + \cdots + \binom{n-1}{x-1} \tag{3}$$

comparing relations (2) and (3), gives:

$$\binom{n-1}{x} + \binom{n-2}{x-1} + \cdots + \binom{n-x+1}{0} =$$

$$\binom{n-1}{x-1} + \binom{n-2}{x-1} + \cdots + \binom{x-1}{x-1} \tag{4}$$

six terms of mentioned binomial) we can write:



taking in account the properties of numbers in the "arithmetical triangle", we have:

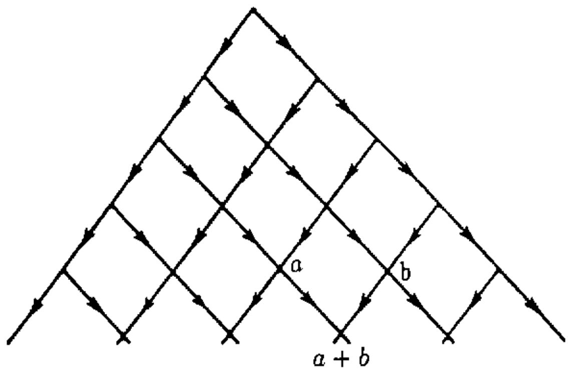
$$\binom{n}{x}=\binom{n-1}{x}+\binom{n-1}{x-1}$$

Summing of Numbers in the Arithmetical Triangle:
Supposing the construction method of the triangle, we can define some important properties of numbers on (n, x):

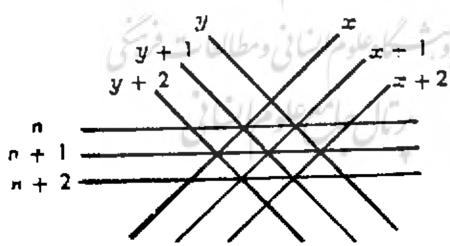
(1): The sum of n terms on the horizontal lines is 2^n and given by:

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$$

(2): Since the arithmetical triangle is symmetric with respect to (0, 0), first six terms of $(A + B)^n$, is:



Now, if we consider the lines, having the same direction of arrows and direccted to the right as "y" lines and the lines which are monodirectional with arrows and directed to the left as "x" lines and finaly if we call the horizontal lines passing through the intersection of the lines as n, we get:



Similarly using "y" and "x" lines and n, we can easily determine the position of each number in the arithmetical triangle.
For instance, any number in the triangle by coordinates (n, x) is th intersection point of the one of "x" lines with "y" lines, where "x" lies between 0 and n.
≪n≫ denotes the number of the raws in the arithmetical triangle.
Accodingly, for the first six raws of the arithmetical triangle (the first

the role of Islamic scientists, especialy al-Karaji and Khayyam in finding and use of this triangle.

(a) Formation Method of the Arithmetical Triangle:

Before talking about the history of the arithmetical triangle let us express its construction method and some of its important properties to those who do not have enough information in subject.

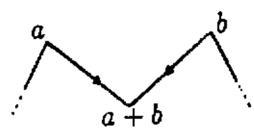
Consider binomial $(A+B)^n$, where $n \geq 0$, for different values of n, we have:

$$\begin{aligned}
(A+B)^0 &= 1\\
(A+B)^1 &= A+B\\
(A+B)^2 &= A^2+2AB+B^2\\
(A+B)^3 &= A^3+3A^2B+3AB^2+B^3\\
(A+B)^4 &= A^4+4A^3B+6A^2B^2+4AB^3+B^4\\
(A+B)^5 &= A^5+5A^4B+10A^3B^2+10A^2B^3+5AB^4+B^5
\end{aligned}$$

Writing the coefficients of each binomial in the same order, the "arithmetical triangle" is formed:

$$\begin{array}{c}
1\\
1 \quad 1\\
1 \quad 2 \quad 1\\
1 \quad 3 \quad 3 \quad 1\\
1 \quad 4 \quad 6 \quad 4 \quad 1\\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1\\
\vdots
\end{array}$$

It is obvious that, each number in it is the sum of two adjacent numbers above it in the preceding row. That is:



Therefore, the arithmetical triangle obtainded from the expansion of the

Omar Khayyam and the Arithmetical Triangle

Jafar Aghayani Chavoshi
Epistemologist and Historian of Sciences
Faculty of Philosophy of Sciences
Sharif University of Technology

Translated by: Maryam Boozarjmehr
Faculty of Physics
University of Tabriz

Introduction:

To those who are concerned with the calculus of probability and the combinatorial analysis, the arithmetical triangle, so called "Pascal's triangle", is a familiar name. The discoveror of this triangle is not exactly known, but it can be found in Islamic, European and Chiness literature. According to the most ancient handwrittings we find that al-Karaji (Died 999A. C.) was the first Islamic mathematician who used the arithmetical triangle in order to find the coefficients of the binomial expansion.

Omar Khayyam, living a century after al-Karaji, clamied that he has found a novel method for the extraction of roots with positive integer numbers.

Has he rediscovered this triangle independent of al-Karaji?

However we have not found any book of Khayyam in subject, but considering his claim and also the arithmetical work of Nasir al-Din-al-Tusi (D. 1251 A. C.) the historians of sciences conclude that Khayyam has formed the arithmetical triangle indepependent of al-Karaji and has used it for the extraction of roots with positive integer numbers.

In this paper we will argue that Pascal (D. 1662 A. C.) was not the first who discovered the arithmetical triangle, meanwhile, we will introduce